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The dual of number sequences, Riordan polynomials, and Sheffer polynomials

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Abstract: In this paper we introduce different families of numerical and polynomial sequences by using Riordan pseudo involutions and Sheffer polynomial sequences. Many examples are given including dual of Hermite numbers and polynomials, dual of Bell numbers and polynomials, among other. The coefficients of some of these polynomials are related to the counting of different families of set partitions and permutations. We also studied the dual of Catalan numbers and dual of Fuss-Catalan numbers, giving several combinatorial identities.

Keywords: Riordan array, Dual number sequence, Bernoulli numbers and polynomials, Fuss-Catalan numbers, Sheffer sequence, Dual Sheffer sequence

MSC: 05A15, 05A19, 11B83.

1 Introduction

Let $\mathcal{A} = \{s_n\}_{n \geq 0}$ be a sequence of complex numbers. The *binomial transform* of $\mathcal{A}$ is the new sequence $\mathcal{A}^* = \{s_n^*\}_{n \geq 0}$ defined by (cf. [1, 2])

$$s_n^* = \sum_{k=0}^n (-1)^k \binom{n}{k} s_k.$$

The new sequence $\mathcal{A}^*$ will be called here the *dual sequence* of $\mathcal{A}$. If $s_n^* = s_n$ for all $n \geq 0$, then $\mathcal{A}$ is called *self-dual sequence* [18]. For example, the sequence of Bernoulli numbers $\{(-1)^n B_n\}_n$ is a self-dual sequence, where the Bernoulli number B_n is defined by the exponential generating function $\sum_{n \geq 0} B_n \frac{t^n}{n!} = t/(e^t - 1)$.

Recently, Zheng and the first author [6] studied several combinatorial identities for the numbers B_n^* and the polynomials $B_n^*(x)$ defined by the binomial transform

$$B_n^* = \sum_{k=0}^n (-1)^k \binom{n}{k} B_k \quad \text{and} \quad B_n^*(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} B_k^*.$$

These new sequences are called the *dual Bernoulli numbers* and *dual Bernoulli polynomials*, respectively. Many new properties for these sequences can be obtained through a matrix approach, in particular by means of Riordan arrays [6].

The concept of Riordan arrays was introduced in 1991 by Shapiro et al. [14]. Riordan arrays are infinite lower triangular matrices defined by the generating function of their columns. More precisely, an infinite lower triangular matrix is called a *Riordan array* if its k -th column has the generating function $d(t)h(t)^k$ for $k \geq 0$, where $d(t)$ and $h(t)$ are formal power series with $d(0) \neq 0$, $h(0) = 0$, and $h'(0) \neq 0$, ($h'(t)$ is the formal

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derivative of $h(t)$). The matrix corresponding to the pair $h(t)$, $d(t)$ is denoted by $(d(t), h(t))$. For example, the Pascal matrix is a Riordan array given by

$$\left[\binom{n}{k} \right]_{n,k \geq 0} = \left(\frac{1}{1-t}, \frac{t}{1-t} \right).$$

The product of two Riordan arrays is defined by the equality

$$(d_1(t), h_1(t)) * (d_2(t), h_2(t)) = (d_1(t)d_2(h_1(t)), h_2(h_1(t))). \quad (1)$$

We recall that the set of all Riordan arrays is a group under the product “ $*$ ”, and it is called *the Riordan group* [14]. The Riordan array $I = (1, t)$ is everywhere 0 except the entries on the main diagonal, which are 1. I acts as the identity for this product. From these facts, we deduce a formula for the inverse Riordan array

$$(d(t), h(t))^{-1} = \left(\frac{1}{d(\bar{h}(t))}, \bar{h}(t) \right), \quad (2)$$

where $\bar{h}(x)$ is the compositional inverse of $h(x)$. If we multiply $(d(t), h(t))$ by a column vector $(f_0, f_1, \dots)^T$ with the generating function $f(t)$, then the resulting column vector has the generating function $d(t)f(h(t))$.

In the Riordan group there are not order 2 matrices with nonnegative integer entries, then we are lead to consider Riordan matrices with pseudo order 2, that is, an element $(d(t), h(t))$ has pseudo order 2 if $(d(t), h(t)) * (1, -t)$ has order 2. Those matrices are called *pseudo-Riordan involutions* or briefly *pseudo-involutions*. According to [3], a Riordan array $(d(t), h(t))$ is a pseudo-involution if and only if it satisfies $(d(t), h(t))^{-1} = (d(-t), -h(-t))$.

Let $R = (d(t), h(t))$ be a pseudo-involution. Since $R^{-1} = M * R * M$, where $M = (1, -t)$, it follows that $(\pm R * M)^2 = (\pm M * R)^2 = I$. Thus the following statements are equivalent:

- (i) the Riordan array $(d(t), h(t))$ is a pseudo-involution.
- (ii) $\pm(d(t), h(t)) * (1, -t) = (\pm d(t), -h(t))$ are involutions.
- (iii) $\pm(1, -t) * (d(t), h(t)) = (\pm d(-t), h(-t))$ are involutions.

For instance, the Pascal matrix and its inverse $(1/(1+t), t/(1+t)) = \left[\binom{n}{k} (-1)^{n-k} \right]_{n,k \geq 0}$ are Riordan pseudo-involutions. Hence, from above (ii) and (iii), generates the following four Riordan involutions, denoted by R_1 , R_2 , R_3 , and R_4 , respectively [6].

$$\begin{aligned} R_1 &= \left(\frac{1}{1-t}, \frac{t}{1-t} \right) * (1, -t) = \left[\binom{n}{k} (-1)^k \right]_{n,k \geq 0}, \\ R_2 &= - \left(\frac{1}{1-t}, \frac{t}{1-t} \right) * (1, -t) = \left[\binom{n}{k} (-1)^{k+1} \right]_{n,k \geq 0}, \\ R_3 &= (1, -t) * \left(\frac{1}{1-t}, \frac{t}{1-t} \right) = \left[\binom{n}{k} (-1)^n \right]_{n,k \geq 0}, \\ R_4 &= -(1, -t) * \left(\frac{1}{1-t}, \frac{t}{1-t} \right) = \left[\binom{n}{k} (-1)^{n+1} \right]_{n,k \geq 0}. \end{aligned} \quad (3)$$

