Research Article

Mehdi Mohammadzadeh Karizaki* and Zahra Niazi Moghani

Idempotent operator and its applications in Schur complements on Hilbert $C^*$-module

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Abstract: The present study proves that $T$ is an idempotent operator if and only if $\mathcal{R}(I - T^*) \oplus \mathcal{R}(T) = X$ and $(T^*T)' = (T')^2T$. Based on the equivalent conditions of an idempotent operator and related results, it is possible to obtain an explicit formula for the Moore-Penrose inverse of 2-by-2 block idempotent operator matrix. For the 2-by-2 block operator matrix, Schur complements and generalized Schur complement are well known and studied. The range inclusions of operators and idempotency of operators are used to obtain new conditions under which we can compute the Moore-Penrose inverse of Schur complements and generalized Schur complements of operators.

Keywords: idempotent operator, Moore-Penrose inverse, Schur complements

MSC 2020: 47A05, 15A09, 46C05

1 Introduction

Hilbert $C^*$-modules are generalizations of Hilbert spaces by allowing inner products to take values in a $C^*$-algebra rather than in the field of real or complex numbers. Some fundamental properties of inner product spaces are no longer valid in inner product $C^*$-modules in their complete generality. Consequently, when we are studying inner product $C^*$-modules, it is always of interest under which conditions the results analogous to those for Hilbert spaces can be regained, as well as which more general situations might appear. The work by Lance [13] is used as a standard reference source.

Idempotent operators are a topic of considerable research with a variety of applications, including signal processing and estimation theory, see [3].

Determining the idempotent elements is not an easy task in general. The existence of idempotent elements, is of interest in the study of the structure of a non-associative algebra [6]. In ring theory, idempotents are certain basic elements, specifically in a clean ring [5]. Idempotent elements play an important role in the creation of units [19].

The notion of Schur complement was first introduced by Schur in [20]; the term Schur complement was named by Haynsworth [9], who studied the inertia of partitioned Hermitian matrices [4]. Let $T$ be the block matrix with form $T = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$. If $A$ is invertible, then the Schur complement of $A$ in $T$ is defined as $D - CA^{-1}B$, and if $D$ is invertible, then the Schur complement of $D$ in $T$ is defined as $A - BD^{-1}C$. The notion of Schur complement was extended to the case, where $A^{-1}$ was replaced by the Moore-Penrose inverse of $A$, by Albert [1] who studied the positive definiteness and nonnegative definiteness for symmetric matrices using this formula. If $A$ has closed range, then the generalized Schur complement of $A$ in $T$ is defined as $D - CA'B$, and if $D$ has closed range, then the generalized Schur complement of $D$ in $T$ is defined as $A - BD'C$. The Schur

* Corresponding author: Mehdi Mohammadzadeh Karizaki, Department of Computer Engineering, University of Torbat Heydari, Torbat Heydari, Iran, e-mail: m.mohammadzadeh@torbath.ac.ir
Zahra Niazi Moghani: Department of Computer Engineering, University of Torbat Heydari, Torbat Heydari, Iran, e-mail: zahra_nm79@yahoo.com
The present study proves that $T$ is an idempotent operator if and only if $(T^*T)' = (T')^2T$ if and only if $\mathcal{R}(I - T^*) \oplus \mathcal{R}(T) = \mathcal{X}$, equivalent conditions of characterization idempotent operators and related results give explicit formulas for the $T^*, T^+$, Moore-Penrose inverse of 2-by-2 block idempotent operator matrix and Schur complements. We find that a relationship between two sets of the spectrum of operators that satisfy in the special conditions, the communication of spectrum of Moore-Penrose inverse of idempotent with relative operators is presented, which is derived through Moore-Penrose inverse properties and the block matrix technique of operators. Using the range inclusions of operators, we obtain new conditions under which Moore-Penrose inverse of Schur complement and generalized Schur complement of $D$ in $T$ are computed. In addition, we state conditions under which the Moore-Penrose inverse of Schur complement $D$ in $T$ is computed if and only if $A'$ is inner inverse of $BD^{-1}C$ in the Hilbert $C^*$-module setting.

## 2 Preliminaries

Throughout the rest of this article, $\mathcal{A}$ is a $C^*$-algebra. An inner-product $\mathcal{A}$-module is a linear space $\mathcal{X}$ that is a right $\mathcal{A}$-module together with a map $(x, y) \rightarrow \langle x, y \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{A}$ such that, for any $x, y, z \in \mathcal{X}, \alpha, \beta \in \mathbb{C}$, and $a \in \mathcal{A}$, the following conditions hold:

(i) $(x, \alpha y + \beta z) = a(x, y) + a(y, z)$;

(ii) $(x, y a) = (x, y) a$;

(iii) $(\gamma, x) = (\gamma, y)^*$;

(iv) $(x, x) \geq 0$, and $(x, x) = 0 \Leftrightarrow x = 0$.

An inner-product $\mathcal{A}$-module $\mathcal{X}$ that is complete with respect to the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$ for $x \in \mathcal{X}$ is called a (right) Hilbert $\mathcal{A}$-module. A closed submodule $\mathcal{Z}$ of a Hilbert $\mathcal{A}$-module $\mathcal{X}$ is said to be orthogonally complemented if $\mathcal{X} = \mathcal{Z} \oplus \mathcal{Z}^*$, where

$$\mathcal{Z}^* = \{x \in \mathcal{X} : \langle x, y \rangle = 0 \text{ for any } y \in \mathcal{Z}\}.$$  

Now suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two Hilbert $\mathcal{A}$-modules. Let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the set of operators $T : \mathcal{X} \rightarrow \mathcal{Y}$ for which there is an operator $T^* : \mathcal{Y} \rightarrow \mathcal{X}$ such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \text{ for any } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}.$$  

It is known that any element $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(ax) = a(Tx)$ for $a \in \mathcal{A}$ and $x \in \mathcal{X}$. We call $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$. For any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the spectrum, the spectral radius, the range, and the null space of $T$ are represented by $\sigma(T)$, $\rho(T)$, $\mathcal{R}(T)$, and $\mathcal{N}(T)$, respectively. In case $\mathcal{X} = \mathcal{Y}$, the space $\mathcal{L}(\mathcal{X}, \mathcal{X})$, which is abbreviated to $\mathcal{L}(\mathcal{X})$, is a $C^*$-algebra.

An operator $T$ is said to be idempotent when $T^2 = T$. The term orthogonal projection will be reserved for $T$, which is self-adjoint and idempotent; in other words, $T^2 = T = T^*$. An operator $T$ is said to have a Moore-Penrose inverse if there exists $X$ such that

$$TXT = T, \quad XTX = X, \quad (TX^*)^* = TX, \quad (XT^*)^* = XT.$$  

It can be proved that if $T \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ has a Moore-Penrose inverse, then $X$ satisfying (2.1) is unique, and we write $X = T^\dagger$. If $X$ satisfies the first condition of (2.1), then $X$ is called an inner inverse of $T$.

