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Quotients of Strongly Realcompact Groups

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Abstract: A topological group is *strongly realcompact* if it is topologically isomorphic to a closed subgroup of a product of separable metrizable groups. We show that if H is an invariant Čech-complete subgroup of an ω-narrow topological group G, then G is strongly realcompact if and only if G/H is strongly realcompact. Our proof of this result is based on a thorough study of the interaction between the P-modification of topological groups and the operation of taking quotient groups.

Keywords: Extension of groups; Strongly realcompact; Strongly Dieudonné complete; P-modification; P-group; Quotient group

1 Introduction

The concepts of strong realcompactness and strong Dieudonné completeness arise as an adaptation of the well-known topological notions of realcompactness and Dieudonné completeness. We say that a topological group G is *strongly realcompact* if it is topologically isomorphic to a closed subgroup of a product of second countable topological groups. Similarly, G is *strongly Dieudonné complete* if it is topologically isomorphic to a closed subgroup of a product of metrizable topological groups (see [5, 8]).

It is clear that strongly Dieudonné complete groups are ω-balanced, while strongly realcompact groups are ω-narrow. Also, strong realcompactness implies strong Dieudonné completeness. Evidently, any discrete group is metrizable and hence strongly Dieudonné complete, but it is strongly realcompact if and only if it is countable. Therefore uncountable discrete groups are examples of strongly Dieudonné complete groups that fail to be strongly realcompact. Let us note that both classes of topological groups are productive and stable with respect to taking closed subgroups.

In this paper we analyze whether the classes of strongly realcompact or strongly Dieudonné complete groups are closed with respect to taking quotients and P-modifications. Our arguments are based on the study of the question of when the P-modification of a quotient group G/H is naturally equivalent to the quotient group G_ω/H_ω corresponding to the P-modifications of the groups G and H. It turns out that the groups $(G/H)_\omega$ and G_ω/H_ω are topologically isomorphic provided that H is completely metrizable (Lemma 4.3) or if G is ω-balanced and H is Čech-complete (Theorem 5.2).

It is shown in Theorem 4.5 that both classes of topological groups contain quotients with respect to Čech-complete subgroups. In fact, we prove a more symmetric result: If G is an ω-narrow (ω-balanced) topological group and H is an invariant Čech-complete subgroup of G, then G is strongly realcompact (strongly Dieudonné-complete) if and only if the quotient group G/H is strongly realcompact (strongly Dieudonné-complete). We also show in Example 4.6 that an extension of a compact group by a separable metrizable group can fail to be strongly realcompact or even Dieudonné complete. It turns out that there exists a pseudocompact non-compact Abelian topological group G containing a closed separable metrizable subgroup H such that the quotient group G/H is compact.

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In Section 5 we present conditions on a topological group G implying that every closed metrizable invariant subgroup H of G such that $(G/H)_\omega \cong G_\omega/H_\omega$ is completely metrizable, i.e. the completeness of H in Lemma 4.3 is necessary provided H is metrizable. In particular we show in Corollary 5.5 that if H is a closed, invariant, metrizable subgroup of a pseudocompact group G, then $(G/H)_\omega \cong G_\omega/H_\omega$ if and only if H is compact. We also prove that the topological isomorphism $(G/H)_\omega \cong G_\omega/H_\omega$ holds true provided a closed invariant subgroup H of G is either Čech-complete or pseudocompact (see Theorem 5.2 and Corollary 5.6).

2 Notation and terminology

We use $\mathbb{I}$ for the unit interval $[0, 1]$, $\mathbb{T}$ for the unit circle, $\mathbb{N}$ for the set of positive integers, $\mathbb{Z}$ for the integers, $\mathbb{Q}$ for the rational numbers, and $\mathbb{R}$ for the set of real numbers.

Let X be a space. As usual, we denote by $w(X)$, $nw(X)$, $\chi(X)$, $\psi(X)$, $d(X)$ the weight, network weight, character, pseudocharacter, and density of X, respectively.

The boolean two-element group is denoted by $\mathbb{Z}_2$, while ω and $\mathfrak{c}$ stand for the cardinality of $\mathbb{N}$ and $\mathbb{R}$, respectively.

Let G be a topological group with identity element e. We denote by $\mathcal{N}_G(e)$ the family of open sets U in G with $e \in U$. We use ϱG to denote the Raĭkov completion of G. The union of a family of G_δ-sets in G is said to be *G_δ-open*. A *G_δ-closed* set is the complement to a G_δ-open set. We denote by $cl_\delta(A)$ the *G_δ-closure* of a subset A of X, i.e. the set of all $x \in X$ such that every G_δ-open set in X containing x intersects A. We say that a subset A of a space X is *G_δ-dense* in X if $cl_\delta(A) = X$. The G_δ-closure of a topological group G in ϱG is denoted by $\varrho_\omega G$. Clearly $\varrho_\omega G$ is a subgroup of ϱG.