The dual sequence $\{s_n^*\}_n$ of a given sequence $\{s_n\}_n$ can be interpreted as the product of the Riordan pseudo-involution R_1 and the column vector $(s_0, s_1, \dots)^T$. From this point of view we introduce several new families of sequences and polynomials by interchanging the matrix R_1 for any Riordan pseudo-involution. From these new sequences we give several (probably new) combinatorial identities involving the Catalan numbers and the Fuss-Catalan numbers. We also introduce and discuss dual Sheffer sequences. We give several new families of polynomials, whose coefficients are related to the counting of different families of set partitions and permutations. Finally, we define some polynomials from different families of Sheffer subgroups.

2 Dual Riordan polynomial sequences

In this section we generalize the concept of dual sequences to dual sequences with respect to Riordan pseudo-involutions, which generalize the special Riordan pseudo-involutions shown in (3).

Definition 2.1. Let $(d(t), h(t)) = [d_{n,k}]_{n,k \geq 0}$ be a Riordan pseudo-involution, and let $\{s_n\}_n$ be a sequence of complex numbers. The sequence $\{s_{1,n}^*\}_n$ defined by

$$s_{1,n}^* = \sum_{k=0}^n d_{n,k}(-1)^k s_k$$

is called the generalized dual sequence of $\{s_n\}_n$. By simplicity we denote $s_n^* \equiv s_{1,n}^*$.

Notice that the additional sequences defined by

$$s_{2,n}^* = \sum_{k=0}^n d_{n,k}(-1)^{k+1} s_k, \quad s_{3,n}^* = \sum_{k=0}^n d_{n,k}(-1)^n s_k, \quad \text{and} \quad s_{4,n}^* = \sum_{k=0}^n d_{n,k}(-1)^{n+1} s_k$$

satisfies the basic relations $s_{2,n}^* = -s_n^*$ and $s_{3,n}^* = -s_{4,n}^*$. Denote the generating functions of the sequences $\{s_n\}_{n \geq 0}$ and $\{s_{\ell,n}^*\}_{n \geq 0}$ ($\ell = 1, 2, 3, 4$) by $S(t)$ and $S_\ell(t)$ ($\ell = 1, 2, 3, 4$), respectively. The definition of the generalized dual sequences is equivalent to the equalities

$$\begin{aligned} S_1(t) &= (d(t), -h(t))S(t), & S_2(t) &= (-d(t), -h(t))S(t), \\ S_3(t) &= (d(-t), h(-t))S(t), & S_4(t) &= (-d(-t), h(-t))S(t). \end{aligned}$$

From Definition 2.1 we have the following result.

Proposition 2.2. Let $(d(t), h(t)) = [d_{n,k}]_{n,k \geq 0}$ be a Riordan pseudo-involution, and let $\{s_n\}_n$ be a sequence of complex numbers. The generalized dual sequence $\{s_n^*\}_n$ satisfies $s_n = \sum_{k=0}^n d_{n,k}(-1)^k s_k^*$ and $s_n^{**} = s_n$. Additionally, we have the equalities $s_n = \sum_{k=0}^n d_{n,k}(-1)^{k+1} s_{2,k}^*$, $s_n = \sum_{k=0}^n d_{n,k}(-1)^n s_{3,k}^*$, and $s_n = \sum_{k=0}^n d_{n,k}(-1)^{n+1} s_{4,k}^*$. Moreover, $s_{\ell,n}^{**} = s_n$ for $\ell = 2, 3, 4$.

Proof. Since $\pm(d(t), h(t)) * (1, -t) = (\pm d(t), -h(t))$ and $\pm(1, -t) * (d(t), h(t)) = (\pm d(-t), h(-t))$ are Riordan array involutions, we have

$$\begin{aligned} S(t) &= (d(t), -h(t))S_1(t), & S(t) &= (-d(t), -h(t))S_2(t), \\ S(t) &= (d(-t), h(-t))S_3(t), & S(t) &= (-d(-t), h(-t))S_4(t), \end{aligned}$$

which imply the desired result. Moreover, we have $s_{\ell,n}^{**} = s_n$ for all integer $n \geq 0$ and $\ell = 1, 2, 3, 4$ because

$$[d_{n,k}(-1)^k]_{n,k \geq 0}^2 = [d_{n,k}(-1)^{k+1}]_{n,k \geq 0}^2 = [d_{n,k}(-1)^n]_{n,k \geq 0}^2 = [d_{n,k}(-1)^{n+1}]_{n,k \geq 0}^2 = I,$$

where $I = (1, t)$ is the identity Riordan array. □

2.1 Some applications

The Fuss-Catalan numbers are defined by

$$F_m(n, r) := \frac{r}{mn+r} \binom{mn+r}{n}, \quad (4)$$

which are named after N. I. Fuss and E. C. Catalan (see for example [2]). The Fuss-Catalan numbers have many combinatorial interpretations (cf. [7]). The generating function $F_m(t)$ for the Fuss-Catalan numbers $\{F_m(n, 1)\}_{n \geq 0}$ is called the generalized binomial series in [2], and it satisfies the functional equation

$$F_m(t) = 1 + tF_m(t)^m. \quad (5)$$

Hence from Lambert's formula for the Taylor expansion of the powers of $F_m(t)$ (see [2, Ec. (5.60)])

$$F_m^r \equiv F_m(t)^r = \sum_{n \geq 0} \frac{r}{mn+r} \binom{mn+r}{n} t^n = \sum_{n \geq 0} F_m(n, r) t^n.$$

For the particular cases $m = 1, 2, 3$, we have the generating functions $F_1 = 1/(1-t)$, $F_2 = C(t) = (1 - \sqrt{1-4t})/2t$, and $F_3 = T(t)$, where $C(t)$ and $T(t)$ are the generating functions of Catalan number and ternary number.