From the definition of Moore-Penrose inverse, it can be proved that the Moore-Penrose inverse of an operator (if it exists) is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections into $\mathcal{R}(T^*)$ and $\mathcal{R}(T)$, respectively. Clearly, $T$ is Moore-Penrose invertible if and only if $T^\dagger$ is Moore-Penrose invertible, and in this case, $(T^\dagger)^\dagger = (T^\dagger)^*$.
Xu and Sheng [21] showed that a bounded adjointable operator between two Hilbert $C^*$-modules admits a bounded Moore-Penrose inverse if and only if that operator has closed range. By the definition of the Moore-Penrose inverse, we have

$$\mathcal{R}(T) = \mathcal{R}(TT^*), \quad \mathcal{R}(T^*) = \mathcal{R}(T^*T) = \mathcal{R}(T^*),$$

and by [13, Theorem 3.2], we have

$$\mathcal{N}(T) = \mathcal{N}(T^*), \quad \mathcal{N}(T^*) = \mathcal{N}(TT^*) = \mathcal{N}(T).$$

Moreover, if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range, then $(T^*T)^\dagger = T^*(T^*)^\dagger$, $T^* = T^*TT^*$ and $T^* = T^*(TT^*)^\dagger$.

The reader is referred to [13,15,16] and references therein for more details.

Throughout this article, $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented; however, it is proved that certain submodules are orthogonally complemented as described below.

**Theorem 2.1.** [13, Theorem 3.2]. Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then,

- $\mathcal{N}(T)$ is orthogonally complemented in $\mathcal{X}$ with complement $\mathcal{R}(T^*)$;
- $\mathcal{R}(T)$ is orthogonally complemented in $\mathcal{Y}$ with complement $\mathcal{N}(T^*)$;
- the map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.

Let us bring a useful lemma that its proof can be found in [10].

**Lemma 2.2.** Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and that $T \in \mathcal{L}(\mathcal{X})$ has closed range. Then, the operator $T$ has the following matrix representations with respect to the orthogonal sums of subspaces:

(i) $T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{R}(T^*), \mathcal{N}(T) \rightarrow \mathcal{R}(T), \mathcal{N}(T^*)$, where $T_1$ is invertible. Moreover,

$$T^* = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{R}(T), \mathcal{N}(T^*) \rightarrow \mathcal{R}(T^*), \mathcal{N}(T).$$

(ii) $T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \mathcal{R}(T), \mathcal{N}(T^*) \rightarrow \mathcal{R}(T), \mathcal{N}(T^*)$, where $D = T_1T_2^* + T_2T_2^* \in \mathcal{L}(\mathcal{R}(T))$ is positive and invertible. Moreover,

$$T^* = \begin{bmatrix} T_2^*D^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} : \mathcal{R}(T), \mathcal{N}(T^*) \rightarrow \mathcal{R}(T), \mathcal{N}(T^*).$$

### 3 Characterization of idempotent operator

The purpose of this section is characterizing the idempotent operator and finding the inverse of some special operators relative of idempotent operator by applying the Moore-Penrose inverse. If $T$ is idempotent, then $T^*$ and $I - T$ are idempotent operators. We use these facts in the proof of the following results.

Let us begin with an auxiliary lemma.

**Lemma 3.1.** Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and let $T \in \mathcal{L}(\mathcal{X})$. Then, the following statements are equivalent:

1. $T^2 = T$;
2. $(T^*)^2 = T^*T^*$;
3. $(T^*)^2 = T^*T^*$;
4. $T^* = PQ$ for two projections $P$ and $Q$;
5. $|T^*x|^2 = \langle T^*x, x \rangle$ for all $x \in (\mathcal{N}(T^*))^\perp$;
Proof. (1) ⇔ (2) ⇔ ⋯ ⇔ (6) In [17, Theorem 2.3], replacing $T^\dagger$ with $T$ and \textit{vice versa} implies that (1) ⇔ (2) ⇔ ⋯ ⇔ (6) are equivalent.

(6) ⇔ (7) Evident.

(6) ⇔ (8) Replace $T$ with $T^*$ and \textit{vice versa}.

(8) ⇔ (9) Replace $T$ with $I - T$ and \textit{vice versa}.

(9) ⇔ (10) Evident.

(10) ⇔ (11) Since $T^*(I - T)^\dagger = 0$, then $R((I - T)^*) \subseteq N(T^*) = N(T^\dagger)$, which implies that $T^\dagger(I - T)^\dagger = 0$. The converse is similar.

In the following theorem, we obtain the Moore-Penrose inverse of an idempotent operator.

**Theorem 3.2.** Let $X$ be a Hilbert $\mathcal{A}$-module, and suppose that $T \in \mathcal{L}(X)$ is an idempotent operator. Then,

$$T^\dagger = T^\dagger(T^\dagger - T^\star)^2T^\star.$$  

**Proof.** Letting $K = I + (T - T^\star)(T - T^\star)^\dagger$ implies that $K$ is positive. Hence, $K = I + (T - T^\star)^\dagger(T - T^\star) = (I - T - T^\star)^2$ is an invertible operator. Since $T$ is idempotent, [13, Corollary 3.3] indicates that $T$ has closed range. Therefore, $T^\dagger(T^\dagger - T^\star) = -T$. Invertibility of $I - T - T^\star$ concludes that

$$TT^\dagger = -(I - T - T^\star)^{-1}.$$  

(3.1)

Now, replacing $T$ with $T^*$ in (3.1), we obtain

$$T^\dagger T = -T^\star(I - T - T^\star)^{-1} = -(I - T - T^\star)^{-1}T.$$  

(3.2)

Equivalently, (1) and (3) in Lemma 3.1 and equalities (3.1) and (3.2) imply that

$$(T^\star)^\dagger = T(T^\dagger)^2T = (-T^\star(I - T - T^\star)^{-1})(-(I - T - T^\star)^{-1}T) = T(I - T - T^\star)^{-2}T.$$  

By taking adjoint, the desired result follows.

**Theorem 3.3.** Suppose that $T \in \mathcal{L}(X)$ is an idempotent operator. Then, $I - T - T^\star$ is invertible, and its inverse is $I - TT^\dagger - T^\dagger T$.

