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Chaohui Chen*, Jiarong Peng, and Tianyuan Chen

The A_α -spectral radius of complements of bicyclic and tricyclic graphs with n vertices

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Abstract: Recently, the extremal problem of the spectral radius in the class of complements of trees, unicyclic graphs, bicyclic graphs and tricyclic graphs had been studied widely. In this paper, we extend the largest ordering of A_α -spectral radius among all complements of bicyclic and tricyclic graphs with n vertices, respectively.

Keywords: Adjacency spectrum; A_α -matrix; A_α -spectral radius; extremal graph

MSC: 05C50; 15A18

1 Introduction

Throughout this paper, we only concern with simple undirected graphs. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, and let $d_G(v_i)$ be the degree of vertex v_i of G . We use $\overline{G}$ and $\Delta(G)$ to denote the complement and maximum degree of graph G , respectively. The adjacency matrix and diagonal matrix of graph G is denoted by $A(G)$ and $D(G)$, respectively. The A_α -**matrix** $A_\alpha(G)$ of graph G is denoted by $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$, where $\alpha \in [0, 1]$. The maximum eigenvalue of $A_\alpha(G)$ is called the A_α -**spectral radius** of G , and denoted by $\rho_\alpha(G)$. In particular, $\rho(G) = \rho_0(G)$ is equal to the spectral radius (maximum eigenvalue) of $A(G)$. If G has the maximum A_α -spectral radius among all graphs of a given graph category $\mathcal{G}$, then we call G as a ρ_α -**maximal graph** of $\mathcal{G}$.

As usual, let P_n and $K_{1,n-1}$ be the path and star with n vertices, respectively. If G has no $2P_2$, P_4 or C_4 (cycle with four vertices) as an induced subgraph, then G is called a **split graph** (see [6]). Let $T_1, T_2, U_1, U_2, B_1, B_2, B_3, B_4, S_1, S_2, \dots, S_7$ be the 15 graphs with n vertices as shown in Figure 1.1, and let $\mathbb{G}(n; c)$ be the class of graphs, each of which is the complement of a c -cyclic graphs with n vertices, that is,

$$\mathbb{G}(n; c) = \{\overline{G} : G \text{ is a } c\text{-cyclic graph with } n \text{ vertices}\}.$$

Denoted by $\overline{\mathbb{G}}(n, k; c)$ the class of graphs, each of which is the complement of a c -cyclic graphs with n vertices and maximum degree k , that is,

$$\overline{\mathbb{G}}(n, k; c) = \{\overline{G} : G \text{ is a } c\text{-cyclic graph with } n \text{ vertices and maximum degree } k\}.$$

Recently, the extremal problem of the spectral radius in the class of complements of trees, unicyclic graphs, bicyclic graphs and tricyclic graphs had been studied in [2, 7, 8]. It had been shown that

Theorem 1.1. *Let $\overline{G}$ be a ρ -maximal graph of $\mathbb{G}(n; c)$.*

***Corresponding Author: Chaohui Chen:** Department of Mathematics, South China Agricultural University, Guangzhou, 510642, China, E-mail: cch956833210@163.com

Jiarong Peng, Tianyuan Chen: Department of Mathematics, South China Agricultural University, Guangzhou, 510642, China
The three authors contribute equally to this paper.

- (1) [2] If $c = 1$, then $G = U_1$;
 (2) [7] If $c = 2$ and $n \geq 5$, then $G = B_1$;
 (3) [8] If $c = 3$ and $n \geq 8$, then $G = S_1$.

Later, the unique ρ_α -maximal graph among the class of complements of trees, unicyclic graphs and bicyclic graphs was determined by Zhang and Guo [9], that is,

Theorem 1.2. [9] Let $0 \leq \alpha < 1$ and $\overline{G}$ be a ρ_α -maximal graph of $\overline{\mathbb{G}}(n; c)$.

- (1) If $c = 0$, then $G = K_{1, n-1}$;
 (2) If $c = 1$, then $G = U_1$;
 (3) If $c = 2$ and $n \geq 5$, then $G = B_1$.

From Theorems 1.1 and 1.2, we can easily see that each of those extremal maximal graphs is the complement of a graph with maximum degree $n - 1$. In [3], the present authors and their cooperators extended this fact to the general cases.

Theorem 1.3. [3] Let $0 \leq \alpha < 1$, $\overline{G}_1$ and $\overline{G}_2$ be a ρ_α -maximal graph of $\overline{\mathbb{G}}(n, s; c)$ and $\overline{\mathbb{G}}(n, t; c)$, respectively. If $0 \leq c \leq n - 4$ and $2 \leq s < t \leq n - 1$, then $\rho_\alpha(\overline{G}_2) > \rho_\alpha(\overline{G}_1)$.

Based on Theorem 1.3, the present authors and their cooperators improved Theorem 1.2 (1) and (2) to the following Theorem 1.4, and they also determined the ρ_α -maximal graphs in the class of complements of tricyclic graphs, that is, Theorem 1.5.

Theorem 1.4. [3] Let $0 \leq \alpha < 1$ and $n \geq 4$.

- (1) If $\overline{G} \in \overline{\mathbb{G}}(n; 0) \setminus \{\overline{T}_1, \overline{T}_2\}$, then $\rho_\alpha(\overline{G}) < \rho_\alpha(\overline{T}_2) < \rho_\alpha(\overline{T}_1)$;
 (2) If $\overline{G} \in \overline{\mathbb{G}}(n; 1) \setminus \{\overline{U}_1, \overline{U}_2\}$, then $\rho_\alpha(\overline{G}) < \rho_\alpha(\overline{U}_2) < \rho_\alpha(\overline{U}_1)$.

Theorem 1.5. [3] Let $0 \leq \alpha < 1$, $n \geq 7$ and $\overline{G}$ be a ρ_α -maximal graph of $\overline{\mathbb{G}}(n; 3)$.

- (1) If $0 \leq \alpha < 0.5$, then $G = S_1$;
 (2) If $\alpha = 0.5$, then $G \in \{S_1, S_2\}$ and $\rho_{0.5}(\overline{S}_1) = \rho_{0.5}(\overline{S}_2)$;
 (3) If $0.5 < \alpha < 1$, then $G = S_2$.

By an observation to the above results, it is an interesting problem that: Is it difficult or not to extend those corresponding ordering? Actually, as shown in the later, different from the ordering of largest spectral radius of trees, unicyclic graphs, bicyclic graphs and tricyclic graphs [5], we find the solution to this problem is rather difficult. The main results of this paper are as follows:

Theorem 1.6. Let $0 \leq \alpha \leq 0.87$ and $n \geq 23$. If $\overline{G} \in \overline{\mathbb{G}}(n; 2) \setminus \{\overline{B}_1, \overline{B}_2, \overline{B}_3\}$, then

$$\rho_\alpha(\overline{G}) < \rho_\alpha(\overline{B}_3) < \rho_\alpha(\overline{B}_2) < \rho_\alpha(\overline{B}_1).$$

Theorem 1.7. Let $\overline{G} \in \overline{\mathbb{G}}(n; 3) \setminus \{\overline{S}_1, \overline{S}_2, \dots, \overline{S}_6\}$. If $n \geq 35$ and $0 \leq \alpha \leq 0.8$, then

$$\rho_\alpha(\overline{G}) < \rho_\alpha(\overline{S}_6) < \rho_\alpha(\overline{S}_5) < \rho_\alpha(\overline{S}_4) < \rho_\alpha(\overline{S}_3) < \min\{\rho_\alpha(\overline{S}_2), \rho_\alpha(\overline{S}_1)\}.$$

The condition $\alpha \leq 0.87$ of Theorem 1.6 is necessary, as we have $\rho_\alpha(\overline{B}_3) > 20.86307 > 20.86298 > \rho_\alpha(\overline{B}_2)$ when $n = 23$ and $\alpha = 0.88$. Furthermore, the condition $0 \leq \alpha \leq 0.8$ of Theorem 1.7 is also necessary, as we have $\rho_\alpha(\overline{S}_6) > 32.8447 > 32.844 > \rho_\alpha(\overline{S}_5)$ if $n = 35$ and $\alpha = 0.81$.

2 The proof of Theorem 1.6

The following lemma will play an important role in the proof of Theorem 1.6.

Figure 1.1: The graphs $T_1, T_2, U_1, U_2, B_1, B_2, B_3, B_4, S_1, S_2, \dots, S_7$ with n vertices.

Lemma 2.1. [3] Let $\bar{G}$ be a ρ_α -maximal graph of $\bar{\mathbb{G}}(n, n-2; c)$ with $d_G(u_0) = n-2$, $u_0v_0 \notin E(G)$ and $S_0 = V(G) \setminus \{u_0\}$. If $0 \leq c \leq n-4$, then $G[S_0]$ has unique component H_0 with at least two vertices such that H_0 is a split graph with $c+1$ edges. Furthermore, if $d_{H_0}(v_0) \geq 2$, then x is adjacent to every vertex of $V(H_0)$ for each vertex $x \in N_{H_0}(v_0)$.