From Theorem 1.1 of Shapiro and the first author [5], we have the following result.

Theorem 2.3. *Let $F_m(t)$ be the generating function defined by (5), and let $\{F_m(n, r)\}_{n \geq 0}$ be the coefficients defined by (4). Then the relationships*

$$\begin{aligned} s_{1,n}^* &= \sum_{k=0}^n F_m(n-k, (2m-1)k+1)(-1)^k s_k, \\ s_{2,n}^* &= \sum_{k=0}^n F_m(n-k, (2m-1)k+1)(-1)^{k+1} s_k, \\ s_{3,n}^* &= \sum_{k=0}^n F_m(n-k, (2m-1)k+1)(-1)^n s_k, \\ s_{4,n}^* &= \sum_{k=0}^n F_m(n-k, (2m-1)k+1)(-1)^{n+1} s_k, \end{aligned} \quad (6)$$

generate four dual sequences of $\{s_n\}_{n \geq 0}$.

Proof. It is sufficient to show that the first equation of (6) defines a dual sequence of $\{s_n\}_{n \geq 0}$. Other equations can be proved with a similar argument. Since

$$F_m(n-k, (2m-1)k+1)(-1)^k = [t^{n-k}]F_m^{(2m-1)k+1}(t)(-1)^k = [t^n]F_m(t) \left(-tF_m^{2m-1}(t)\right)^k,$$

from the definition of the generalized dual sequence we only need to show that $(F_m(t), -tF_m^{2m-1}(t))$ is a Riordan involution, or equivalently, $(F_m(t), tF_m^{2m-1}(t))$ is a pseudo-involution. In fact, this is the consequence of Theorem 1.1 of [5], which shows that the Riordan array $(F_m^l(t), tF_m^r(t))$ is a pseudo involution if and only if $r = 2m - 1$. $\square$

Corollary 2.4. *The sequences $\{F_m(n-1, 1)\}_{n \geq 1}$ and $(-F_m(n-1, 2m-1))_{n \geq 1}$ are generalized dual each other with respect to the dual relationship given in (6), that is,*

$$s_n^* = \sum_{k=0}^n \frac{(2m-1)k+1}{mn+(m-1)k+1} \binom{mn+(m-1)k+1}{n-k} (-1)^k s_k. \quad (7)$$

Proof. Let $s_n = F_m(n-1, 1) = [t^{n-1}]F_m(t)$. Then the right-hand side of (7) can be written as

$$\begin{aligned} [t^n](F_m(t), -tF_m^{2m-1}(t))(tF_m(t)) &= [t^n]F_m(t) \left(-tF_m^{2m-1}(t)F_m(-tF_m^{2m-1}(t))\right) \\ &= [t^n] \left(-tF_m^{2m-1}(t)\right) \left(F_m(t)F_m(-tF_m^{2m-1}(t))\right) \\ &= [t^n] \left(-tF_m^{2m-1}(t)\right) \\ &= -F_m(n-1, 2m-1) = s_n^*. \end{aligned}$$

Notice that we use that $-tF_m(t)^r$ is an involution if and only if $r = 2m - 1$, see [5, Theorem 1.1]. So $F_m(t)F_m(-tF_m^{2m-1}(t)) = 1$.

Conversely, if $s_n = -F_m(n-1, 2m-1) = [t^n](-tF_m^{2m-1}(t))$, then the right-hand side of (7) becomes

$$[t^n](F_m(t), -tF_m^{2m-1}(t))(-tF_m^{2m-1}(t)) = [t^n]F_m(t) \left(tF_m^{2m-1}(t)F_m(-tF_m^{2m-1}(t))\right)$$

$$= [t^n] (tF_m(t)).$$

□

For example, if $m = 2$ then the sequences $\{C(n-1)\}_{n \geq 1}$ and $\{-F_2(n-1, 3)\}_{n \geq 1}$, where $C(n) = \frac{1}{n+1} \binom{2n}{n}$ and $F_2(n-1, 3) = \frac{3}{2n+3} \binom{2n+3}{n}$, are dual each other with respect to the relationship

$$s_n^* = \sum_{k=1}^n \frac{3k+1}{2n+k+1} \binom{2n+k+1}{n-k} (-1)^k s_k.$$

Therefore we obtain the combinatorial identities

$$-\frac{3}{2n+1} \binom{2n+1}{n-1} = \sum_{k=1}^n \frac{3k+1}{k(2n+k+1)} \binom{2n+k+1}{n-k} \binom{2(k-1)}{k-1} (-1)^k, \quad n \geq 0$$

and

$$-\frac{1}{n} \binom{2(n-1)}{n-1} = \sum_{k=0}^n \frac{3(3k+1)}{(2n+k+1)(2k+1)} \binom{2n+k+1}{n-k} \binom{2k+1}{k-1} (-1)^k, \quad n \geq 1.$$

For $m = 3$ we obtain the combinatorial identities

$$-\frac{5}{3n+1} \binom{3n+2}{n-1} = \sum_{k=0}^n \frac{5k+1}{(3k-2)(3n+2k+1)} \binom{3n+2k+1}{n-k} \binom{3k-2}{k-1} (-1)^k, \quad n \geq 0$$

and

$$-\frac{1}{3n-2} \binom{3n-2}{n-1} = \sum_{k=0}^n \frac{5(5k+1)}{(3k+2)(3n+2k+1)} \binom{3n+2k+1}{n-k} \binom{2k+1}{k-1} (-1)^k, \quad n \geq 0.$$

We now consider a special class of Riordan array defined by Mu, Mao, and Wang in [8] by

$$R(p) := \left(\frac{d(t)}{1-ph(t)}, h(t) \right). \quad (8)$$

Thus, the entries of $R(p)$ are given by

$$d_{n,k} = [t^n] \frac{d(t)}{1-ph(t)} h(t)^k = \sum_{j \geq 0} p^j [t^n] d(t) h(t)^{k+j} = \sum_{j \geq 0} p^j \tilde{d}_{n,k+j} = \sum_{j=k}^n p^{j-k} \tilde{d}_{n,j},$$

where $\tilde{d}_{n,j}$ are the entries of the Riordan array $(d(t), h(t))$.