**Proof.** Put $K = I - T - T^\star$ and $H = I - TT^\dagger - T^\dagger T$. Then,

$$KH = (I - T - T^\star)(I - TT^\dagger - T^\dagger T)$$

$$= I - TT^\dagger - T^\dagger T + T^2T^\dagger + TT^\dagger T - T^\star + T^\star TT^\dagger + T^\star T^\dagger T$$

$$= I - TT^\dagger - T^\dagger T + TT^\dagger + T - T^\star + (TT^\dagger T^\dagger + T^3TT^\dagger)^\star$$

$$= I.$$  

Therefore, $KH = I$. Since $K$ and $H$ are self-adjoint, then $(KH)^\star = HK = I$. Hence, $I - T - T^\star$ is invertible, and its inverse is $I - TT^\dagger - T^\dagger T$.

**Corollary 3.4.** Suppose that $T \in \mathcal{L}(X)$ is an idempotent operator. Then,

(1) $T^\dagger = T^\dagger(T^\dagger - P_{\mathcal{R}(T^\dagger) - \mathcal{R}(T^\star)})^2T^\star$;

(2) $T^\star = (I - P_{\mathcal{R}(T^\star) - \mathcal{R}(T^\dagger)})^2T^\dagger$. 


Proof. (1) By Theorem 3.3, equality (1) is obvious.
(2) By applying Lemma 3.1 part (8), we infer that
\[ (I - T - T^*)^2(T^*) = (I - T - T^*) T^* = T^* TT^* + T^* T^* + T^* T^* T^* = T^* T^* T^* = T^*. \]
Using Theorem 3.3, the desired result follows. □

Corollary 3.5. Suppose that \( T \in \mathcal{L}(X) \) is an idempotent operator. Then,
\[ (I - T)^* = I + (T^*)^* - T^*T - TT^*. \]

Proof. Using Theorem 3.2, we conclude that
\[ (I - T)^* = (I - T^*(I - TT^* - T^*T)^2(I - T^*). \]

Applying part (1) of Corollary 3.4, we obtain that
\[ (I - T)^* = (I - T^*(I - TT^* - T^*T)^2(I - T^*). \] (3.3)
Now, since \( T \) is an idempotent operator, part (3) of Lemma 3.1 implies that \((T^*)^* = T(T^*)^2T; \) hence, we from
\[ (I - TT^* - T^*T)^2 = I - TT^* - T^*T + T^* + (T^*)^* - T^*T - TT^* + (T^*)^* - T^*T - TT^* \] (3.4)

From equations (3.3) and (3.4) and items (7) and (8) in Lemma 3.1, we know that \( T^*T^* = T^* \).
Hence, we obtain
\[ (I - T)^* = (I - TT^* - T^*T + T^* + (T^*)^* - T^*T^* + T^*T^* + TT^* - T^*T^* - TT^* + (TT^*)^* - TT^* + (TT^*)^* - TT^* + (TT^*)^*) \]
\[ = I + (T^*)^* - T^*T - TT^*. \] □

In the following theorem, we will characterize idempotent operators through the Moore-Penrose inverse and matrix decomposition in Hilbert C*-modules.

Theorem 3.6. Let \( X \) be a Hilbert \( \mathcal{A} \)-module and let \( T \in \mathcal{L}(X) \). Then, the following assertions are equivalent:
(1) \( T^2 = T; \)
(2) \( (T^*)^* = T(T^*)^2T; \)
(3) \( TT^* = T^*; \)
(4) \( T^*(I - T)^* = 0; \)
(5) \( T = \begin{bmatrix} 1 & T_2 \\ 0 & 0 \end{bmatrix}; \)
(6) \( \mathcal{R}(T) = \mathcal{N}(T^*); \)
(7) \( (TT^*)^* = T(T^*)^2; \)
(8) \( ((I - T)(I - T)^*)^* = (I - T)((I - T)^*)^2; \)
(9) \( P_{\mathcal{R}(I - T)} = I - T^*T; \)
(10) \( \mathcal{R}(I - T) \oplus \mathcal{R}(T^*) = X; \)
(11) \( \mathcal{R}(I - T^*) \oplus \mathcal{R}(T) = X. \)

Proof. (1) \( \iff \) (2)\( \iff \cdots \iff \) (4) They are proved in Lemma 3.1.
By Lemma 2.2 part (ii), we consider 
\[
T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \frac{\mathcal{R}(T)}{\mathcal{N}(T)} \to \frac{\mathcal{R}(T)}{\mathcal{N}(T)}.
\]
Since \(T^*T = T\), then
\[
\begin{bmatrix} T_1D^{-1} & 0 \\ T_2D^{-1} & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1D^{-1} & 0 \\ T_2D^{-1} & 0 \end{bmatrix} = \begin{bmatrix} T_1D^{-1}T_1 & 0 \\ T_2D^{-1}T_1 & 0 \end{bmatrix}.
\]
where \(D = T_1T_1^* + T_2T_2^* \in \mathcal{L}(\mathcal{R}(T))\) is invertible. Thereby,
\[
T_1D^{-1}T_1 = T_1D^{-1}, \tag{3.5}
\]
\[
T_2D^{-1}T_1 = T_2D^{-1}. \tag{3.6}
\]
By multiplying \(T_1\) and \(T_2\) on the left of equations (3.5) and (3.6), respectively, we obtain
\[
T_1T_1D^{-1}T_1^* = T_1T_1D^{-1} \\
T_2T_2D^{-1}T_1^* = T_2T_2D^{-1}.
\]
By summing obtained equations, it follows that \(T_1 = I\).

(5) \(\Rightarrow\) (1) Evident.

(4) \(\Rightarrow\) (6) By Corollary 3.5, we conclude that \((I - T)^* = I + (T^*)^* - T^*T - TT^*\). Then, by multiplying \(T^*\) on the left side and applying part (4), we have
\[
0 = T^*(I - T)T^* \\
= T^* + (T^*)^* - T^*T - TT^* \\
= T^* + (T^*)^* - (T^*)^* - T^*T - TT^* \\
= (T^*)^* - (T^*)^* T^* + TT^* T^* \\
= (T(T^*)^* - (T^*)^* T)^* T^* + TT^* T^*.
\]
which implies that \((T^*)^* = (T^*)^* T^*\).

(6) \(\Rightarrow\) (2) Multiplying \(T\) on the left \((T^*)^* = (T^*)^* T\) yields \(TT^* (T^*)^* = T(T^*)^* T\), i.e., \((T^*)^* = T(T^*)^* T\).

(6) \(\Leftrightarrow\) (7) Replace \(T\) with \(T^*\) and vice versa.

(7) \(\Leftrightarrow\) (8) Replace \(T\) with \(I - T\) and vice versa.