By an elementary computation with the application of Equitable partitions (see [1]), $\rho_\alpha(\bar{B}_2)$, $\rho_\alpha(\bar{B}_3)$ and $\rho_\alpha(\bar{B}_4)$ are equal to the maximum root of $\Phi_1(x)$, $\Phi_2(x)$ and $\Phi_3(x)$, respectively, where

$$\begin{aligned}\Phi_1(x) &= x^2 - (\alpha(n-1) + n-4)x + (n^2 - 3n - 2)\alpha + 8 - 2n, \\ \Phi_2(x) &= x^5 - ((3\alpha+1)n - 6\alpha - 5)x^4 + (3\alpha(\alpha+1)n^2 - (11\alpha^2 + 18\alpha + 3)n + (2\alpha+1)(2\alpha+1))x^3 \\ &\quad - (\alpha^2(\alpha+3)n^3 - 2\alpha(2\alpha^2 + 10\alpha + 3)n^2 - (3\alpha^3 - 29\alpha^2 - 41\alpha + 1)n + 6\alpha^3 + 24\alpha^2 - 78\alpha + 7)x^2 \\ &\quad + (\alpha^3n^4 + \alpha^2(\alpha^2 - 6\alpha - 3)n^3 - \alpha(7\alpha^3 + 5\alpha^2 - 29\alpha + 2)n^2 + (10\alpha^4 + 68\alpha^3 - 73\alpha^2 + \alpha + 4)n \\ &\quad - (\alpha+1)(58\alpha^2 - 61\alpha + 19))x - (n-5)((n-2)\alpha^2 + \alpha - 1)((n^2 - 3n - 2)\alpha^2 - (2n-9)\alpha - 1), \\ \Phi_3(x) &= x^4 - ((2\alpha+1)n - 3\alpha - 5)x^3 + (\alpha(\alpha+2)n^2 - (2\alpha^2 + 10\alpha + 3)n - 2\alpha^2 + 8\alpha + 11)x^2 \\ &\quad - (\alpha^2n^3 + \alpha(\alpha^2 - 4\alpha - 3)n^2 - (4\alpha^3 + 7\alpha^2 - 20\alpha + 1)n + 34\alpha^2 - 37\alpha + 5)x \\ &\quad + (n-4)(\alpha^3n^2 - \alpha(3\alpha^2 + \alpha + 1)n - 2\alpha^3 + 9\alpha^2 - 3\alpha + 2).\end{aligned}$$

Lemma 2.2. If $n \geq 23$ and $0 \leq \alpha \leq 0.87$, then $\rho_\alpha(\bar{B}_3) < \rho_\alpha(\bar{B}_2)$.

Proof. From the formulas of $\Phi_1(x)$ and $\Phi_2(x)$, it follows that

$$\Phi_2(x) = \Phi_1(x)\Psi_1(x) + Y_1(x), \quad (2.1)$$

where $\Psi_1(x) = x^3 - ((2n-5)\alpha - 1)x^2 + (\alpha^2n^2 - \alpha(4\alpha+1)n - \alpha^2 + 5\alpha - 1)x - \alpha^3n^2 + (6\alpha^3 + 3\alpha^2 - 4\alpha + 2)n - 5\alpha^3 - 15\alpha^2 + 21\alpha - 11$ and $Y_1(x) = (1-\alpha)\Psi_2(x)$ and $\Psi_2(x) = ((\alpha-1)(\alpha-2)n^2 + (\alpha^3 - 9\alpha^2 + 23\alpha - 17)n - 5\alpha^3 + 20\alpha^2 - 40\alpha + 33)x - \alpha(\alpha-1)(\alpha-2)n^3 + (8\alpha^3 - 21\alpha^2 + 11\alpha + 4)n^2 - (13\alpha^3 - 14\alpha^2 - 26\alpha + 39)n - 10\alpha^3 + 80\alpha^2 - 147\alpha + 93$.

By (2.1), to complete the proof of this lemma, it suffices to show the following two claims:

Claim 1. If $x \geq \rho_\alpha(\overline{B_2})$, then $Y_1(x) > 0$.

Proof. Define $\varphi_1(n) = (\alpha - 1)(\alpha - 2)n^2 + (\alpha^3 - 9\alpha^2 + 23\alpha - 17)n - 5\alpha^3 + 20\alpha^2 - 40\alpha + 33$. Since $\varphi_1''(n) = 2(\alpha - 1)(\alpha - 2) > 0$, $\varphi_1'(n) \geq \varphi_1'(23) = \alpha^3 + 37\alpha^2 - 115\alpha + 75 = \varphi_2(\alpha)$. By Matlab, the two smallest roots of $\varphi_2(\alpha)$ are about -39.9273 and 0.95 , we can conclude that $\varphi_1'(n) \geq \varphi_2(\alpha) > 0$, as $0 \leq \alpha \leq 0.87$. Thus, $\varphi_1(n) \geq \varphi_1(23) = 2(9\alpha^3 + 171\alpha^2 - 549\alpha + 350) = \varphi_3(\alpha) > 0$, as the two smallest roots of $\varphi_3(\alpha)$ are about -21.8705 and 0.9044 .

Since $\Phi_1(n - 3) = (n - 5)(\alpha - 1) < 0$, we have $\rho_\alpha(\overline{B_2}) > n - 3$. When $x \geq n - 3$, since $\varphi_1(n) > 0$, we have

$$\Psi_2(x) \geq \Psi_2(n - 3) = \varphi_4(n),$$

where $\varphi_4(n) = (2 - \alpha)(1 - \alpha)^2 n^3 - (1 - \alpha)(9\alpha^2 - 24\alpha + 19)n^2 - (21\alpha^3 - 61\alpha^2 + 83\alpha - 45)n + 5\alpha^3 + 20\alpha^2 - 27\alpha - 6$.

Since $\varphi_4'''(n) = (2 - \alpha)(1 - \alpha)^2 > 0$, we have $\varphi_4''(n) \geq \varphi_4''(23) = 2(1 - \alpha)(60\alpha^2 - 183\alpha + 119) > 0$, and thus $\varphi_4'(n) \geq \varphi_4'(23) = -1194\alpha^3 + 4891\alpha^2 - 6040\alpha + 2345 = \varphi_5(\alpha) > 0$, as the smallest root of $\varphi_5(\alpha)$ is about 0.9021 and $0 \leq \alpha \leq 0.87$. Thus, $\Psi_2(x) \geq \varphi_4(n) \geq \varphi_4(23) = 2(-3942\alpha^3 + 16317\alpha^2 - 20012\alpha + 7656) = \varphi_6(\alpha) > 0$, as the smallest root of $\varphi_6(\alpha)$ is about 0.8701 . Combining this with $Y_1(x) = (1 - \alpha)\Psi_2(x)$, we have $Y_1(x) > 0$, and thus Claim 1 holds. $\square$

Claim 2. If $x \geq \rho_\alpha(\overline{B_2})$, then $\Psi_1(x) > 0$.

Proof. When $x > n - 3$, since $\Psi_1''(x) = 2(3x + 5\alpha - 2n\alpha + 1) \geq 2((3 - 2\alpha)n + 5\alpha - 8) \geq 2(23(3 - 2\alpha) + 5\alpha - 8) = 2(61 - 41\alpha) > 0$, we have $\Psi_1'(x) \geq \Psi_1'(n - 3) = \varphi_7(n)$, where $\varphi_7(n) = (\alpha - 1)(\alpha - 3)n^2 - (4\alpha^2 - 21\alpha + 16)n - \alpha^2 - 25\alpha + 20$. Note that $\varphi_7''(n) = 2(\alpha - 1)(\alpha - 3) > 0$. Thus, $\varphi_7'(n) \geq \varphi_7'(23) = 42\alpha^2 - 163\alpha + 122 > 0$, which implies that $\varphi_7(n) \geq \varphi_7(23) = 436\alpha^2 - 1658\alpha + 1239 > 0$. Therefore,

$$\Psi_1(x) \geq \Psi_1(n - 3) = \varphi_8(n),$$

where $\varphi_8(n) = (1 - \alpha)^2 n^3 + (1 - \alpha)(\alpha^2 + 8\alpha - 8)n^2 + (6\alpha^3 + 14\alpha^2 - 44\alpha + 22)n - 5\alpha^3 - 12\alpha^2 + 51\alpha - 26$.

Since $\varphi_8'''(n) = 6(1 - \alpha)^2 > 0$, we have $\varphi_8''(n) \geq \varphi_8''(23) = 2(1 - \alpha)(\alpha^2 - 61\alpha + 61) > 0$, and thus $\varphi_8'(n) \geq \varphi_8'(23) = -40\alpha^3 + 1279\alpha^2 - 2482\alpha + 1241 = \varphi_9(\alpha) > 0$, as the smallest root of $\varphi_9(\alpha)$ is about 0.9733 . This implies that $\Psi_1(x) \geq \varphi_8(n) \geq \varphi_8(23) = -396\alpha^3 + 8774\alpha^2 - 16831\alpha + 8415 = \varphi_{10}(\alpha) > 0$, as the smallest root of $\varphi_{10}(\alpha)$ is about 0.9538 . Now, we can conclude that Claim 2 holds. $\square$

By $\Phi_2(x)$ and $\Phi_3(x)$, we can deduce the following Lemma 2.3.

Lemma 2.3. If $n \geq 23$ and $0 \leq \alpha \leq 0.87$, then $\rho_\alpha(\overline{B_4}) < \rho_\alpha(\overline{B_3})$.