Theorem 2.5. *The Riordan array $R(p)$ is a pseudo involution if and only if for some constant $p \neq 0$*

$$d(t)d(-h(t)) = 1 + pt - ph(t) - p^2 th(t) \quad \text{and} \quad h(-h(t)) = -t. \quad (9)$$

Let $\tilde{d}_{n,k}$ be the entries of the Riordan array $(d(t), h(t))$. A generalized dual sequence of $\{s_n\}_{n \geq 0}$ can be found from

$$s_n^*(p) = \sum_{k=0}^n \left(\sum_{j=k}^n \tilde{d}_{n,j} p^{j-k} \right) (-1)^k s_k,$$

where $d(t)$ and $h(t)$ satisfies (9) for some $p \neq 0$.

Proof. Suppose that $R(p)$ is a pseudo involution. Then

$$\left(\frac{d(t)}{1-ph(t)}, -h(t) \right) * \left(\frac{d(t)}{1-ph(t)}, -h(t) \right) = (1, t).$$

The above equation holds if and only if $-h(-h(t)) = t$ and

$$\frac{d(t)}{1-ph(t)} \frac{d(-h(t))}{1-ph(-h(t))} = 1,$$

or equivalently, $d(t)d(-h(t)) = (1-ph(t))(1+pt)$. Thus, we obtain the desired result. □

Notice that for $d(t) = 1/(1-t)$ and $h(t) = t/(1-t)$, Equation (9) holds if and only if

$$pt(1-t) - pt - p^2t^2 = -p(p+1)t^2 = 0$$

Therefore $p = -1$. For example, the Fibonacci sequence $\{F_n\}_n$ can be considered as self-dual with respect to the relationship given in the Theorem 2.5 (with $p = -1$, $d(t) = 1/(1-t)$, and $h(t) = t/(1-t)$) in the following sense

$$-F_{n-2} = \sum_{k=0}^n \left(\sum_{j=k}^n \binom{n}{j} (-1)^j F_k \right), \quad n \geq 1.$$

For example, the Motzkin numbers M_n , defined by $M_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C(k)$, are the generalized dual sequence of the Catalan numbers in the following sense

$$(-1)^{n+1} M_{n-1} = \sum_{k=0}^n \left(\sum_{j=k}^n \binom{n}{j} (-1)^j C(k) \right), \quad n \geq 1.$$

2.2 A generalization of dual polynomial sequences

A sequence $\{p_n(x)\}_n$ of polynomials is called a *binomial type sequence* if $p_n(x+y) = \sum_{\ell=0}^n \binom{n}{\ell} p_\ell(x) p_{n-\ell}(y)$, for all integer $n \geq 0$, with $p_0(x) = 1$, and the additional condition that for any positive integer n , $p_n(x)$ is a polynomial of degree n with $p_n(0) = 0$.

We now extend the definition of generalized dual polynomial sequences respect to the binomial type sequences.

Definition 2.6. Let $\{s_n^*\}_{n \geq 0}$ be the generalized dual sequence of $\{s_n\}_{n \geq 0}$ with respect to the Riordan array $(d(t), h(t)) = [d_{n,k}]_{n,k \geq 0}$, and let $p_n(x)$ be a binomial type polynomial sequence. The polynomials defined by

$$S_n(x) := \sum_{k=0}^n d_{n,k} p_{n-k}(x) s_k \quad \text{and} \quad S_n^*(x) := \sum_{k=0}^n d_{n,k} p_{n-k}(x) s_k^*$$

are called the pair of dual of polynomials of n -th order.

For example, if $d(t) = F_m(t)$ and $h(t) = tF_m(t)^{2m-1}$, then the corresponding pair of dual polynomial sequences $\{S_n(x)\}_{n \geq 0}$ and $\{S_n^*(x)\}_{n \geq 0}$ are given by the combinatorial expressions

$$\begin{aligned} S_n(x) &= \sum_{k=0}^n F_m(n-k, (2m-1)k+1) p_{n-k}(x) s_k \\ &= \sum_{k=0}^n \frac{(2m-1)k+1}{mn + (m-1)k+1} \binom{mn + (m-1)k+1}{n-k} p_{n-k}(x) s_k, \\ S_n^*(x) &= \sum_{k=0}^n F_m(n-k, (2m-1)k+1) p_{n-k}(x) s_k^* \\ &= \sum_{k=0}^n \frac{(2m-1)k+1}{mn + (m-1)k+1} \binom{mn + (m-1)k+1}{n-k} p_{n-k}(x) s_k^*. \end{aligned}$$

From Corollary 2.4, if $s_k = [t^k]tC(t) = 1/k \binom{2k-2}{k-1}$ and $s_k^* = [t^k](-tC(t))^3 = \frac{-3}{2k+1} \binom{2k+1}{k-1}$, then we have a pair of dual of Catalan polynomial sequences given by

$$\begin{aligned} S_n(x) &:= \sum_{k=0}^n \frac{3k+1}{k(2n+k+1)} \binom{2n+k+1}{n-k} \binom{2k-2}{k-1} x^{n-k}, \\ S_n^*(x) &:= \sum_{k=0}^n \frac{-3(3k+1)}{(2k+1)(2n+k+1)} \binom{2n+k+1}{n-k} \binom{2k+1}{k-1} x^{n-k}. \end{aligned}$$

3 Dual sequences of Sheffer sequences

A sequence $\{p_n(x)\}_n$ is a *Sheffer polynomial sequence* (or Sheffer sequence for abbreviation) for the pair $(g(t), f(t))$ if and only if its exponential generating function has the form

$$\sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)}, \quad (10)$$

where $\bar{f}(t)$ denotes the compositional inverse of $f(t)$. From the definitions of Riordan array and Sheffer sequences, one may establish a one-to-one correspondence between them (cf. [4, 16]).