(1) \(\Rightarrow\) (9) By Corollary 3.5 and part (6) of Lemma 3.1, we have
\[
P_{\mathcal{R}(I-T)}(I - T)(I - T)^* = (I - T)(I + (T^*)^* - T^*T - TT^*) \\
= I + (T^*)^* - T^*T - TT^* - T(T^*)^* + TT^* + TTT^* \\
= I - T^* T.
\]

(9) \(\Rightarrow\) (1) Since (9) holds that is \((I - T)(I - T)^* = I - T^* T\), by multiplying \((I - T)\) on the right, we obtain
\[
\]
Then, \(T^* T = T^* T^* T\), again multiplying \(T\), implies \(TT^* T = TT^* T^* T\). Thus, \(T = T^*\) as desired.

(9) \(\Rightarrow\) (10) Since \(P_{\mathcal{R}(I-T)} + P_{\mathcal{R}(I-T)} = I\), so \(\mathcal{R}(I - T) + \mathcal{R}(T) = X\). It is sufficient to prove that \(\mathcal{R}(T^*)^* = \mathcal{R}(I - T)\) and \(\mathcal{R}(T^*)^* \cap \mathcal{R}(I - T) = \{0\}\).

From the equivalence of (1) and (9), we know that \(T\) is an idempotent operator. Hence, \(T^*\) is idempotent. By applying part (1) \(\Rightarrow\) (4), we conclude that \((T^*)^* = (T^*)^* T\). Therefore, \((I - T)^* T^* = 0\). Consequently, for all \(x_1, x_2 \in X\), we have
\[
\langle (I - T)(I - T)^* y, T^*T^* x \rangle = \langle T^*T(I - T)(I - T)^* y, T^*T^* x \rangle = \langle x_1, (I - T)(I - T)^* T^*T^* x_2 \rangle = 0.
\]
Then, \(\mathcal{R}(T^*)^* = \mathcal{R}(I - T)\).

We prove that \(\mathcal{R}(T^*)^* \cap \mathcal{R}(I - T) = \{0\}\). If \(y \in \mathcal{R}(T^*)^* \cap \mathcal{R}(I - T)\), then there are \(x_3, x_4 \in X\) such that \(y = T^* x_3\) and \(y = (I - T)^* x_4\). Then, \(-y = -T^* x_3\) and \(-y = (I - T)^* x_4\), and by summing the last equations, we obtain \(0 = (I - T - T^*)y\). Invertibility of \(I - T - T^*\) with Theorem 3.3 concludes that \(y = 0\).

(10) \(\Rightarrow\) (1) For all \(x \in X\), we have
\begin{align*}
\langle (T - T^2)x, x \rangle = \langle (I - T)x, T^*x \rangle.
\end{align*}

Since \((I - T)x \in \mathcal{R}(I - T)\) and \(T^*x \in \mathcal{R}(T^*)\), then
\begin{align*}
\langle (T - T^2)x, x \rangle = 0.
\end{align*}

We thus obtain \(T - T^2 = 0\).

(10) \iff (11) Evident. \qed

The following theorem states that if \(T\) has inner inverse, then we can obtain the Moore-Penrose inverse of \(T\) based on its inner inverse and orthogonal projections.

**Theorem 3.7.** Let \(T, S \in \mathcal{L}(X)\) such that \(TST = T\). Then,

1. \(T^\dagger = P_{R(\mathcal{ST}/S)}P_{R(TS)}\);
2. \((TS)^\dagger = P_{R(TS)/P_{R(TS)}}\).

**Proof.** (1): Put \(X = P_{R(\mathcal{ST}/S)}P_{R(TS)} = (ST)^\dagger(ST)(ST)^\dagger(TS)^\dagger\). Since \(TST = T\), then \(X = (ST)^\dagger STS(TS)^\dagger\). Hence,
\begin{align*}
XT &= ((ST)^\dagger STS(TS)^\dagger)^T \\
&= (ST)^\dagger (TS)(TS)^\dagger (TS)T \\
&= (ST)^\dagger STS(TS)^\dagger \\
&= (ST)^\dagger (ST) = P_{R(\mathcal{ST})}.
\end{align*}
Similarly, \(TX = (TS)(TS)^\dagger = P_{R(TS)}\). Furthermore,
\begin{align*}
TXT &= T(ST)^\dagger(ST) = T(ST)(ST)^\dagger(ST) = TST = T
\end{align*}
and
\begin{align*}
XTX &= (ST)^\dagger(ST)((ST)^\dagger STS(TS)^\dagger) = (ST)^\dagger STS(TS)^\dagger = X.
\end{align*}
The uniqueness of the Moore-Penrose inverse implies that
\(T^\dagger = (ST)^\dagger STS(TS)^\dagger = P_{R(\mathcal{ST}/S)}P_{R(TS)}\).

(2): Since \(TS\) is an idempotent operator, then Lemma 3.1 implies that \((TS)^\dagger = ((TS)^\dagger TS)(TS)^\dagger = P_{R(TS)/P_{R(TS)}}\). \qed

In the following results, we discover the relationships between two sets of the spectrum of operators that satisfy the special conditions.

**Theorem 3.8.** Let \(S, T \in \mathcal{L}(X)\) with \(ST = TS\) and \(STS = (ST)^2\). Then,
\begin{align*}
\sigma(S) \cup \{0\} = \sigma(ST) \cup \sigma(S - ST).
\end{align*}

**Proof.** For every \(\lambda \in \mathbb{C}\), we have
\begin{align*}
(\lambda - ST)(\lambda - S - ST) &= \lambda^2 - \lambda S + \lambda ST - \lambda S T - STS - (ST)^2 = \lambda(\lambda - S),
\end{align*}
i.e., \(\sigma(S) \cup \{0\} = \sigma(ST) \cup \sigma(S - ST)\). \qed

**Corollary 3.9.** [12, Lemma 4.1.] Suppose that \(S \in \mathcal{L}(X)\) commutes with an idempotent operator \(T \in \mathcal{L}(X)\). Then,
\begin{align*}
\sigma(S) \cup \{0\} = \sigma(ST) \cup \sigma(S(I - T)).
\end{align*}

**Proof.** By Theorem 3.8, we conclude the desired result. \qed
3.1 Application in spectrum of operator

The Moore-Penrose inverse of operators in $\mathcal{L}(X)$ plays an important role in our investigation. The next results link Moore-Penrose properties and spectrum of special operators relative of an idempotent operator. We recall that, if $P$ is an orthogonal projection, then $\sigma(P) \subseteq \{0, 1\}$ and $\sigma(-P) \subseteq \{0, -1\}$.