Proof. By the formulas of $\Phi_2(x)$ and $\Phi_3(x)$, we have

$$\Phi_2(x) + Y_2(x) = (x - n + 2)\Phi_3(x), \text{ and } Y_2(x) = ((\alpha - 1)(n - 3) - 1)\Phi_3(x) + Y_3(x), \quad (2.2)$$

where $Y_2(x) = ((\alpha - 1)(n - 3) - 1)x^4 - ((2\alpha^2 - \alpha - 1)n^2 - (9\alpha^2 + \alpha - 7)n + 6\alpha^2 + 10\alpha - 10)x^3 + (\alpha(\alpha + 2)(\alpha - 1)n^3 - (5\alpha^3 + 12\alpha^2 - 11\alpha - 3)n^2 + (\alpha^3 + 34\alpha^2 - 7\alpha - 17)n + 6\alpha^3 - 14\alpha^2 - 25\alpha + 24)x^2 - (\alpha^2(\alpha - 1)n^4 + \alpha(\alpha^3 - 8\alpha^2 + 3\alpha + 3)n^3 - (7\alpha^4 - 8\alpha^3 - 29\alpha^2 + 27\alpha - 1)n^2 + (10\alpha^4 + 50\alpha^3 - 134\alpha^2 + 77\alpha - 5)n - 66\alpha^3 + 107\alpha^2 - 44\alpha - 1)x + \alpha^3(\alpha - 1)n^4 - \alpha(10\alpha^3 - 8\alpha^2 - 1)n^3 + (29\alpha^4 - 9\alpha^3 - 10\alpha^2 - \alpha - 2)n^2 - (16\alpha^4 + 58\alpha^3 - 75\alpha^2 + 30\alpha - 13)n - 20\alpha^4 + 116\alpha^3 - 137\alpha^2 + 74\alpha - 21$, and $Y_3(x) = \alpha(3\alpha - 1)x^3 - (\alpha(\alpha + 1)(3\alpha - 1)n - 14\alpha^2 + 8\alpha - 2)x^2 + (\alpha^2(3\alpha - 1)n^2 + (2\alpha^4 - 3\alpha^3 - 11\alpha^2 + 8\alpha - 2)n - 36\alpha^3 + 72\alpha^2 - 45\alpha + 11)x - (2\alpha - 1)(\alpha^2(\alpha + 1)n^2 - (3\alpha^3 + 4\alpha^2 + 4\alpha - 1)n - 2\alpha^3 + 3\alpha^2 + 16\alpha - 5)$.

Let β_1 be the maximum root of $Y_2(x)$.

Claim 3. $Y_2(n - 2.26) > 0$.

Proof. Let $6250000Y_2(n - 2.26) = 1625000(1 - \alpha)^3 n^4 - 2500(1 - \alpha)^2(1850\alpha^2 - 22869\alpha + 18757)n^3 - 50(1 - \alpha)(397500\alpha^3 - 7724750\alpha^2 + 12555294\alpha - 5285491)n^2 + (41250000\alpha^4 + 618672500\alpha^3 - 1835793650\alpha^2 +$

$1656883211\alpha - 481324711)n - 125000000\alpha^4 - 15715000\alpha^3 + 641079100\alpha^2 - 724756083\alpha + 225411222 = \varphi_1(n)$. Since $\varphi_1^{(4)}(n) = 39000000(1-\alpha)^3 > 0$, we have $\varphi_1'''(n) \geq \varphi_1'''(23) = -15000(1-\alpha)^2(1850\alpha^2 + 36931\alpha - 41043) > 0$, and thus $\varphi_1''(n) \geq \varphi_1''(23) = 100(1-\alpha)(5985000\alpha^3 + 25599200\alpha^2 - 75255594\alpha + 43728841) = 100(1-\alpha)\varphi_2(\alpha)$. Since the two smallest roots of $\varphi_2(\alpha)$ are -6.4149 and 1.0102 , we have $\varphi_2(\alpha) > 0$, as $0 \leq \alpha \leq 0.87$. This implies that $\varphi_1'(n) > 0$. Now, we have $\varphi_1'(n) \geq \varphi_1'(23) = -6384375000\alpha^4 + 8264505000\alpha^3 + 18841020050\alpha^2 - 37063869789\alpha + 16342407089 = \varphi_3(\alpha) > 0$, as the two smallest roots of $\varphi_3(\alpha)$ are -1.9871 and 0.9977 . Thus, $6250000Y_2(n-2.26) = \varphi_1(n) \geq \varphi_1(23) = -44934750000\alpha^4 + 152801172500\alpha^3 - 158999173550\alpha^2 + 37969822020\alpha + 13156757319 = \varphi_4(\alpha) > 0$, as the two smallest roots of $\varphi_4(\alpha)$ are -0.1821 and 0.9959 . $\square$

Claim 4. If $x \geq n-2$, then $Y_2(x) < 0$.

Proof. Since $Y_2^{(4)}(x) = 24((\alpha-1)(n-3)-1) < 0$, we have $Y_2'''(x) \leq Y_2'''(n-2) = -6(1-\alpha)(3-2\alpha)n^2 + (54\alpha^2 - 114\alpha + 54)n - 36\alpha^2 + 84\alpha - 36 = \varphi_5(n)$. Note that $\varphi_5''(n) = -12(1-\alpha)(3-2\alpha) < 0$. Thus, $\varphi_5'(n) \leq \varphi_5'(23) = -498\alpha^2 + 1266\alpha - 774 < 0$, and so $Y_2''(x) \leq \varphi_5(n) \leq \varphi_5(23) = -5142\alpha^2 + 13332\alpha - 8316 < 0$.

This follows that $Y_2'(x) \leq Y_2'(n-2) = -2(3-\alpha)(1-\alpha)^2n^3 + 2(1-\alpha)(5\alpha^2 - 22\alpha + 12)n^2 + (2\alpha^3 - 76\alpha^2 + 106\alpha - 34)n + 12\alpha^3 + 44\alpha^2 - 74\alpha + 24 = \varphi_6(n)$. Since $\varphi_6'''(n) = -12(3-\alpha)(1-\alpha)^2 < 0$, we have $\varphi_6''(n) \leq \varphi_6''(23) = 4(\alpha-1)(64\alpha^2 - 254\alpha + 195) < 0$, and thus $\varphi_6'(n) \leq \varphi_6'(23) = 2716\alpha^3 - 13462\alpha^2 + 19196\alpha - 8452 = \varphi_7(\alpha) < 0$, as the smallest root of $\varphi_7(\alpha)$ is about 1.0051 . Thus, $Y_2''(x) \leq \varphi_6(n) \leq \varphi_6(23) = 19102\alpha^3 - 94808\alpha^2 + 136730\alpha - 61064 = \varphi_8(\alpha) < 0$, as the smallest root of $\varphi_8(\alpha)$ is about 1.0099 .

Thus, $Y_2'(x) \leq Y_2'(n-2) = (1-\alpha)\varphi_9(n)$, where $\varphi_9(n) = -(1-\alpha)^2n^4 - (1-\alpha)(\alpha^2 + 8\alpha - 5)n^3 - (7\alpha^3 + 21\alpha^2 - 42\alpha + 17)n^2 + (10\alpha^3 + 52\alpha^2 - 98\alpha + 45)n + 3(1-\alpha)(14\alpha - 13)$. Since $\varphi_9^{(4)}(n) = -24(1-\alpha)^2 < 0$, we have $\varphi_9'''(n) \leq \varphi_9'''(23) = 6(\alpha-1)(\alpha^2 - 84\alpha + 87) < 0$, and thus $\varphi_9''(n) \leq \varphi_9''(23) = 124\alpha^3 - 5424\alpha^2 + 10986\alpha - 5692 = \varphi_{10}(\alpha) < 0$, as the smallest root of $\varphi_{10}(\alpha)$ is about 1.0136 . This implies that $\varphi_9'(n) \leq \varphi_9'(23) = 1275\alpha^3 - 38473\alpha^2 + 78539\alpha - 41470 = \varphi_{11}(\alpha) < 0$, as the smallest root of $\varphi_{11}(\alpha)$ is about 1.0293 . Therefore, $\varphi_9(n) \leq \varphi_9(23) = 8694\alpha^3 - 204627\alpha^2 + 421556\alpha - 227003 = \varphi_{12}(\alpha) < 0$, as the smallest root of $\varphi_{12}(\alpha)$ is about 1.0456 .

Now, we can conclude that $Y_2(x) \leq Y_2(n-2) = (1-\alpha)^2\varphi_{13}(n)$, where $\varphi_{13}(n) = -(1-\alpha)(6-\alpha)n^3 + (5\alpha^2 - 51\alpha + 37)n^2 + (4\alpha^2 + 96\alpha - 64)n - 20\alpha^2 - 32\alpha + 25$. Since $\varphi_{13}'''(n) = -6(1-\alpha)(6-\alpha) < 0$, we have $\varphi_{13}''(n) \leq \varphi_{13}''(23) = -128\alpha^2 + 864\alpha - 754 < 0$, and thus $\varphi_{13}'(n) \leq \varphi_{13}'(23) = -1353\alpha^2 + 8859\alpha - 7884 < 0$. This follows that $Y_2(x) \leq (1-\alpha)^2\varphi_{13}(n) \leq (1-\alpha)^2\varphi_{13}(23) = (1-\alpha)^2(-9450\alpha^2 + 60366\alpha - 54876) < 0$. $\square$

Claim 5. For $n \geq 24$ and $0.81 \leq \alpha \leq 0.87$, $n-2.16 < \beta_1 < n-2$.