One of the most famous classes of polynomial sequences is the class of Sheffer sequences, which contains sequences of Bernoulli polynomials, Euler polynomials, Abel polynomials, Hermite polynomials, Laguerre polynomials, and many other important polynomials. Rota, Roman, et al. (cf. [10–12]) studied the Sheffer sequences systematically by using the theory of modern umbral calculus.

In analogy with the generalized dual sequence, we introduce one more generalization respect to the Sheffer sequences.

Definition 3.1. Let $\{s_n(x)\}_n$ be a Sheffer polynomial sequence for the pair $(g(t), f(t))$. The dual Sheffer numbers respect to the sequence $\{s_n(x)\}_n$ are defined by

$$s_n^*(a) := \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell s_\ell(a),$$

where a is a real number. Analogously, the dual Sheffer polynomials are defined by

$$s_n^*(a, x) := \sum_{\ell=0}^n \binom{n}{\ell} p_{n-\ell}(x) s_\ell^*(a),$$

where $p_n(x)$ is the binomial type sequence associated to $f(t)$.

Example 3.2. The generalized Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x)$ are defined by the exponential generating function

$$\sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x) \frac{t^n}{n!} = \left(\frac{t}{e^t - 1} \right)^\alpha e^{xt}.$$

They are a Sheffer sequence for the pair $\left((e^t - 1)/t, t \right)$. In this case the binomial type sequence associated to $f(t) = t$ is $p_n(x) = x^n$. Then $\mathfrak{B}_n^{(\alpha)}(x) = \sum_{\ell=0}^n \binom{n}{\ell} B_\ell^{(\alpha)} x^{n-\ell}$, where $B_n^{(\alpha)}(0) = B_n^{(\alpha)}$. The generalized dual Bernoulli polynomials (for $a = 0$) $\mathfrak{B}_n^{(\alpha)*}(x)$ are given by

$$\mathfrak{B}_n^{(\alpha)*}(0, x) = \mathfrak{B}_n^{(\alpha)*}(x) = \sum_{\ell=0}^n \binom{n}{\ell} B_\ell^{(\alpha)*} x^{n-\ell},$$

where $B_n^{(\alpha)*}$ are the dual Bernoulli numbers defined by $B_n^{(\alpha)*}(0) = B_n^{(\alpha)*} = \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell B_\ell^{(\alpha)}$. By setting $\alpha = 1$ we recover the sequences studied in [6].

A Sheffer sequence for $(g(t), t)$ is called an *Appell sequence* for $g(t)$. For example, the generalized Bernoulli polynomials are an Appell sequence for $g(t) = (e^t - 1)/t$. The dual of any Appell sequence $s_n(x)$ is given by $s_n^*(a, x) := \sum_{\ell=0}^n \binom{n}{\ell} s_\ell^*(a) x^{n-\ell}$, where $s_n^*(a) := \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell s_\ell(a)$.

Theorem 3.3. Let $S^*(a, x)$, $S^*(a)$, $S(x)$, and $D[t]$ denote the matrices defined by

$$\begin{aligned} S^*(a, x) &:= (s_0^*(a, x), s_1^*(a, x), s_2^*(a, x), \dots)^T, & S^*(a) &:= (s_0^*(a), s_1^*(a), s_2^*(a), \dots)^T, \\ S(x) &:= (s_0(x), s_1(x), s_2(x), \dots)^T, & D[t] &:= (1, t, t^2, \dots)^T, \end{aligned}$$

where $s_n(x)$ is an Appell sequence, then $S^*(a, x) = D[-1]S(a - x - 1)$. In particular, $S^*(a) = D[-1]S(a - 1)$.

Proof. From definition of dual Sheffer polynomials and numbers, it is clear that

$$S^*(a, x) = \mathbb{P}[x]S^*(a) \quad \text{and} \quad S^*(a) = D[-1]\mathbb{P}[-1]S(a),$$

where $\mathbb{P}[x]$ is the Pascal matrix function defined by $\mathbb{P}[x] := \left[\binom{n}{k} x^{n-k} \right]_{n,k \geq 0}$. Since $\mathbb{P}[x+y] = \mathbb{P}[x]\mathbb{P}[y]$ for any $x, y \in \mathbb{R}$, we have

$$\begin{aligned} S^*(a, x) &= \mathbb{P}[x]S^*(a) = \mathbb{P}[x]D[-1]\mathbb{P}[-1]S(a) \\ &= D[-1]\mathbb{P}[-x]\mathbb{P}[-1]S(a) = D[-1]\mathbb{P}[-x-1]S(a) = D[-1]S(a-x-1). \end{aligned}$$

□

From above theorem we obtain the relation

$$s_n^*(a, x) = (-1)^n s_n(a-x-1), \quad (11)$$

for any Appell sequence.

Theorem 3.4. Let $s_n(x)$ be an Appell sequence for $g(t)$. Then the exponential generating function of the dual Sheffer polynomials $s_n^*(a, x)$ is

$$\sum_{n=0}^{\infty} s_n^*(a, x) \frac{t^n}{n!} = \frac{1}{g(-t)} e^{-t(a-x-1)}.$$

Proof. From (11) and (10) we have

$$\sum_{n=0}^{\infty} s_n^*(a, x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n s_n(a-x-1) \frac{t^n}{n!} = \frac{1}{g(-t)} e^{-t(a-x-1)}.$$

□

Example 3.5. The generating function of the dual generalized Bernoulli polynomials is

$$\sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)*}(x) \frac{t^n}{n!} = \frac{t^\alpha e^{(x+1)t}}{(1-e^{-t})^\alpha}.$$

In particular, the generating function of the dual Bernoulli polynomials is

$$\sum_{n=0}^{\infty} \mathfrak{B}_n^*(x) \frac{t^n}{n!} = \frac{e^{(x+1)t}}{1 + \left(\frac{1-t-e^{-t}}{t} \right)}.$$

Theorem 3.4 can be generalized to any Sheffer sequence.