**Theorem 3.10.** Let $X$ be a Hilbert $\mathcal{A}$-module, and suppose that $T \in \mathcal{L}(X)$ is an idempotent operator. Then, the following statements hold:

1. $\sigma((T^*)' + T - I) \cup \{0\} = \sigma((T^*)') \cup \{0, -1\}$;
2. $\sigma(T^* + T - I) \cup \{0\} = \sigma(T') \cup \{0, -1\}$;
3. If $\lambda \in \mathbb{C} \setminus \{0, 1\}$, then $\lambda \in \sigma(T - (T^*))$ if and only if $\lambda \in \sigma(I - (T^*))$;
4. If $\lambda \in \mathbb{C} \setminus \{0, 1\}$, then $\lambda \in \sigma(T + (T^*))$ if and only if $\lambda \in \sigma(I + (T^*))$;
5. If $\lambda \in \mathbb{C} \setminus \{0, -1\}$, then $\lambda \in \sigma(-I + TT^* + T'T)$ if and only if $\lambda^2 \in \sigma((T^*))$;
6. If $\lambda \in \mathbb{C} \setminus \{0, -1\}$, then $\|(I - T - T^*)^{-1}\| = \sup|\lambda| \in \sigma(-I + TT^* + T'T)$;
7. $\|T + T^* - I\| = \|I - T\| = \|T\|$.

**Proof.** (1): Lemma 3.1 implies that

$$
((T^*)' + T - I)T = T((T^*)' + T - I) = (T^*)'.
$$

In addition,

$$
((T^*)' + T - I)(I - T) = (T - I).
$$

Now, Corollary 3.9 implies that

$$
\sigma((T^*)' + T - I) \cup \{0\} = \sigma((T^*)') \cup \sigma(T - I).
$$

Since $\sigma(T - I) \subseteq \{0, 1\}$, then $\sigma((T^*)' + T - I) \cup \{0\} = \sigma((T^*)') \cup \{0, -1\}$.

(2): Applying part (1) and replacing $T^*$ with $T$ and vice versa, we conclude that (2) is satisfied.

(3): Since $\sigma(I - T) \subseteq \{0, 1\}$, for every $\lambda \in \mathbb{C} \setminus \{0, 1\}$, Lemma 3.1 implies that

$$(\lambda - (I - T))(\lambda - (T - (T^*))' = (\lambda^2 + \lambda(T^*)' - \lambda) = \lambda(\lambda - (I - (T^*)')).$$

Consequently, $\lambda \in \sigma(T - (T^*))$ if and only if $\lambda \in \sigma(I - (T^*))$.

(4): Since $\sigma(I - T) \subseteq \{0, 1\}$, for every $\lambda \in \mathbb{C} \setminus \{0, 1\}$, we have

$$(\lambda - (I - T))(\lambda - (T + (T^*))' = (\lambda^2 - \lambda(T^*)' - \lambda) = \lambda(\lambda - (I + (T^*)')).$$

This means that if $\lambda \in \mathbb{C} \setminus \{0, 1\}$, then $\lambda \in \sigma(T + (T^*))$ if and only if $\lambda \in \sigma(I + (T^*))$.

(5): Since $\sigma(-TT^*) \subseteq \{0, -1\}$ and $\sigma(-T'T) \subseteq \{0, -1\}$, for every $\lambda \in \mathbb{C} \setminus \{0, -1\}$, we conclude that

$$(\lambda - (-TT^*))(\lambda - (-I + TT^* + T'T)(\lambda - (-T'T)) = (\lambda + TT^*)((\lambda + I) - (TT^* + T'T))(\lambda + T'T) = (\lambda^2 + \lambda - TT^* - (T^*)')(\lambda + T'T) = \lambda^2 + \lambda - (T^*)' - (T^*)' = (\lambda + I)(\lambda^2 - (T^*)').$$

That is, $\lambda \in \sigma(-I + TT^* + T'T)$ if and only if $\lambda^2 \in \sigma((T^*))$.

(6): Since $\lambda \in \mathbb{C} \setminus \{0, -1\}$, by applying part (5) and Theorem 3.3, we obtain

$$
\|(I - T - T^*)^{-1}\| = \|I - TT^* - T'T\| = \|(I - TT^* - T'T)\| = \sup|\lambda| : \lambda \in \sigma(-I + TT^* + T'T).
$$

(7): Applying Theorem 3.6 concludes that

$$
T = \begin{bmatrix} I & T \\ 0 & 0 \end{bmatrix} : \mathcal{R}(T) \rightarrow \mathcal{R}(T).
$$
The matrix form of $T$ leads to

$$(T + T^* - I)^2 = \begin{bmatrix} I + T_2T_2^* & 0 \\ 0 & I + T_2T_2^* \end{bmatrix}, \quad TT^* = \begin{bmatrix} I + T_2T_2^* & 0 \\ 0 & 0 \end{bmatrix}. $$

$$(I - T^*)(I - T) = \begin{bmatrix} 0 & 0 \\ 0 & I + T_2T_2^* \end{bmatrix}. $$

Since $\sigma(I + T_2T_2^*) = \sigma(I + T_2T_2^*)$, we have

$$\sigma((T + T^* - I)^2) \cup \{0\} = \sigma((I - T^*)(I - T)). $$

From $r(xx^*)^\frac{1}{2} = \|x^*\|^2 = \|x\|$, we conclude that

$$\|T + T^* - I\| = \|I - T\| = \|T\|. $$

## 4 Schur complements

In this section, we compute the Moore-Penrose inverse of the 2-by-2 block operator matrix when it is idempotent. In this case, we find the Moore-Penrose inverse of its Schur complements. We use the range inclusions of operators to obtain new conditions under which we can compute the Moore-Penrose inverse of the Schur complement and the generalized Schur complement of $D$ in $T$. We also obtain some conditions under which the Moore-Penrose inverse of Schur complement of $D$ in $T$ is obtained if and only if $A^\dagger$ is inner inverse of $B^\dagger C$.

In the following theorem, we give an explicit formula for the Moore-Penrose inverse of the 2-by-2 block operator matrix.

**Theorem 4.1.** Suppose that $A, B, C, D \in \mathcal{L}(X)$ such that $A = A^2 + BC$, $B = AB + BD$, $C = CA + DC$, and $D = D^2 + CB$. Then,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} (\tilde{K})^{-1} & - (\tilde{K})^{-1}K_5K_4^{-1} \\ -K_4^{-1}K_5(\tilde{K})^{-1} & K_4^{-1} + K_4^{-1}K_5(\tilde{K})^{-1}K_4^{-1} \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix},$$

where

- $K_1 = I - A - A^* + AA^* + A^*A + BB^* + C^*C$,
- $K_2 = AC^* + A^*B - B + BD^* - C^* + C^*D$,
- $K_3 = K_3$,
- $K_4 = I - D - D^* + DD^* + D^*D + B^*B + CC^*$,
- $\tilde{K} = K_1 - K_2K_5^{-1}K_3$.