Proof. Combining with Claim 4 and $(\alpha-1)(n-3)-1 < 0$, we just need to show that $390625Y_2(n-2.16) = 62500(1-\alpha)^3n^4 - 625(1-\alpha)^2(525\alpha^2 - 5191\alpha + 4298)n^3 + (1515625\alpha^4 - 24753125\alpha^3 + 60460700\alpha^2 - 52863625\alpha + 15640425)n^2 + 4(546875\alpha^4 + 9252500\alpha^3 - 27051725\alpha^2 + 24249539\alpha - 6999289)n - 7812500\alpha^4 + 560000\alpha^3 + 34870225\alpha^2 - 39924418\alpha + 12333237 = \varphi_{14}(n) > 0$. Since $\varphi_{14}^{(4)}(n) = 1500000(1-\alpha)^3 > 0$, we have $\varphi_{14}'''(n) \geq \varphi_{14}'''(24) = -3750(1-\alpha)^2(525\alpha^2 + 4409\alpha - 5302) > 0$, and thus $\varphi_{14}''(n) \geq \varphi_{14}''(24) = 50(1-\alpha)(884375\alpha^3 - 719300\alpha^2 - 1688728\alpha + 1529217) = 50(1-\alpha)\varphi_{15}(\alpha) > 0$, as the two smallest roots of $\varphi_{15}(\alpha)$ are about -1.4102 and 1.0124 . This implies that $\varphi_{14}'(n) \geq \varphi_{14}'(24) = -492062500\alpha^4 + 2133140000\alpha^3 - 3259493300\alpha^2 + 2081504156\alpha - 463096756 = \varphi_{16}(\alpha) > 0$, as the two smallest roots of $\varphi_{16}(\alpha)$ are about 0.5078 and 0.9987 . Now, we can conclude that $390625Y_2(n-2.16) = \varphi_{14}(n) \geq \varphi_{14}(24) = -3618312500\alpha^4 + 19817240000\alpha^3 - 36899932175\alpha^2 + 28750263326\alpha - 8049433707 > 0$. Thus, $n-2.16 < \beta_1 < n-2$. $\square$

Claim 6. If $0 \leq \alpha < \frac{113}{139} \approx 0.8129$, then $Y_3(n-2.26) > 0$.

Proof. It suffices to show that $125000Y_3(n-2.26) = \varphi_{17}(n) > 0$, where $\varphi_{17}(n) = 2500\alpha(1-\alpha)(113 - 139\alpha)n^2 + 50(3700\alpha^4 - 98857\alpha^3 + 183333\alpha^2 - 107838\alpha + 18700)n + 500000\alpha^4 + 9170000\alpha^3 - 19355391\alpha^2 + 12297797\alpha - 2455600$. Since $\varphi_{17}''(n) = 5000\alpha(1-\alpha)(113 - 139\alpha) > 0$, we have $\varphi_{17}'(n) \geq \varphi_{17}'(23) =$

$50(3700\alpha^4 + 220843\alpha^3 - 396267\alpha^2 + 152062\alpha + 18700) = \varphi_{18}(\alpha) > 0$, as the second and third roots of $\varphi_{18}(\alpha)$ are about -0.0971 and 0.8289 . Thus, $125000Y_3(n - 2.26) = \varphi_{17}(n) \geq \varphi_{17}(23) = 4755000\alpha^4 + 79311950\alpha^3 - 141792441\alpha^2 + 37726597\alpha + 19049400 = \varphi_{19}(\alpha) > 0$, as the second and third roots of $\varphi_{19}(\alpha)$ are about -0.2462 and 0.8432 . $\square$

Claim 7. $Y_3(\beta_1) > 0$.

Proof. Let $\varphi_{20}(\alpha) = \alpha(3\alpha - 1)$. From Claims 3 and 4, we have

$$n - 2.26 < \beta_1 < n - 2. \quad (2.3)$$

Case 1. $\alpha = 0$. Then $Y_3(x) = 2x^2 - (2n - 11)x + n - 5$. When $x \geq n - 2.26$, since $Y'_3(x) \geq Y'_3(n - 2.26) = 2n + 1.96 > 0$, we have $625Y_3(x) \geq 625Y_3(n - 2.26) = 4675n - 12278 > 0$. Since $\beta_1 > n - 2.26$ by (2.3), $Y_3(\beta_1) > 0$.

Case 2. $0 < \alpha < \frac{1}{3}$. We first show that $Y_3(n - 2) = (1 - \alpha)\varphi_{21}(n) > 0$, where $\varphi_{21}(n) = 2\alpha(1 - \alpha)n^2 - (2\alpha^3 - 35\alpha^2 + 35\alpha - 8)n + (1 - 2\alpha)(2\alpha^2 + 35\alpha - 19)$. Since $\varphi'_{21}(n) = 4\alpha(1 - \alpha) > 0$, we have $\varphi'_{21}(n) \geq \varphi'_{21}(23) = -2\alpha^3 - 57\alpha^2 + 57\alpha + 8 > 0$. This follows that $\varphi_{21}(n) \geq \varphi_{21}(23) = -50\alpha^3 - 321\alpha^2 + 326\alpha + 165 > 0$, and thus $Y_3(n - 2) > 0$.

We second show that either $Y_3(7) < 0$ or $Y_3(\alpha n) < 0$.

If $0 < \alpha \leq 0.3$, then $Y_3(7) = -2\alpha^2(\alpha^2 - 10\alpha + 3)n^2 + (20\alpha^4 - 163\alpha^3 - 171\alpha^2 + 99\alpha - 13)n + 4\alpha^4 - 260\alpha^3 + 2190\alpha^2 - 1024\alpha + 170 = \varphi_{22}(n)$. Since $\varphi'_{22}(n) = -4\alpha^2(\alpha^2 - 10\alpha + 3) < 0$, we have $\varphi'_{22}(n) \leq \varphi'_{22}(23) = -72\alpha^4 + 757\alpha^3 - 447\alpha^2 + 99\alpha - 13 < 0$, and thus $Y_3(7) = \varphi_{22}(n) \leq \varphi_{22}(23) = -594\alpha^4 + 6571\alpha^3 - 4917\alpha^2 + 1253\alpha - 129 < 0$.

If $0.3 < \alpha < \frac{1}{3}$, then $Y_3(\alpha n) = -\alpha(1 - \alpha)(2\alpha^3 + 11\alpha^2 - 9\alpha + 2)n^2 - (30\alpha^4 - 77\alpha^3 + 41\alpha^2 - 5\alpha - 1)n - (1 - 2\alpha)(2\alpha^3 - 3\alpha^2 - 16\alpha + 5) = \varphi_{23}(n)$. Since $\varphi'_{23}(n) = -2\alpha(1 - \alpha)(2\alpha^3 + 11\alpha^2 - 9\alpha + 2) < 0$, we have $\varphi'_{23}(n) \leq \varphi'_{23}(23) = 92\alpha^5 + 384\alpha^4 - 843\alpha^3 + 465\alpha^2 - 87\alpha + 1 < 0$, and thus $Y_3(\alpha n) = \varphi_{23}(n) \leq \varphi_{23}(23) = 1058\alpha^5 + 4075\alpha^4 - 8817\alpha^3 + 4847\alpha^2 - 917\alpha + 18 < 0$.

Combining this with $\varphi_{20}(\alpha) = \alpha(3\alpha - 1) < 0$ and (2.3), we have $\max\{7, \alpha n\} < n - 2.26 < \beta_1 < n - 2$, and thus $Y_3(\beta_1) > 0$, as $Y_3(n - 2.26) > 0$ by Claim 6.

Case 3. $\alpha = \frac{1}{3}$. Since $\alpha = \frac{1}{3}$, $81Y_3(x) = 4(18x^2 - (13n - 54)x + n^2 - 6n + 4) = 4\varphi_{24}(x)$. If $x \geq n - 2.26$, then $25\varphi'_{24}(x) = 25(36x - 13n + 54) \geq 25\varphi'_{24}(n - 2.26) = 575n - 684 > 0$, and thus $1250\varphi_{24}(x) \geq 1250\varphi_{24}(n - 2.26) = 7500n^2 - 4975n - 32629 > 0$. Combining this with $\beta_1 > n - 2.26$ by (2.3), we have $Y_3(\beta_1) > 0$.

Case 4. $\frac{1}{3} < \alpha < \frac{113}{139} \approx 0.8129$. Since $\beta_1 > n - 2.26$ by (2.3), it suffices to show that

$$\text{If } x \geq n - 2.26, \text{ then } Y_3(x) > 0. \quad (2.4)$$

Since $Y'''_3(x) = 6\varphi_{20}(\alpha) > 0$, we have $25Y'''_3(x) \geq 25Y'''_3(n - 2.26) = 50\alpha(3\alpha - 1)(2 - \alpha)n - 317\alpha^2 - 61\alpha + 100 = \varphi_{25}(n) \geq \varphi_{25}(23) = -3450\alpha^3 + 7733\alpha^2 - 2361\alpha + 100 > 0$, and thus $2500Y'_3(x) \geq 2500Y'_3(n - 2.26) = 2500\alpha(3\alpha - 1)(1 - \alpha)n^2 + (5000\alpha^4 + 26400\alpha^3 - 36600\alpha^2 + 2600\alpha + 5000)n - 90000\alpha^3 + 136721\alpha^2 - 60407\alpha + 4900 = \varphi_{26}(n)$. Note that $\varphi'_{26}(n) = 5000\alpha(3\alpha - 1)(1 - \alpha) > 0$. Thus, $\varphi'_{26}(n) \geq \varphi'_{26}(23) = 200(25\alpha^4 - 1593\alpha^3 + 2117\alpha^2 - 562\alpha + 25) > 0$, which implies that $2500Y'_3(x) \geq \varphi_{26}(n) \geq \varphi_{26}(23) = 115000\alpha^4 - 3450300\alpha^3 + 4584921\alpha^2 - 1323107\alpha + 119900 > 0$. Now, we can conclude that $125000Y_3(x) \geq 125000Y_3(n - 2.26) > 0$ by Claim 6 and thus (2.4) holds.