Theorem 3.6. Let $s_n(x)$ be a Sheffer sequence for the pair $(g(t), f(t))$. The exponential generating function of the dual Sheffer sequence $s_n^*(a, x)$ is

$$\sum_{n=0}^{\infty} s_n^*(a, x) \frac{t^n}{n!} = \frac{e^{t+a\bar{f}(-t)+x\bar{f}(t)}}{g(\bar{f}(-t))}.$$

3.1 Some applications

In this section we show that the dual Sheffer numbers and polynomials of some well-known families of special polynomials have combinatorial interpretations by using set partitions and permutations. Remember that a *partition* of a set $[n] := \{1, 2, \dots, n\}$ is a collection of nonempty disjoint subsets, called blocks, whose union is $[n]$. The total number of partitions of $[n]$ is counted by the Bell number.

3.1.1 The dual Hermite polynomials

The Hermite polynomials $H_n^{(v)}(x)$ are defined by the exponential generating function $\sum_{n=0}^{\infty} H_n^{(v)}(x) \frac{t^n}{n!} = e^{xt - vt^2/2}$. They are Appell for $g(t) = e^{vt^2/2}$. The dual Hermite polynomials $H_n^{(v)*}(a, x)$ are given by

$$H_n^{(v)*}(a, x) = \sum_{\ell=0}^n \binom{n}{\ell} H_{\ell}^{(v)*}(a) x^{n-\ell},$$

where $H_n^{(v)*}(a)$ are the dual Hermite numbers defined by $H_n^{(v)*}(a) = \sum_{\ell=0}^n \binom{n}{\ell} (-1)^{\ell} H_{\ell}^{(v)}(a)$. From Theorem 3.4 the generating function of the dual Hermite polynomials is

$$\sum_{n=0}^{\infty} H_n^{(v)*}(a, x) \frac{t^n}{n!} = e^{t(x+1-a) - vt^2/2}.$$

The first few terms of the dual Hermite numbers $H_n^{(1)*}(a)$ are

$$\begin{aligned} H_0^{(1)*}(a) &= 1, & H_1^{(1)*}(a) &= -a + 1, & H_2^{(1)*}(a) &= a^2 - 2a, & H_3^{(1)*}(a) &= -a^3 + 3a^2 - 2, \\ H_4^{(1)*}(a) &= a^4 - 4a^3 + 8a - 2, & H_5^{(1)*}(a) &= -a^5 + 5a^4 - 20a^2 + 10a + 6. \end{aligned}$$

In particular, if $a = 0$ and $v = 1$ we have the sequence A001464 from the OEIS [17].

$$\{H_n^{(1)*}(0)\}_n = \{1, -1, 0, 2, -2, -6, 16, 20, -132, -28, 1216, \dots\}.$$

Moreover, $H_0(t) := \sum_{n=0}^{\infty} H_0^{(1)*}(0) \frac{t^n}{n!} = e^{-t-t^2/2}$. Note that $H_0(-t)$ is the exponential generating function of the number of even involutions minus the number of odd involutions [13]. If $a = 0$ and $v = -1$ we have

$$\{H_n^{(-1)*}(0)\}_n = \{1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, \dots\} \quad (\text{A000085})$$

Moreover, $H_1(t) := \sum_{n=0}^{\infty} H_1^{(-1)*}(0) \frac{t^n}{n!} = e^{t+t^2/2}$. This is the generating function of the total number of involutions of a set with n elements or equivalently the sequence of set partitions with blocks of size at most 2.

3.1.2 The dual lower factorial polynomials

The lower factorial polynomials $(x)_n$ are defined by $(x)_n := x(x-1)\cdots(x-n+1)$ for $n \geq 1$ and $(x)_0 = 1$. Their generating function is $\sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!} = (1+t)^x$. The dual lower factorial polynomials $D_n^*(a, x)$ are given by

$$D_n^*(a, x) = \sum_{\ell=0}^n \binom{n}{\ell} (x)_{n-\ell} D_{\ell}^*(a)^*,$$

where $D_n(a)^* = \sum_{\ell=0}^n \binom{n}{\ell} (-1)^{\ell} (a)_{\ell}$. From Theorem 3.6 the generating function of the dual lower factorial polynomials is

$$\sum_{n=0}^{\infty} D_n^*(a, x) \frac{t^n}{n!} = e^t (1-t)^a (1+t)^x.$$

The first few terms of the dual lower factorial numbers $D_n^*(a)$ are

$$\begin{aligned} D_0^*(a) &= 1, & D_1^*(a) &= -a + 1, & D_2^*(a) &= a^2 - 3a + 1, & D_3^*(a) &= -a^3 + 6a^2 - 8a + 1, \\ D_4^*(a) &= a^4 - 10a^3 + 29a^2 - 24a + 1, & D_5^*(a) &= -a^5 + 15a^4 - 75a^3 + 145a^2 - 89a + 1. \end{aligned}$$

In particular, if $a = -1$ we obtain the sequence

$$\{D_n^*(-1)\}_n = \{1, 2, 5, 16, 65, 326, 1957, 13700, 109601, 986410, 9864101, \dots\} \quad (\text{A000522}).$$

Moreover,

$$D_0(t) := \sum_{n=0}^{\infty} D_n^*(-1) \frac{t^n}{n!} = \frac{e^t}{1-t}.$$

The generating function $D_0(t)$ counts the total number of arrangements of a set with n elements (cf. [9]).