**Proof.** Letting $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and using the hypotheses conclude that $T$ is an idempotent operator. Then, Theorem 3.2 leads to $T^\dagger$, which has the following form:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} I - A - A^* & -B - C^* \\ -C - B^* & I - D - D^* \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix},$$

where

- $K_1 = I - A - A^* + AA^* + A^*A + BB^* + C^*C$,
- $K_2 = AC^* + A^*B - B + BD^* - C^* + C^*D$,
- $K_3 = (AC^* + A^*B - B + BD^* - C^* + C^*D)^*$,
- $K_4 = I - D - D^* + DD^* + D^*D + B^*B + CC^*$. 
Since \((I - T - T^*)^2 = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}\) is positive and invertible, then [14, Corollary 4.1] implies
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A' & C' \\ B' & D' \end{bmatrix} \begin{pmatrix} (\tilde{K})^{-1} & - (\tilde{K})^{-1}K_4^{-1} \\ -K_4^{-1}K_3(\tilde{K})^{-1} & K_3^{-1} + K_4^{-1}K_3(\tilde{K})^{-1}K_4^{-1} \end{pmatrix} \begin{bmatrix} A & C' \\ B' & D' \end{bmatrix}
\]
such that \(\tilde{K} = K_1 - K_2K_4^{-1}K_3\).

Following theorem calculates the Moore-Penrose inverse of Schur complements of the 2-by-2 block idempotent operator matrix.

**Theorem 4.2.** Suppose that \(A, B, C, D \in \mathcal{L}(\mathcal{X})\) and that the hypotheses of Theorem 4.1 are satisfied.

1. If \(A\) is an invertible operator, then
   \[
   (D - CA^{-1}B)^\dagger = (D - CA^{-1}B)^\dagger (I - D - D^* + CA^{-1}B + (CA^{-1}B)^\dagger)^{-1} (D - CA^{-1}B)^\dagger.
   \]

2. If \(D\) is an invertible operator, then
   \[
   (A - BD^{-1}C)^\dagger = (A - BD^{-1}C)^\dagger (I - A - A^* + BD^{-1}C + (BD^{-1}C)^\dagger)^{-1} (A - BD^{-1}C)^\dagger.
   \]

**Proof.** (1): By the hypotheses, we conclude that
\[
(D - CA^{-1}B)(D - CA^{-1}B) = D^2 - DCA^{-1}B - CA^{-1}BD + CA^{-1}BCA^{-1}B \\
= D - CB - (C - CA)A^{-1}B - CA^{-1}(B - AB) + CA^{-1}(A - A^2)A^{-1}B \\
= D - CA^{-1}B.
\]
Hence, \(D - CA^{-1}B\) is an idempotent operator. Then, Theorem 3.2 implies that
\[
(D - CA^{-1}B)^\dagger = (D - CA^{-1}B)^\dagger (I - D - D^* + CA^{-1}B + (CA^{-1}B)^\dagger)^{-1} (D - CA^{-1}B)^\dagger.
\]

(2): By the hypotheses, we have
\[
(A - BD^{-1}C)(A - BD^{-1}C) = A^2 - ADB^{-1}C - BD^{-1}CA + BD^{-1}CBD^{-1}C \\
= A - BC - (B - BD)D^{-1}C - BD^{-1}(C - DC) + BD^{-1}(D - D^2)D^{-1}C \\
= A - BD^{-1}C.
\]
Hence, \(A - BD^{-1}C\) is an idempotent operator, and again by applying Theorem 3.2, we have
\[
(A - BD^{-1}C)^\dagger = (A - BD^{-1}C)^\dagger (I - A - A^* + BD^{-1}C + (BD^{-1}C)^\dagger)^{-1} (A - BD^{-1}C)^\dagger.\]

In the following theorem, we compute Moore-Penrose inverse of \(A - X\) in special conditions by using the matrix block technique.

**Theorem 4.3.** Let \(A, X \in \mathcal{L}(\mathcal{X})\), and let \(A\) and \(A - X\) have closed ranges such that \((I - AA^*)X = 0\). Then,
\[
(A - X)^\dagger = (A - X)^\dagger ((I - XX^*)AA^*(I - XX^*)^* + X(I - AA^*)X^*).
\]

**Proof.** Since \(A\) has closed range, part (i) of Lemma 2.2 implies that the operator \(A\) has the following matrix form:
\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.
\]
In addition, \(X\) and \(A - X\) have the following forms:
\[
X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},
\]
\[
A - X = \begin{bmatrix} A_1 - X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.
\]
Since \((I - AA^\dagger)X = \begin{bmatrix} 0 & 0 \\ X_3 & X_4 \end{bmatrix} = 0\), then \(X_3 = 0\) and \(X_4 = 0\). The matrix forms imply that
\[
(I - XA^\dagger)AA^\dagger(I - XA^\dagger)^\dagger = \begin{bmatrix} A_1A_1^\dagger - A_1X_1^\dagger - X_1A_1^\dagger + X_1X_1^\dagger & 0 \\ 0 & 0 \end{bmatrix},
\]
and
\[
X(I - AA^\dagger)X^\dagger = \begin{bmatrix} X_2X_2^\dagger & 0 \\ 0 & 0 \end{bmatrix}.
\]
On the other hand, we have
\[
(A - X)^\dagger = (A - X)^\dagger((A - X)(A - X)^\dagger)^\dagger
= (A - X)^\dagger\begin{bmatrix} A_1 - X_1 & -X_2 \\ 0 & 0 \end{bmatrix}^\dagger
= (A - X)^\dagger\begin{bmatrix} (A_1 - X_1)^\dagger & 0 \\ -X_2^\dagger & 0 \end{bmatrix}
= (A - X)^\dagger((I - XA^\dagger)AA^\dagger(I - XA^\dagger)^\dagger + X(I - AA^\dagger)(X)^\dagger).
\]
Then, the desired result follows.

**Corollary 4.4.** Let \(A, B, C, D \in \mathcal{L}(X)\) and let \(A\) and \(D\) have closed ranges such that \((I - AA^\dagger)BD^\dagger C = 0\). Then,
\[
(A - BD^\dagger C)^\dagger = (A - BD^\dagger C)^\dagger((I - BD^\dagger CA^\dagger)AA^\dagger(I - BD^\dagger CA^\dagger)^\dagger + BD^\dagger C(I - AA^\dagger)(BD^\dagger C)^\dagger)^\dagger.
\]
Using the range inclusions of operators, we obtain Moore-Penrose inverse of the Schur complement and the generalized Schur complement of \(D\) in \(T\) under some new conditions.