Case 5. $0.81 < \alpha \leq 0.87$ and $n \geq 24$. By Claim 5, it suffices to show that

$$\text{If } x \geq n - 2.16, \text{ then } Y_3(x) > 0. \quad (2.5)$$

Since $Y'''_3(x) = 6\varphi_{20}(\alpha) > 0$, we have $25Y'''_3(x) \geq 25Y'''_3(n - 2.16) = 50\alpha(3\alpha - 1)(2 - \alpha)n - 272\alpha^2 - 76\alpha + 100 = \varphi_{27}(n) \geq \varphi_{27}(24) = -3600\alpha^3 + 8128\alpha^2 - 2476\alpha + 100 > 0$, and thus $625Y'_3(x) \geq 625Y'_3(n - 2.16) = 625\alpha(3\alpha - 1)(1 - \alpha)n^2 + 25(50\alpha^4 + 249\alpha^3 - 331\alpha^2 + 16\alpha + 50)n - 22500\alpha^3 + 33444\alpha^2 - 15273\alpha + 1475 = \varphi_{28}(n)$. Since $\varphi'_{28}(n) = 1250\alpha(3\alpha - 1)(1 - \alpha) > 0$, we have $\varphi'_{28}(n) \geq \varphi'_{28}(24) = 25(50\alpha^4 - 3351\alpha^3 + 4469\alpha^2 - 1184\alpha + 50) > 0$, and thus $625Y'_3(x) \geq \varphi_{28}(n) \geq \varphi_{28}(24) = 30000\alpha^4 - 953100\alpha^3 + 1274844\alpha^2 - 365673\alpha + 31475 > 0$.

This follows that $15625Y_3(x) \geq 15625Y_3(n - 2.16) = 1250\alpha(27 - 31\alpha)(1 - \alpha)n^2 + 25(1050\alpha^4 - 24073\alpha^3 + 44962\alpha^2 - 26907\alpha + 4800)n + 62500\alpha^4 + 1090000\alpha^3 - 2334917\alpha^2 + 1499264\alpha - 303575 = \varphi_{29}(n)$. Since

$\varphi''_{29}(n) = 2500\alpha(27 - 31\alpha)(1 - \alpha) > 0$, we have $\varphi'_{29}(n) \geq \varphi'_{29}(24) = 25(1050\alpha^4 + 50327\alpha^3 - 94238\alpha^2 + 37893\alpha + 4800) > 0$, and thus $15625Y_3(x) \geq \varphi_{29}(n) \geq \varphi_{29}(24) = 692500\alpha^4 + 8966200\alpha^3 - 17117717\alpha^2 + 4795064\alpha + 2576425 > 0$. This completes the proof of (2.5).

Case 6. $0.81 < \alpha \leq 0.87$ and $n = 23$. Since $(\alpha - 1)(n - 3) - 1 < 0$ and $625Y_2(20.8) = -2(2409750\alpha^4 - 11557450\alpha^3 + 19284595\alpha^2 - 13475955\alpha + 3339258) > 0$, we have $\beta_1 > 20.8$, and thus it suffices to show that

$$\text{If } x \geq 20.8, \text{ then } Y_3(x) > 0. \quad (2.6)$$

Since $Y_3'''(x) = 6\varphi_{18}(\alpha) > 0$, we have $5Y_3''(x) \geq 5Y_3''(20.8) = -690\alpha^3 + 1552\alpha^2 - 474\alpha + 20 > 0$, and thus $25Y_3'(x) \geq 25Y_3'(20.8) = 1150\alpha^4 - 34710\alpha^3 + 46314\alpha^2 - 13373\alpha + 1205 > 0$, which implies that $125Y_3(x) \geq 125Y_3(20.8) = 2(2550\alpha^4 + 34465\alpha^3 - 63984\alpha^2 + 16868\alpha + 9705) > 0$. Thus, (2.6) holds. $\square$

Claim 8. If $x \geq n - 2.5$, then $\Phi_3'(x) > 0$.

Proof. Since $\Phi_3'''(x) = 24x - 6(2\alpha + 1)n + 18\alpha + 30 \geq \Phi_3'''(n - 2.5) = (18 - 12\alpha)n + 18\alpha - 30 \geq (18 - 12\alpha)23 + 18\alpha - 30 = 6(64 - 43\alpha) > 0$, we have $\Phi_3''(x) \geq \Phi_3''(n - 2.5) = 2(1 - \alpha)(3 - \alpha)n^2 - (4\alpha^2 - 28\alpha + 21)n - 4\alpha^2 - 29\alpha + 22 = \varphi_{30}(n)$. Note that $\varphi_{30}'(n) = 4(1 - \alpha)(3 - \alpha) > 0$. Thus, $\varphi_{30}'(n) \geq \varphi_{30}'(23) = 88\alpha^2 - 340\alpha + 255 > 0$, and so $\Phi_3'(x) \geq \varphi_{30}(n) \geq \varphi_{30}(23) = 962\alpha^2 - 3617\alpha + 2713 > 0$, which implies that $4\Phi_3'(x) \geq 4\Phi_3'(n - 2.5) = 4(1 - \alpha)^2n^3 + 4(1 - \alpha)(\alpha^2 + 6\alpha - 6)n^2 + (16\alpha^3 + 52\alpha^2 - 146\alpha + 77)n - 96\alpha^2 + 213\alpha - 115 = \varphi_{31}(n)$. Since $\varphi_{31}'(n) = 8(1 - \alpha)(3(1 - \alpha)n + \alpha^2 + 6\alpha - 6) \geq \varphi_{31}'(23) = 8(1 - \alpha)(\alpha^2 - 63\alpha + 63) > 0$, we have $\varphi_{31}'(n) \geq \varphi_{31}'(23) = -168\alpha^3 + 5480\alpha^2 - 10634\alpha + 5321 > 0$, and so $\varphi_{31}(n) \geq \varphi_{31}(23) = -1748\alpha^3 + 39188\alpha^2 - 75089\alpha + 37628 > 0$. $\square$

By (2.2) and Claim 7, we have $(1 + (1 - \alpha)(n - 3))\Phi_3(\beta_1) = Y_3(\beta_1) > 0$, and thus $\Phi_3(\beta_1) > 0$. Combining this with $\beta_1 > n - 2.26 > n - 2.5$ by Claim 3, if $x \geq \beta_1$, then $\Phi_3(x) \geq \Phi_3(\beta_1) > 0$ by Claim 8, which implies that $\beta_1 > \rho_\alpha(\overline{B_4})$.

On the other hand, we have $\Phi_2(\beta_1) = (\beta_1 - n + 2)\Phi_3(\beta_1) < 0$ (as $\beta_1 < n - 2$ by Claim 4), which implies that $\beta_1 < \rho_\alpha(\overline{B_3})$. Now, we can conclude that $\rho_\alpha(\overline{B_4}) < \beta_1 < \rho_\alpha(\overline{B_3})$. $\square$

Proof of Theorem 1.6. By Lemma 2.1, either $\overline{B_3}$ or $\overline{B_4}$ is the ρ_α -maximal graph of $\mathbb{G}(n, n - 2; 2)$. Since B_1 and B_2 are all bicyclic graphs with n vertices and maximum degree $n - 1$, by Theorems 1.2(3) and 1.3, to complete the proof of Theorem 1.6, it suffices to show that: If $n \geq 23$ and $0 \leq \alpha \leq 0.87$, then $\rho_\alpha(\overline{B_4}) < \rho_\alpha(\overline{B_3}) < \rho_\alpha(\overline{B_2}) < \rho_\alpha(\overline{B_1})$. Now, the result follows from Lemmas 2.2–2.3. $\square$

3 The proof of Theorem 1.7

This section is dedicated to the proof of Theorem 1.7.

Lemma 3.1. [3] If $n \geq 7$ and $0 \leq \alpha < 1$, then $\rho_\alpha(\overline{S_5}) < \rho_\alpha(\overline{S_4}) < \rho_\alpha(\overline{S_3}) < \min\{\rho_\alpha(\overline{S_2}), \rho_\alpha(\overline{S_1})\}$.

By an elementary computation, $\rho_\alpha(\overline{S_7})$, $\rho_\alpha(\overline{S_6})$, and $\rho_\alpha(\overline{S_5})$ are equal to the maximum root of $\Phi_6(x)$, $\Phi_7(x)$ and $\Phi_8(x)$, respectively, where

$$\begin{aligned} \Phi_6(x) &= x^5 - \left((3\alpha + 1)n - 6\alpha - 6 \right) x^4 + \left(3\alpha(\alpha + 1)n^2 - (11\alpha^2 + 20\alpha + 4)n + 4\alpha^2 + 26\alpha + 17 \right) x^3 \\ &\quad - \left(\alpha^2(\alpha + 3)n^3 - \alpha(4\alpha^2 + 21\alpha + 8)n^2 - (3\alpha^3 - 28\alpha^2 - 55\alpha - 1)n + 6\alpha^3 + 36\alpha^2 - 96\alpha - 2 \right) x^2 \\ &\quad + \left(\alpha^3n^4 + \alpha^2(\alpha^2 - 6\alpha - 4)n^3 - \alpha^2(7\alpha^2 + 8\alpha - 36)n^2 + (10\alpha^4 + 81\alpha^3 - 77\alpha^2 - 15\alpha + 5)n \right. \\ &\quad \left. - 56\alpha^3 - 33\alpha^2 + 78\alpha - 25 \right) x - \alpha(n - 4)(n - 5)((n - 2)\alpha^2 + \alpha - 1)((n + 1)\alpha - 3), \\ \Phi_7(x) &= x^5 - \left((3\alpha + 1)n - 7\alpha - 5 \right) x^4 + \left(3\alpha(\alpha + 1)n^2 - (13\alpha^2 + 19\alpha + 3)n + 5\alpha^2 + 28\alpha + 12 \right) x^3 \end{aligned}$$

$$\begin{aligned}
& - \left(\alpha^2(\alpha+3)n^3 - \alpha(5\alpha^2+22\alpha+6)n^2 - (3\alpha^3-34\alpha^2-48\alpha+2)n + 7\alpha^3+39\alpha^2-110\alpha+14 \right) x^2 \\
& + \left(\alpha^3n^4 + \alpha^2(\alpha^2-7\alpha-3)n^3 - \alpha(8\alpha^3+5\alpha^2-35\alpha+3)n^2 + (12\alpha^4+89\alpha^3-103\alpha^2+5\alpha+5)n \right. \\
& \left. - 68\alpha^3-8\alpha^2+68\alpha-28 \right) x - (n-6) \left((n-2)\alpha^2 + \alpha - 1 \right) \left(\alpha^2n^2 - \alpha(3\alpha+2)n - 4\alpha^2 + 12\alpha - 2 \right), \\
& \text{and } \Phi_8(x) = x^2 - \left((\alpha+1)n - \alpha - 4 \right) x + (n+1)(n-4)\alpha + 10 - 2n.
\end{aligned}$$

In what follows, we shall prove that $\rho_\alpha(\overline{S_7}) < \rho_\alpha(\overline{S_6}) < \rho_\alpha(\overline{S_5})$.