3.1.3 The dual Bell polynomials

The Bell polynomials $\mathfrak{B}_n(x)$ are defined by the generating function $\sum_{n=0}^{\infty} \mathfrak{B}_n(x) \frac{t^n}{n!} = e^{x(e^t-1)}$. They are Sheffer for $(1, \ln(1+t))$. Note that $\mathfrak{B}_n(0) = \mathfrak{B}_n$ are the Bell number. This sequence counts the number of set partitions of $[n]$. The dual Bell polynomials $\mathfrak{B}_n^*(a, x)$ are given by

$$\mathfrak{B}_n^*(a, x) = \sum_{\ell=0}^n \binom{n}{\ell} \mathfrak{B}_\ell^*(a) \mathfrak{B}_{n-\ell}(x),$$

where $\mathfrak{B}_n^*(a) = \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell \mathfrak{B}_\ell(a)$. From Theorem 3.4 the generating function of the dual Bell polynomials is

$$\sum_{n=0}^{\infty} \mathfrak{B}_n^*(a, x) \frac{t^n}{n!} = e^{t+a(e^{-t}-1)+x(e^t-1)}.$$

The first few terms of the dual Bell numbers $\mathfrak{B}_n^*(a)$ are

$$\begin{aligned} \mathfrak{B}_0^*(a) &= 1, & \mathfrak{B}_1^*(a) &= -a + 1, & \mathfrak{B}_2^*(a) &= a^2 - a + 1, & \mathfrak{B}_3^*(a) &= -a^3 - a + 1, \\ \mathfrak{B}_4^*(a) &= a^4 + 2a^3 + a^2 - a + 1, & \mathfrak{B}_5^*(a) &= -a^5 - 5a^4 - 5a^3 - a + 1. \end{aligned}$$

In particular, if $a = -1$ we have

$$\{\mathfrak{B}_n^*(1)\}_n = \{1, 0, 1, -1, 4, -11, 41, -162, 715, -3425, 17722, \dots\} \quad (\text{A000296}).$$

Moreover,

$$\mathfrak{B}_0(t) := \sum_{n=0}^{\infty} \mathfrak{B}_n^*(1) \frac{t^n}{n!} = e^{e^{-t}+t-1}.$$

The generating function $\mathfrak{B}_0(-t)$ counts the total number of set partitions of a set with n elements without singletons (cf. [9]).

3.2 The dual Mittag-Leffler polynomials

The Mittag-Leffler polynomials $M_n(x)$ are defined by the generating function $\sum_{n=0}^{\infty} M_n(x) \frac{t^n}{n!} = \left(\frac{1+t}{1-t}\right)^x$. They are Sheffer for $(1, \frac{e^t-1}{e^t+1})$. The dual Mittag-Leffler polynomials $M_n^*(a, x)$ are given by $M_n^*(a, x) = \sum_{\ell=0}^n \binom{n}{\ell} M_\ell^*(a) M_{n-\ell}(x)$, where $M_n^*(a) = \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell M_\ell(a)$. From Theorem 3.4 the generating function of the dual Mittag-Leffler polynomials is

$$\sum_{n=0}^{\infty} M_n^*(a, x) \frac{t^n}{n!} = e^t (1-t)^{a-x} (1+t)^{x-a}.$$

4 Sheffer-type polynomial sequences

In [4], the isomorphism between the Sheffer group and the Riordan group has been established. Hence, Sheffer subgroups can be found with the aid of Riordan subgroups. More precisely, for any Riordan subgroup $\{(d(t), h(t))\}$ the corresponding isomorphic Sheffer subgroups $\{p_n(x)\}_{n \geq 0}$ are given by

$$p_n(x) := \theta(d(t), h(t)) \equiv \left(\frac{1}{d(\bar{h})}, \bar{h}\right) e^{xt} \equiv \sum_{k \geq 0} \bar{d}_{n,k} \frac{x^k}{k!}, \quad (12)$$

where $\bar{d}_{n,k} = [t^n](\bar{h}(t))^k / d(\bar{h}(t))$. For example, the Appell subgroup $(d(t), t)$ in the Riordan group implies the Appell subgroup $\{p_n(x) = [t^n]g(t)e^{xt}\}$, $g(t) = 1/d(t)$, in the Sheffer group, which includes Bernoulli polynomials and Euler polynomials as special cases. Similarly, from the subgroups of Riordan group shown in Shapiro [15], we have the following subgroups of the Sheffer group:

- (1) The c -Appell subgroup $\{[t^n]g(t)e^{cxt}\}_{n \geq 0}$ corresponds to the Appell subgroup $\{(d(t), t)\}$ of the Riordan group, where $g(t) = 1/d(t)$. When $c = 1$, c -Appell subgroup is the Appell subgroup.
- (2) The associated subgroup $\{[t^n]e^{xf(t)}\}_{n \geq 0}$ corresponds to the associated subgroup $\{(t, h(t))\}$ of the Riordan group, where $f(t) = \bar{h}(t)$.
- (3) The c -Bell subgroup $\{[t^n]g(t)e^{cxtg(t)} : c \neq 0\}_{n \geq 0} = \{[t^n](f(t)/t)e^{cf(t)} : c \neq 0\}_{n \geq 0}$ corresponds to the Bell subgroup $\{(d(t), td(t))\}$, where $g(t) = 1/d(t)$. When $c = 1$, c -Bell subgroup is called the Bell group.
- (4) The checkerboard subgroup $\{[t^n]g(t)e^{xf(t)} : g(-t) = g(t), f(-t) = -f(t)\}_{n \geq 0}$ corresponds to the checkerboard subgroup $\{(d(t), h(t)) : d(-t) = d(t), h(-t) = -h(t)\}$, where $g(t) = 1/d(t)$ and $f(t) = \bar{h}(t)$.
- (5) The hitting time subgroup $\{[t^n](tf'(t)/f(t))e^{xf(t)}\}_{n \geq 0}$ corresponds to the hitting time subgroup $\{(d(t), h(t)) : d(t) = th'(t)/h(t)\}$, where $f(t) = \bar{h}(t)$.

From the above observation, we can construct numerous Sheffer-type polynomials sequences related to a given number sequence in a subgroup of the Sheffer group.