**Theorem 4.5.** Suppose that \(A, B, C, D \in \mathcal{L}(X)\) and that \(\mathcal{R}(A)\) is closed such that \(\mathcal{R}(B) \subset \mathcal{R}(A)\) and \(\mathcal{R}(C^\dagger) \subset \mathcal{R}(A^\dagger)\). Assume that \(D\) and \(D - CA^\dagger B\) are invertible operators. Then, \((A - BD^\dagger C)^\dagger\) exists and
\[
(A - BD^\dagger C)^\dagger = A^\dagger + A^\dagger B(D - CA^\dagger B)^{-1}CA^\dagger.
\]  

**Proof.** Since \(\mathcal{R}(B) \subset \mathcal{R}(A)\) and \(\mathcal{R}(C^\dagger) \subset \mathcal{R}(A^\dagger)\), then \(AA^\dagger B = B\) and \(CA^\dagger A = C\). We note that the existence of the Moore-Penrose inverse of the operator ensures closedness range of operator. Let \(E = A^\dagger + A^\dagger B(D - CA^\dagger B)^{-1}CA^\dagger\). Then,
\[
(A - BD^\dagger C)E = (A - BD^\dagger C)(A^\dagger + A^\dagger B(D - CA^\dagger B)^{-1}CA^\dagger)
= AA^\dagger + AA^\dagger B(D - CA^\dagger B)^{-1}CA^\dagger - BD^\dagger CA^\dagger - BD^\dagger CA^\dagger B(D - CA^\dagger B)^{-1}CA^\dagger
= AA^\dagger + AA^\dagger B(D - CA^\dagger B)^{-1}CA^\dagger - BD^\dagger CA^\dagger B(D - CA^\dagger B)^{-1}CA^\dagger
= AA^\dagger.
\]
Similarly, \(E(A - BD^\dagger C) = A^\dagger A\). In addition, by a straightforward computation, we obtain that
\[
((A - BD^\dagger C)E(A - BD^\dagger C) = (A - BD^\dagger C), \quad E(A - BD^\dagger C)E = E,
\]
i.e., \((A - BD^\dagger C)^\dagger E = E\).

**Theorem 4.6.** Suppose that \(A, B, C, D \in \mathcal{L}(X)\) and that \(\mathcal{R}(A), \mathcal{R}(D), \mathcal{R}(D - CA^\dagger B)\) are closed such that \(\mathcal{R}(B) \subset \mathcal{R}(A), \mathcal{R}(C^\dagger) \subset \mathcal{R}(A^\dagger), \mathcal{R}(C) \subset \mathcal{R}(D - CA^\dagger B), \mathcal{R}((D - CA^\dagger B)^\dagger) \subset \mathcal{R}(D^\dagger),\) and \(\mathcal{R}(D - CA^\dagger B) \subset \mathcal{R}(D)\). Then, \((A - BD^\dagger C)^\dagger\) exists and
\[
(A - BD^\dagger C)^\dagger = A^\dagger + A^\dagger B(D - CA^\dagger B)^{-1}CA^\dagger.
\]  

**Proof.** Define \(X = A^\dagger + A^\dagger B(D - CA^\dagger B)^{-1}CA^\dagger\). We prove that \(X\) is Moore-Penrose inverse of \(A - BD^\dagger C\). Since \(\mathcal{R}(B) \subset \mathcal{R}(A), \mathcal{R}(C^\dagger) \subset \mathcal{R}(A^\dagger), \mathcal{R}(C) \subset \mathcal{R}(D - CA^\dagger B), \mathcal{R}((D - CA^\dagger B)^\dagger) \subset \mathcal{R}(D^\dagger),\) we obtain \(AA^\dagger B = B, CA^\dagger A = C, (D - CA^\dagger B)(D - CA^\dagger B)^{\dagger}C = C,\) and \(D^\dagger (D - CA^\dagger B)^\dagger = (D - CA^\dagger B)^\dagger\). Hence,
\[(A - BD' C)X = (A - BD' C)(A' + A'B(D - CA'B)'CA') \]
\[= AA' - AA'B(D - CA'B)'CA' - BD'CA' - BD'CA'B(D - CA'B)'CA' \]
\[= AA' - AA'B(D - CA'B)'CA' - BD'(D - CA'B)(D - CA'B)'CA' - BD'CA'B(D - CA'B)'CA' \]
\[= AA' - AA'B(D - CA'B)'CA' - BD'(D - CA'B)(D - CA'B)'CA' + BD'CA'B(D - CA'B)'CA' \]
\[= AA'. \]

Since \( \mathcal{R}(D - CA'B) \subset \mathcal{R}(D) \), we obtain \( DD'((D - CA'B)') = ((D - CA'B)') \). Taking adjoint implies \( (D - CA'B)'DD' = (D - CA'B)' \). Hence, we have
\[(A - BD' C)X = AA'(A' + A'B(D - CA'B)'CA') = X. \]

Then, the desired result follows. \( \square \)

The following theorem states conditions under which the Moore-Penrose inverse of Schur complement \( D \) in \( T \) is computed if and only if \( A' \) is inner inverse of \( B^{-1}C \).

**Theorem 4.7.** Suppose that \( A, B, C \in \mathcal{L}(X), D \in \mathcal{L}(X) \) is an invertible operator and \( \mathcal{R}(A) \) is closed such that \( \mathcal{R}(C^*) \subset \mathcal{R}(A^*) \) and \( \mathcal{R}(B) \subset \mathcal{R}(A) \). Then, the following statements are equivalent:
1. \( BD^{-1}CA'BD^{-1}C = BD^{-1}C \),
2. \( (A - BD^{-1}C)' = (I - ((A'B^{-1}C')*)(A'B^{-1}C'))(A' - (BD^{-1}CA')^*)(BD^{-1}CA') \).

**Proof.** (1) \( \Rightarrow \) (2) Since \( \mathcal{R}(C^*) \subset \mathcal{R}(A^*) \) and \( \mathcal{R}(B) \subset \mathcal{R}(A) \), we obtain \( CA'A = C \) and \( AA'B = B \). Since (1) holds, then \( (A - BD^{-1}C)A'(A - BD^{-1}C) = A - BD^{-1}C \). Therefore, part (1) of Theorem 3.7 implies that \( (A - BD^{-1}C)' \) exists and
\[(A - BD^{-1}C)' = P_{\mathcal{R}(A'(A - BD^{-1}C))}A'P_{\mathcal{R}(A - BD^{-1}C)} \].

Now, we calculate some quantities involved in \( P_{\mathcal{R}(A'(A - BD^{-1}C))} \). Straightforward computations yield
(i) \( (A'A - A'B^{-1}C)(A'A - A'B^{-1}C) = A'A - A'B^{-1}C \), i.e., \( (A'A - A'B^{-1}C) \) is an idempotent operator,
(ii) \( (I - 2A'A)^2 = I \),
(iii) \( (A'A - I)(I - A'B^{-1}C - (A'B^{-1}C)^*) = A'A - I \),
(iv) \( (A'B^{-1}C) + (A'B^{-1}C)^* = ((A'B^{-1}C) + (A'B^{-1}C)^*)(2A'A - I) \).