Lemma 3.2. *If $n \geq 35$ and $0 \leq \alpha \leq 0.8$, then $\rho_\alpha(\overline{S_6}) < \rho_\alpha(\overline{S_5})$.*

Proof. It can be checked that

$$\Phi_7(x) = \Phi_8(x)\Psi_3(x) + Y_4(x), \quad (3.1)$$

where $\Psi_3(x) = x^3 - \left((2n-6)\alpha - 1 \right) x^2 + \left((n^2-5n-1)\alpha^2 - (n-7)\alpha - 2 \right) x - (n-1)(n-6)\alpha^3 + 3(n-6)\alpha^2 - 4(n-7)\alpha + 2n - 16$, and $Y_4(x) = (1-\alpha)\Psi_4(x)$, where $\Psi_4(x) = \left((n-6)\alpha^3 + (n-4)(n-6)\alpha^2 - (3n^2-26n+50)\alpha + (2n-7)(n-8) \right) x - (n-6)(n-4)(n+1)\alpha^3 + (n-6)(3n^2-5n-26)\alpha^2 - (2n^3-16n^2-28n+256)\alpha + 4n^2-54n+172$.

Claim 9. $\rho_\alpha(\overline{S_5}) > n - 2.5$.

Proof. Since $4\Phi_8(n-2.5) = -2(1-\alpha)n - 26\alpha + 25 \leq -70(1-\alpha) - 26\alpha + 25 = 44\alpha - 45 < 0$, we have $\rho_\alpha(\overline{S_5}) > n - 2.5$, as $\Phi_8(+\infty) > 0$. $\square$

Claim 10. *If $x \geq n - 2.5$, then $\Psi_3(x) > 0$.*

Proof. When $x \geq n - 2.5$, since $\Psi_3''(x) = 6x - 4(n-3)\alpha + 2 \geq 6(n-2.5) - 4(n-3)\alpha + 2 = (6-4\alpha)n + 12\alpha - 13 > n - 13 > 0$, $4\Psi_3'(x) \geq 4\Psi_3'(n-2.5) = (4n^2-20n-4)\alpha^2 - (16n^2-84n+92)\alpha + 12n^2-52n+47 = \varphi_1(\alpha)$. Since $5\varphi_1'(\alpha) = 20\left((2n^2-10n-2)\alpha - 4n^2+21n-23\right) \leq 5\varphi_1'(0.8) = 4(-12n^2+65n-123) < 0$, we have $100\Psi_3'(x) \geq 25\varphi_1(\alpha) \geq 25\varphi_1(0.8) = 44n^2+60n-729 > 0$.

Thus, $8\Psi_3(x) \geq 8\Psi_3(n-2.5) = -8(n-1)(n-6)\alpha^3 + (8n^3-60n^2+116n-124)\alpha^2 - (16n^3-120n^2+296n-384)\alpha + 8n^3-52n^2+110n-163 = \varphi_2(\alpha)$. Since $\varphi_2''(\alpha) = (-48n^2+336n-288)\alpha + 16n^3-120n^2+232n-248 \geq 0.8(-48n^2+336n-288) + 16n^3-120n^2+232n-248 = 1.6(10n^3-99n^2+313n-299) > 0$, we have $25\varphi_2'(\alpha) \leq 25\varphi_2'(0.8) = -8(10n^3-27n^2+9n-292) < 0$, and thus $1000\Psi_3(x) \geq 125\varphi_2(\alpha) \geq 125\varphi_2(0.8) = 40n^3+188n^2-2986n+5033 > 0$. $\square$

Claim 11. *If $x \geq n - 2.5$, then $\Psi_4(x) > 0$.*

Proof. When $x \geq n - 2.5$, $\Psi_4'(x) = (n-6)\alpha^3 + (n-4)(n-6)\alpha^2 - (3n^2-26n+50)\alpha + (2n-7)(n-8) = \varphi_3(\alpha)$. Since $\varphi_3''(\alpha) = 2(n-6)(3\alpha+n-4) > 0$, we have $25\varphi_3'(\alpha) \leq 25\varphi_3'(0.8) = -35n^2+298n-578 < 0$, and thus $125\Psi_4'(x) = 125\varphi_3(\alpha) \geq 125\varphi_3(0.8) = 30n^2-1011n+3536 > 0$.

Thus, $2\Psi_4(x) \geq 2\Psi_4(n-2.5) = -(n-6)(2n^2-8n-3)\alpha^3 + (n-6)(8n^2-23n-32)\alpha^2 - (10n^3-99n^2+174n+262)\alpha + 4n^3-48n^2+119n+64 = \varphi_4(\alpha)$. Since $5\varphi_4''(\alpha) = 10(n-6)((-6n^2+24n+9)\alpha + 8n^2-23n-32) \geq 10(n-6)(0.8(-6n^2+24n+9) + 8n^2-23n-32) = 2(n-6)(16n^2-19n-124) > 0$, we have $25\varphi_4'(\alpha) \leq 25\varphi_4'(0.8) = -26n^3+595n^2-2270n+266 < 0$, and thus $250\Psi_4(x) \geq 125\varphi_4(\alpha) \geq 125\varphi_4(0.8) = 12n^3-500n^2+3075n-3992 > 0$. $\square$

By Claims 9–11 and (3.1), when $x \geq \rho_\alpha(\overline{S_5}) > n - 2.5$, we have $\Phi_7(x) = \Phi_8(x)\Psi_3(x) + Y_4(x) > 0$, and thus $\rho_\alpha(\overline{S_5}) > \rho_\alpha(\overline{S_6})$. $\square$

By $\Phi_6(x)$ and $\Phi_7(x)$, we can deduce the following Lemma 3.3.

Lemma 3.3. *If $n \geq 35$ and $0 \leq \alpha \leq 0.8$, then $\rho_\alpha(\overline{S_7}) < \rho_\alpha(\overline{S_6})$.*

Proof. It can be checked that

$$\Phi_6(x) = \Phi_7(x) + Y_5(x), \text{ and } \Phi_7(x) = Y_5(x)\Psi_5(x) + Y_6(x), \quad (3.2)$$

where $Y_5(x) = (1-\alpha)x^4 + ((2\alpha^2 - \alpha - 1)n - \alpha^2 - 2\alpha + 5)x^3 + (\alpha(\alpha + 2)(1-\alpha)n^2 - (3\alpha + 1)(3 - 2\alpha)n + \alpha^3 + 3\alpha^2 - 14\alpha + 16)x^2 + (\alpha^2(\alpha - 1)n^3 + \alpha(\alpha^3 - 3\alpha^2 + \alpha + 3)n^2 - 2\alpha(\alpha^3 + 4\alpha^2 - 13\alpha + 10)n - (1-\alpha)(12\alpha^2 - 13\alpha - 3))x - ((n-2)\alpha^2 + \alpha - 1)(\alpha(\alpha - 1)n^2 - (3\alpha^2 - 3\alpha - 2)n - 4\alpha^2 + 12\alpha - 12)$, $(1-\alpha)^2\Psi_5(x) = x(1-\alpha) - \alpha((1-\alpha)n + 6\alpha - 4)$ and $(1-\alpha)^2Y_6(x) = \Psi_6(x)$ with $\Psi_6(x) = 2(2\alpha - 1)(3\alpha(\alpha - 1) + 2)x^3 - (2(6\alpha^4 - 4\alpha^3 + \alpha^2 + 2\alpha - 1)n + \alpha^5 - \alpha^4 - 48\alpha^3 + 86\alpha^2 - 67\alpha + 17)x^2 + (2\alpha(5\alpha^3 - 6\alpha^2 + 4\alpha - 1)n^2 + (11\alpha^5 - 45\alpha^4 + 54\alpha^3 - 30\alpha^2 - \alpha + 3)n + 2(1-\alpha)(2\alpha^4 + 19\alpha^3 - 35\alpha^2 + 30\alpha - 8))x - 2((5\alpha^3 - 6\alpha^2 + 4\alpha - 1)n - 16\alpha^3 + 30\alpha^2 - 24\alpha + 6)(\alpha + \alpha^2(n-2) - 1)$.

Claim 12. If $x \geq n - 2$, then $Y_5(x) > 0$.

Proof. When $x \geq n - 2$, since $Y_5^{(4)}(x) = 24(1-\alpha) > 0$, we have $Y_5'''(x) \geq Y_5'''(n-2) = 6((2\alpha^2 - 5\alpha + 3)n - \alpha^2 + 6\alpha - 3) \geq 6(35(2\alpha^2 - 5\alpha + 3) - \alpha^2 + 6\alpha - 3) = 6(69\alpha^2 - 169\alpha + 102) > 0$.