Definition 4.1. Let $\mathcal{B} = \{b_n\}_{n \geq 0}$ be a complex sequence such that $\mathcal{B}$ is the row sum sequence of the Riordan array $(A(t), g(t))$. A sequence of polynomials $\{p_n(x)\}_n$ is called the Sheffer-type polynomial sequence associated with $\mathcal{B}$ if $p_n(x) = [t^n]A(t)e^{xg(t)}$ for all $n \geq 0$. If $p_n(x)$ is in a certain subgroup of the Sheffer group, then $\{p_n(x)\}_n$ is called the Sheffer-type polynomial sequence associated with $\mathcal{B}$ in the Sheffer subgroup.

Usually, the Sheffer-type polynomial sequence associated with a given number sequence is not unique. For example, the sequence of Catalan numbers $\mathcal{B} = \{C(n)\}_n$ can be obtained in the following two ways

$$\left(\frac{1}{1-2t}, \frac{1-4t-\sqrt{1-4t}}{2-4t} \right) \frac{1}{1-t} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 4 & -3 & 1 & 0 & 0 & 0 \\ 8 & -6 & 4 & -1 & 0 & 0 \\ 16 & -7 & 9 & -5 & 1 & 0 \\ 32 & 10 & 8 & -13 & 6 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ \vdots \end{bmatrix}$$

and

$$\left(\frac{1}{1-4t}, \frac{1-8t-\sqrt{1-4t}}{2-8t} \right) \frac{1}{1-t} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 & 0 & 0 \\ 16 & -23 & 9 & 0 & 0 & 0 \\ 64 & -134 & 102 & -27 & 0 & 0 \\ 256 & -699 & 781 & -405 & 81 & 0 \\ 1024 & -3434 & 5026 & -3843 & 1512 & -243 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ \vdots \end{bmatrix}.$$

Therefore we have the following two different Sheffer-type polynomial sequences associated to the Catalan numbers

$$p_n(x) = [t^n] \frac{1}{1-2t} \exp \left(x \left(\frac{1-4t-\sqrt{1-4t}}{2-4t} \right) \right) \quad \text{and} \\ q_n(x) = [t^n] \frac{1}{1-4t} \exp \left(x \left(\frac{1-8t-\sqrt{1-4t}}{2-8t} \right) \right).$$

The first few terms of these polynomials are

$$\{p_n(x)\}_n = \left\{ 1, 2-x, \frac{1}{2}(x^2-6x+8), -\frac{x^3}{6}+2x^2-6x+8, \frac{x^4}{24}-\frac{5x^3}{6}+\frac{9x^2}{2}-7x+16, \dots \right\}, \\ \{q_n(x)\}_n = \left\{ 1, 4-3x, \frac{9x^2}{2}-23x+16, -\frac{9x^3}{2}+51x^2-134x+64, \dots \right\}.$$

However, in some subgroups, such as the Appell subgroup, associated subgroup, Bell subgroup, etc., the sequence is unique. We now construct the corresponding Sheffer-type polynomials in the Appell, associated, and Bell subgroups.

Theorem 4.2. Let $\mathcal{B} = \{b_n\}_{n \geq 0}$ be a complex sequence with generating function $P(t)$. Then its corresponding Sheffer-type polynomials in the Appell, associated, and Bell subgroups are respectively $\{[t^n](P(t)-tP(t))e^{xt}\}_{n \geq 0}$, $\{[t^n]e^{x(P(t)-1)/P(t)}\}_{n \geq 0}$ (for $P(t) \neq 0$), and $\{[t^n](P(t)/(1+tP(t)))e^{xtP(t)/(1+P(t))}\}_{n \geq 0}$ (for $P(t) \neq -1/t$).

Proof. It is sufficient to prove the case for the Appell subgroup. The similar argument can be applied to prove the results for the associated and Bell subgroups. Our purpose is to find the matrix $(A(t), t)$ such that it is the Appell decomposition of the given number sequence $\mathcal{B}$ with the generating function $P(t) = \sum_{k \geq 0} b_k t^k$; or equivalently, the row sums of the matrix $(A(t), t)$ are the sequence $\mathcal{B}$. Hence, we need that $(A(t), t)$ satisfies

$$(A(t), t) \frac{1}{1-t} = \frac{A(t)}{1-t} = P(t),$$

which yields $A(t) = P(t) - tP(t)$, completing the proof. $\square$

Example 4.3. Considering the Catalan number $\{1, 1, 2, 5, 14, 42, \dots\}$, with generating function $P(t) = (1 - \sqrt{1-4t})/(2t)$. The corresponding Sheffer-type polynomial sequence $\{p_n(x)\}_n$ in the Appell subgroup, called the Catalan-Sheffer polynomial sequence, is given by

$$p_n(x) = [t^n](1-t)(1 - \sqrt{1-4t})e^{xt}/(2t),$$

which has the first few polynomial expression as follows.

$$p_0(x) = 1, p_1(x) = x, p_2(x) = 1 + \frac{x^2}{2!}, p_3(x) = 3 + x + \frac{x^3}{3!}, p_4(x) = 9 + 3x + \frac{x^2}{2!} + \frac{x^4}{4!}, \dots$$

Example 4.4. For the big Schröder number sequence, $\{1, 2, 6, 22, 90, \dots\}$, with generating function $P(t) = (1 - t - \sqrt{1-6t+t^2})/(2t)$, the corresponding Sheffer-type polynomial sequence $\{p_n(x)\}_n$ in the Appell subgroup, called the big Schröder-Sheffer polynomial sequence, is given by

$$p_n(x) = [t^n](1-t)(3t - 1 + \sqrt{1-6t+t^2})e^{xt}/(2t),$$

which has the first few elements as follows.

$$p_0(x) = 1, p_1(x) = 1 + x, p_2(x) = 4 + x + \frac{x^2}{2!}, p_3(x) = 16 + 4x + \frac{x^2}{2!} + \frac{x^3}{3!}, \\ p_4(x) = 68 + 16x + 2x^2 + \frac{x^3}{3!} + \frac{x^4}{4!}, \dots$$

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