Since \( A'B^{-1}C \) is an idempotent operator, Theorem 3.3 implies that \( (I - A'B^{-1}C - (A'B^{-1}C)^*) \) is an invertible operator.

We recall that
\[P_{\mathcal{R}(A'(A - BD^{-1}C))} = (A'A - A'B^{-1}C)'(A'A - A'B^{-1}C). \] (4.3)

By virtue of Theorem 3.2 and results (i)-(iv), we compute the first term on the right of equation (4.3) with the following form:
\[(A' A - A' BD^{-1} C)' = (A' A - A' BD^{-1} C)(I - 2A'A + A' BD^{-1} C + (A' BD^{-1} C)^{-2}\]
\[(A' A - A' BD^{-1} C)^* = (A' A - A' BD^{-1} C)^*((I - 2A'A) + (A' BD^{-1} C + (A' BD^{-1} C)^2)(2A' A - I)^{-2}\]
\[(A' A - A' BD^{-1} C)^* = (A' A - A' BD^{-1} C)(I - 2A'A - (A' BD^{-1} C)^{-2}(A' A - I)\]
\[(A' A - A' BD^{-1} C) = (A' A - I + I - A' BD^{-1} C)^*(I - A' BD^{-1} C - (A' BD^{-1} C)^{-2}\]
\[(A' A - I - A' BD^{-1} C)^* = (A' A - I)(I - A' BD^{-1} C - (A' BD^{-1} C)^{-2}(A' A - I)\]
\[+ (I - A' BD^{-1} C)^*(I - A' BD^{-1} C - (A' BD^{-1} C)^{-2}(A' A - I)\]
\[+ (I - A' BD^{-1} C)^*(I - A' BD^{-1} C - (A' BD^{-1} C)^{-2}(I - A' BD^{-1} C)^-\]
\[= (A' A - I) + (I - A' BD^{-1} C)^*(I - A' BD^{-1} C - (A' BD^{-1} C)^{-2}(I - A' BD^{-1} C)^*\]
\[= (A' A - I) + I - A' BD^{-1} C)^.\]

Therefore,
\[P_{R(A - BD^{-1} C)} = (I - A' BD^{-1} C)(A' A - A' BD^{-1} C).\] (4.4)

Now, we calculate some quantities involved in \(P_{R(A - BD^{-1} C)}\). Straightforward computations yield

(i) \((A A' - BD^{-1} CA')\), \((A A' - BD^{-1} CA')\) is an idempotent operator,

(ii) \((I - 2A'A)^2 = I^2\),

(iii) \((A A' - I)\), \((I - BD^{-1} CA' - (BD^{-1} CA')^2) = A A' - I\),

(iv) \((BD^{-1} CA') + (BD^{-1} CA')^2 = ((BD^{-1} CA') + (BD^{-1} CA')^2)(2A'A - I)\).

Since \(BD^{-1} CA'\) is an idempotent operator, again applying Theorem 3.3 implies that \((I - BD^{-1} CA' - (BD^{-1} CA')^2)\) is an invertible operator.

Then, by using Theorem 3.2 and items (i)–(iv), we compute \((A A' - BD^{-1} CA')^\dagger\). Hence,
\[\begin{align*}
(A A' - BD^{-1} CA')' &= (A A' - BD^{-1} CA')^2(I - 2A'A + BD^{-1} CA' + (BD^{-1} CA')^2)
\[\quad \times (BD^{-1} CA')^2(I - 2A'A + BD^{-1} CA' + (BD^{-1} CA')^2)(2A'A - I)^{-2}\]
(A A' - BD^{-1} CA')^* &= (A A' - BD^{-1} CA')^*(I - 2A'A + BD^{-1} CA' + (BD^{-1} CA')^2)(2A'A - I)^{-2}\]
(A A' - BD^{-1} CA')^* &= (A A' - BD^{-1} CA')^*(I - 2A'A + BD^{-1} CA' + (BD^{-1} CA')^2)(2A'A - I)^{-2}\]
(A A' - I + I - BD^{-1} CA')^* &= (A A' - I)(I - BD^{-1} CA' - (BD^{-1} CA')^2(2A'A - I)^{-2}\]
(A A' - I + I - BD^{-1} CA')^* &= (A A' - I)(I - BD^{-1} CA' - (BD^{-1} CA')^2(2A'A - I)^{-2}\]
(A A' - I + I - BD^{-1} CA')^* &= (A A' - I)(I - BD^{-1} CA' - (BD^{-1} CA')^2(2A'A - I)^{-2}\]
(A A' - I + I - BD^{-1} CA')^* &= (A A' - I)(I - BD^{-1} CA' - (BD^{-1} CA')^2(2A'A - I)^{-2}\]

Therefore,
\[P_{R(A - BD^{-1} C)} = (A A' - BD^{-1} CA')(I - BD^{-1} CA').\] (4.5)

Equations (4.4) and (4.5) and part (9) of Theorem 3.6 imply that
\[(A - BD^{-1} C)' = P_{R(A - BD^{-1} C)} A P_{R(A - BD^{-1} C)}\]
\[= [(I - A' BD^{-1} C)(A A' - A' BD^{-1} C)A']\]
\[\times [(A A' - BD^{-1} CA')(I - BD^{-1} CA')A']\]
\[= (I - A' BD^{-1} C)(I - A' BD^{-1} C)A'\]
\[\times (I - BD^{-1} CA')(I - BD^{-1} CA')\]
\[= P_{R(I - BD^{-1} C)} A P_{R(I - BD^{-1} C)}\]
\[= (I - (A' BD^{-1} C)^2(A' BD^{-1} C)A'\]
\[\times (I - (BD^{-1} CA')^2(BD^{-1} CA'))\].
(2) ⇒ (1) Since equation (2) holds, we have
\[ \mathcal{R}(BD^{-1}CA)^+ = \mathcal{R}(BD^{-1}CA)^* \subseteq N(A - BD^{-1}C)^t = N(A - BD^{-1}C)^t. \]
Therefore, \[ N(BD^{-1}CA)] \subseteq \mathcal{R}(A - BD^{-1}C)^t. \] Theorem 2.1 implies that \( \mathcal{R}(A - BD^{-1}C) \subseteq N(BD^{-1}CA). \) Then, \( BD^{-1}CA(A - BD^{-1}C) = 0. \) It concludes that (1) holds. \( \square \)

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