Thus, $Y_5''(x) \geq Y_5''(n-2) = 2((1-\alpha)^2(3-\alpha)n^2 - (9\alpha^2 - 17\alpha + 6)n + \alpha^3 + 9\alpha^2 - 26\alpha + 10) = 2\varphi_1(n)$. Since $\varphi_1'(n) = 2(1-\alpha)^2(3-\alpha) > 0$, we have $\varphi_1'(n) \geq \varphi_1'(35) = -70\alpha^3 + 341\alpha^2 - 473\alpha + 204 > 0$, and thus $0.5Y_5''(x) \geq \varphi_1(n) \geq \varphi_1(35) = -1224\alpha^3 + 5819\alpha^2 - 8006\alpha + 3475 > 0$.

This implies that $Y_5'(x) \geq Y_5'(n-2) = (1-\alpha)\varphi_2(n)$, where $\varphi_2(n) = (1-\alpha)^2n^3 - (\alpha^3 + 2\alpha^2 - 8\alpha + 3)n^2 + (2\alpha^3 + 8\alpha^2 - 36\alpha + 20)n - (1-\alpha)(33 - 8\alpha)$. Since $\varphi_2'''(n) = 6(1-\alpha)^2 > 0$, we have $\varphi_2''(n) \geq \varphi_2''(35) = -2\alpha^3 + 206\alpha^2 - 404\alpha + 204 > 0$, and thus $\varphi_2'(n) \geq \varphi_2'(35) = -68\alpha^3 + 3543\alpha^2 - 6826\alpha + 3485 > 0$. Now, we can conclude that $\varphi_2(n) \geq \varphi_2(35) = -1155\alpha^3 + 40697\alpha^2 - 77169\alpha + 39867 > 0$, which implies that $Y_5'(x) > 0$.

Thus, $Y_5(x) \geq Y_5(n-2) = (1-\alpha)^2\varphi_3(n)$, where $\varphi_3(n) = (\alpha^2 - 9\alpha + 10)n^2 + (2\alpha^2 + 25\alpha - 35)n - 8\alpha^2 - 8\alpha + 22$. Since $\varphi_3''(n) = 2(\alpha^2 - 9\alpha + 10) > 0$, $\varphi_3'(n) \geq \varphi_3'(35) = 72\alpha^2 - 605\alpha + 665 > 0$, and so $\varphi_3(n) \geq \varphi_3(35) = 1287\alpha^2 - 10158\alpha + 11047 > 0$. $\square$

Claim 13. $Y_5(n - 2.5) < 0$.

Proof. It can be checked that $16Y_5(n - 2.5) = \varphi_4(n) = 8(\alpha - 1)^3n^3 - 4(1-\alpha)(2\alpha^3 - 45\alpha^2 + 96\alpha - 49)n^2 + (48\alpha^4 + 384\alpha^3 - 1720\alpha^2 + 2046\alpha - 750)n - 128\alpha^4 + 68\alpha^3 + 910\alpha^2 - 1541\alpha + 663$. Since $\varphi_4'''(n) = 48(\alpha - 1)^3 < 0$, we have $\varphi_4''(n) \leq \varphi_4''(35) = 8(\alpha - 1)(2\alpha^3 + 165\alpha^2 - 324\alpha + 161) < 0$. Thus, $\varphi_4'(n) \leq \varphi_4'(35) = 608\alpha^4 + 16624\alpha^3 - 50440\alpha^2 + 49646\alpha - 16430 < 0$. Now, we can conclude that $16Y_5(n - 2.5) = \varphi_4(n) \leq \varphi_4(35) = 11352\alpha^4 + 126208\alpha^3 - 397390\alpha^2 + 388569\alpha - 128487 < 0$. $\square$

Let β_1 be the maximum roots of $Y_5(x)$. By Claims 12 and 13, we have

$$n - 3 < n - 2.5 < \beta_1 < n - 2. \quad (3.3)$$

For the case $35 \leq n \leq 49$, we can slightly improves (3.3) to

Claim 14. If $35 \leq n \leq 49$, then $n - 2.5 < \beta_1 < n - 2.15$.

Proof. From Claim 13, it suffices to show that

$$\text{If } x \geq n - 2.15, \text{ then } Y_5(x) > 0. \quad (3.4)$$

When $x \geq n - 2.15$, since $Y_5^{(4)}(x) = 24(1-\alpha) > 0$, we have $5Y_5'''(x) \geq 5Y_5'''(n - 2.15) = 30(1-\alpha)(3 - 2\alpha)n - 30\alpha^2 + 198\alpha - 108 \geq 1050(1-\alpha)(3 - 2\alpha) - 30\alpha^2 + 198\alpha - 108 = 6(345\alpha^2 - 842\alpha + 507) > 0$.

Thus, $100Y_5''(x) \geq 100Y_5''(n - 2.15) = 200(3-\alpha)(1-\alpha)^2n^2 - 10(198\alpha^2 - 385\alpha + 147)n + 200\alpha^3 + 1890\alpha^2 - 5767\alpha + 2297 = \varphi_5(n)$. Since $\varphi_5''(n) = 400(3-\alpha)(1-\alpha)^2 > 0$, we have $\varphi_5'(n) \geq \varphi_5'(35) = -14000\alpha^3 +$

$68020\alpha^2 - 94150\alpha + 40530 > 0$, and thus $100Y_5''(x) \geq \varphi_5(n) \geq \varphi_5(35) = -244800\alpha^3 + 1157590\alpha^2 - 1586017\alpha + 685847 > 0$.

This implies that $2000Y_5'(x) \geq 2000Y_5'(n-2.15) = \varphi_6(n)$, where $\varphi_6(n) = 2000(1-\alpha)^3n^3 - 200(1-\alpha)(2\alpha-1)(5\alpha^2+14\alpha-39)n^2 - 5(800\alpha^4+2400\alpha^3-18734\alpha^2+24575\alpha-8801)n + 15400\alpha^3 - 103535\alpha^2 + 164437\alpha - 72432$. Note that $\varphi_6'''(n) = 12000(1-\alpha)^3 > 0$. Thus, $\varphi_6''(n) \geq \varphi_6''(35) = 400(1-\alpha)(-10\alpha^3+1027\alpha^2-2008\alpha+1011) > 0$, which implies that $\varphi_6'(n) \geq \varphi_6'(35) = 136000\alpha^4 - 7180000\alpha^3 + 20533670\alpha^2 - 20338875\alpha + 6848005 = 5(27200\alpha^4 - 1436000\alpha^3 + 4106734\alpha^2 - 4067775\alpha + 1369601) > 0$. Thus, $2000Y_5'(x) \geq \varphi_6(n) \geq \varphi_6(35) = 2310000\alpha^4 - 82969600\alpha^3 + 232249915\alpha^2 - 229291188\alpha + 77662743 > 0$.

Now, we can conclude that $160000Y_5(x) \geq 160000Y_5(n-2.15) = \varphi_7(\alpha)$, where $\varphi_7(\alpha) = 8000(17n+80)(n-2)\alpha^4 + 400(60n^3 - 4469n^2 + 8760n + 2729)\alpha^3 - 20(3600n^3 - 244900n^2 + 718474n - 300447)\alpha^2 + (72000n^3 - 4929200n^2 + 16607900n - 10192921)\alpha - 24000n^3 + 1682800n^2 - 6103220n + 4349701$. Since $\varphi_7^{(4)}(\alpha) = 192000(17n+80)(n-2) > 0$ and $35 \leq n \leq 49$, $\varphi_7'''(\alpha) \leq \varphi_7'''(0.8) = 2400(60n^3 - 3381n^2 + 11704n - 7511) < 0$, and so $\varphi_7''(\alpha) \geq \varphi_7''(0.8) = -40(720n^3 - 56500n^2 + 227338n - 185679) > 0$, and so $\varphi_7'(\alpha) \leq \varphi_7'(0.8) = 2880n^3 - 246064n^2 + 1098076n - 1104185 < 0$. This implies that $800000Y_5(x) \geq 5\varphi_7(\alpha) \geq 5\varphi_7(0.8) = -960n^3 + 73072n^2 - 342932n + 378485 > 0$. Thus (3.4) holds. $\square$

Claim 15. $\Phi_7(\beta_1) = Y_6(\beta_1) < 0$.

Proof. Let $\varphi_8(\alpha) = (2\alpha-1)(3\alpha(\alpha-1)+2)$. Since $(1-\alpha)^2Y_6(x) = \Psi_6(x)$ and $\beta_1 > n-3$ by (3.3), to complete the proof, it suffices to show that

$$\text{If } x \geq n-3, \text{ then } \Psi_6(x) < 0. \quad (3.5)$$

Case 1. $0 \leq \alpha < 0.5$. Let $\varphi_9(\alpha) = -12\alpha^4 + 44\alpha^3 - 56\alpha^2 + 38\alpha - 10$. Since $\varphi_9''(\alpha) = -8(2-3\alpha)(7-6\alpha) < 0$, we have $\varphi_9'(\alpha) = 2(-24\alpha^3 + 66\alpha^2 - 56\alpha + 19) > \varphi_9'(0.5) = 9 > 0$, and thus $\varphi_9(\alpha) < \varphi_9(0.5) = -0.25 < 0$. When $x \geq n-3$, since $\Psi_6'''(x) = 12\varphi_8(\alpha) < 0$, we have $0.5\Psi_6''(x) \leq 0.5\Psi_6''(n-3) = \varphi_9(\alpha)n - \alpha^5 + \alpha^4 - 60\alpha^3 + 76\alpha^2 - 59\alpha + 19 \leq 35\varphi_9(\alpha) - \alpha^5 + \alpha^4 - 60\alpha^3 + 76\alpha^2 - 59\alpha + 19 = -\alpha^5 - 419\alpha^4 + 1480\alpha^3 - 1884\alpha^2 + 1271\alpha - 331 < 0$.

Now, we have $\Psi_6'(x) \leq \Psi_6'(n-3) = (9n+2)\alpha^5 - (14n^2 - 29n + 40)\alpha^4 + (40n^2 - 114n + 144)\alpha^3 - (50n^2 - 134n + 100)\alpha^2 + (32n^2 - 95n + 52)\alpha - 8n^2 + 29n - 22 = \varphi_{10}(\alpha)$. Since $\varphi_{10}^{(5)}(\alpha) = 120(9n+2) > 0$ and $n \geq 35$, we have $\varphi_{10}^{(4)}(\alpha) < \varphi_{10}^{(4)}(0.5) = -12(28n^2 - 103n + 70) < 0$, and thus $\varphi_{10}'''(\alpha) > \varphi_{10}'''(0.5) = 3(24n^2 - 67n + 138) > 0$. This implies that $2\varphi_{10}''(\alpha) < 2\varphi_{10}''(0.5) = -(11n+18)(4n-13) < 0$ and so $16\varphi_{10}'(\alpha) > 16\varphi_{10}'(0.5) = 80n^2 - 467n + 650 > 0$. Thus, $32\Psi_6'(x) \leq 32\Psi_6'(n-3) < 32\varphi_{10}(0.5) = -12n^2 + 91n - 174 < 0$.

Now, we can conclude that $\Psi_6(x) \leq \Psi_6(n-3) = (21n-61)\alpha^5 - (2n^3 - 10n^2 + 107n - 263)\alpha^4 + (8n^3 - 26n^2 + 140n - 404)\alpha^3 - (12n^3 - 36n^2 + 64n - 234)\alpha^2 + (8n^3 - 30n^2 + 29n - 63)\alpha - 2n^3 + 10n^2 - 15n + 15 = \varphi_{11}(\alpha)$. Since $\varphi_{11}^{(5)}(\alpha) = 120(21n-61) > 0$, we have $\varphi_{11}^{(4)}(\alpha) \leq \varphi_{11}^{(4)}(0.5) = -12(4n^3 - 20n^2 + 109n - 221) < 0$, and so $\varphi_{11}'''(\alpha) > \varphi_{11}'''(0.5) = 3(8n^3 - 12n^2 - 43n - 61) > 0$, which implies that $2\varphi_{11}''(\alpha) < 2\varphi_{11}''(0.5) = -12n^3 + 48n^2 + 47n - 215 < 0$. From this, it follows that $16\varphi_{11}'(\alpha) > 16\varphi_{11}'(0.5) = 16n^3 - 136n^2 + 369n - 313 > 0$ and thus $32\Psi_6(x) \leq 32\Psi_6(n-3) < 32\varphi_{11}(0.5) = -4n^3 + 44n^2 - 161n + 193 < 0$. In this case, (3.5) holds.

Case 2. $\alpha = 0.5$. Then, $32\Psi_6(x) = (-8n+33)x^2 + (4n^2 - 23n + 24)x - 2(n-4)^2 = \varphi_{12}(x)$. When $x \geq n-3$, since $32\Psi_6'(x) = (66-16n)x + 4n^2 - 23n + 24 \leq (66-16n)(n-3) + 4n^2 - 23n + 24 = -12n^2 + 91n - 174 < 0$, we have $32\Psi_6(x) = \varphi_{12}(x) \leq \varphi_{12}(n-3) = -4n^3 + 44n^2 - 161n + 193 < 0$, and thus (3.5) holds.

Case 3. $0.5 < \alpha \leq 0.8$. To prove Claim (3.5), by Claim 14 and (3.3), since $\Psi_6(-\infty) < 0$ and $\Psi_6(+\infty) > 0$, it suffices to show that $\Psi_6(\alpha) > 0$, $\max\{\Psi_6(n-2.5), \Psi_6(n-2)\} < 0$ for $n \geq 50$ and $\max\{\Psi_6(n-2.5), \Psi_6(n-2.15)\} < 0$ for $35 \leq n \leq 49$.

Since $\varphi_{13}(\alpha) = -\alpha^4 + 13\alpha^3 - 15\alpha^2 + 9\alpha - 2 > 0$, $\Psi_6(\alpha) = (1-\alpha)^2(\varphi_{13}(\alpha)n - \alpha^5 + 7\alpha^4 - 53\alpha^3 + 75\alpha^2 - 52\alpha + 12) \geq (1-\alpha)^2(35\varphi_{13}(\alpha) - \alpha^5 + 7\alpha^4 - 53\alpha^3 + 75\alpha^2 - 52\alpha + 12) = (1-\alpha)^2\varphi_{14}(\alpha) > 0$, as $\varphi_{14}(\alpha) = -\alpha^5 - 28\alpha^4 + 402\alpha^3 - 450\alpha^2 + 263\alpha - 58 > 0$.

It can be checked that $4\Psi_6(n-2.5) = (102n-241)\alpha^5 - (8n^3 - 12n^2 + 382n - 973)\alpha^4 + (32n^3 - 24n^2 + 376n - 1382)\alpha^3 - (48n^3 - 44n^2 + 44n - 803)\alpha^2 + (32n^3 - 56n^2 - 36n - 200)\alpha - 8n^3 + 24n^2 - 12n + 33 = \varphi_{15}(\alpha)$. Since $\varphi_{15}^{(5)}(\alpha) = 120(102n-241) > 0$, we have $\varphi_{15}^{(4)}(\alpha) \leq \varphi_{15}^{(4)}(0.8) = -24(2n+1)(4n^2 - 8n - 9) < 0$, and thus $5\varphi_{15}'(\alpha) \geq 5\varphi_{15}'(0.8) = 12(16n^3 + 36n^2 - 484n + 473) > 0$. This implies that $25\varphi_{15}''(\alpha) \leq 25\varphi_{15}''(0.8) =$

$-2(48n^3 - 812n^2 + 2156n + 285) < 0$, and so $125\varphi'_{15}(\alpha) \geq 125\varphi'_{15}(0.8) = 4(8n^3 - 222n^2 + 1315n - 2172) > 0$. Now, we have $12500\Psi_6(n - 2.5) = 3125\varphi_{15}(\alpha) \leq 3125\varphi_{15}(0.8) = -40n^3 - 40n^2 + 1588n - 3419 < 0$.

If $n \geq 50$, then $\Psi_6(n - 2) = 2(1 - \alpha)^2\varphi_{16}(\alpha)$, where $\varphi_{16}(\alpha) = 15(n - 2)\alpha^3 - (n^3 + 2n^2 + 15n - 52)\alpha^2 + (2n^3 + 3n^2 - 10n - 20)\alpha - n^3 + n^2 + 2n + 4$. Since $\varphi'''_{16}(\alpha) = 90(n - 2) > 0$, we have $\varphi''_{16}(\alpha) \leq \varphi''_{16}(0.8) = -2(n^3 + 2n^2 - 21n + 20) < 0$, and thus $5\varphi'_{16}(\alpha) \geq 5\varphi'_{16}(0.8) = 2n^3 - n^2 - 26n + 28 > 0$. Now, we have $25\varphi_{16}(\alpha) \leq 25\varphi_{16}(0.8) = -n^3 + 53n^2 - 198n + 148 < 0$, and thus $\Psi_6(n - 2) = 2(1 - \alpha)^2\varphi_{16}(\alpha) < 0$.

If $35 \leq n \leq 49$, then $4000\Psi_6(n - 2.15) = \varphi_{17}(n)$, where $\varphi_{17}(n) = -8000(1 - \alpha)^4n^3 + 400(1 - \alpha)(19\alpha^3 - 61\alpha^2 + 4\alpha + 32)n^2 + 20(5730\alpha^5 - 18204\alpha^4 + 14978\alpha^3 + 1888\alpha^2 - 4859\alpha + 485)n - 240090\alpha^5 + 918890\alpha^4 - 1270322\alpha^3 + 771423\alpha^2 - 211319\alpha + 30284$. Since $\varphi'''_{17}(n) = -48000(1 - \alpha)^4 < 0$, we have $\varphi''_{17}(n) \leq \varphi''_{17}(35) = 800(1 - \alpha)(2119\alpha^3 - 6361\alpha^2 + 6304\alpha - 2068) < 0$, and so $\varphi'_{17}(n) \leq \varphi'_{17}(35) = 20(5730\alpha^5 - 1514804\alpha^4 + 6006978\alpha^3 - 8909112\alpha^2 + 5835941\alpha - 1424715) < 0$. This follows that $4000\Psi_6(n - 2.15) = \varphi_{17}(n) \leq \varphi_{17}(35) = 3770910\alpha^5 - 364133910\alpha^4 + 1420414278\alpha^3 - 2087756977\alpha^2 + 1354667381\alpha - 326950216 < 0$. This completes the proof of Claim 15. $\square$

From Claim 15, we have $\rho_\alpha(\overline{S_6}) > \beta_1$. By (3.2), when $x \geq \rho_\alpha(\overline{S_6})$, we can conclude that $\Phi_6(x) > 0$ and thus $\rho_\alpha(\overline{S_7}) < \rho_\alpha(\overline{S_6})$. $\square$

The proof of Theorem 1.7: Let $\overline{G_0}$ be a ρ_α -maximal graph of $\overline{\mathbb{G}}(n, n - 2; 3)$. Then, $G_0 \in \{S_6, S_7\}$ by Lemma 2.1. Note that $\{S_1, S_2, \dots, S_5\}$ are all tricyclic graphs with maximum degree $n - 1$ and n vertices [4]. Thus, the result follows from Lemmas 3.1–3.3 and Theorem 1.3. $\square$

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