Let (X, τ) be a topological space. The topology on X generated by G_δ-subset of (X, τ) is denoted by τ_ω. The space $X_\omega = (X, \tau_\omega)$ is known as the *P-modification* of X.

We say that a property $\mathcal{P}$ is a *three space property* if for every topological group G and every closed invariant subgroup H of G such that H and G/H have $\mathcal{P}$, the group G also has $\mathcal{P}$.

3 *P*-modifications of strongly realcompact and strongly Dieudonné complete groups

Many topological properties are not preserved under taking the P-modification. As examples we can mention density, cellularity, compactness, connectedness, and so on. However it is proved in [4, Theorem 8] that the P-modification of a Raĭkov complete topological group is again Raĭkov complete. In this section we characterize strong realcompactness and strong Dieudonné completeness of topological groups in terms of their P-modifications.

Let us start with a lemma.

Lemma 3.1. *Let G be an ω-balanced topological group. Then the group G_ω, the P-modification of G, is balanced.*

Proof. Let U be an open neighborhood of the identity element e in G_ω. We can assume that $U = \bigcap_{n\in\omega} U_n$, where U_n is an open subset of G, for each $n \in \omega$. As G is ω-balanced, for each $n \in \omega$ there is a countable family γ_n of open neighborhoods of e in G such that for each $x \in G$ one can find $O \in \gamma_n$ satisfying $xOx^{-1} \subset U_n$.

Let $V_n = \bigcap \gamma_n$. For each $n \in \omega$, V_n is a G_δ-set in G with the property that for every $x \in G$, $xV_nx^{-1} \subset U_n$. Let $V = \bigcap_{n\in\omega} V_n$. It is clear that V is a G_δ-set in G containing e. Hence V is an open neighborhood of e in G_ω satisfying $xVx^{-1} \subset U$ for each $x \in G$. Thus the group G_ω is balanced. □

Let us note that there exists a topological group G such that G_ω is balanced, but G fails to be ω-balanced. Indeed, Pestov in [6] presents an example of a group G of countable pseudocharacter which is not ω-balanced. Clearly the P-modification of G is discrete and hence balanced.

It was proved in [4, Theorem 8] that if a topological group X is Raĭkov complete, then X_ω is also Raĭkov complete. In the lemma below we describe the Raĭkov completion of the P-modification of a given topological group.

Lemma 3.2. *Let X be a topological group. Then $\varrho(X_\omega)$ is topologically isomorphic to the P-modification of the G_δ-closure of X in ϱX, i.e. $\varrho(X_\omega) \cong (\varrho_\omega X)_\omega$.*

Proof. It is clear that X is G_δ-dense in $\varrho_\omega X$ and $\varrho_\omega X$ is G_δ-closed in ϱX. Hence X_ω is dense in $(\varrho_\omega X)_\omega$ while the latter group is closed in $(\varrho X)_\omega$.

According to [4, Theorem 8] the group $(\varrho X)_\omega$ is Raĭkov complete, and so is $(\varrho_\omega X)_\omega$. As X_ω is dense in $(\varrho_\omega X)_\omega$, we conclude that $\varrho(X_\omega) = (\varrho_\omega X)_\omega$. □

The next two results show that the Raĭkov completeness of G_ω is an essential component in our characterizations of strongly Dieudonné complete and strongly realcompact groups G.

Theorem 3.3. *An ω-balanced group X is strongly Dieudonné complete if and only if X_ω is Raĭkov complete.*

Proof. Suppose that X is strongly Dieudonné complete. By [8, Theorem 3.3], the equality $X = \varrho_\omega X$ is valid. It follows from Lemma 3.2 that $\varrho(X_\omega) \cong (\varrho_\omega X)_\omega = X_\omega$. Hence the group X_ω is Raĭkov complete.

Conversely, if X_ω is Raĭkov complete, then $X_\omega = \varrho(X_\omega) = (\varrho_\omega X)_\omega$, whence it follows that $X = \varrho_\omega X$. Finally, as X is ω-balanced, [8, Theorem 3.3] implies that X is strongly Dieudonné complete. □

Unlike the case of ω-balancedness, the P-modification of an ω-narrow group can fail to be ω-narrow. The topological group $\mathbb{R}$ with the usual topology is ω-narrow, but $\mathbb{R}_\omega$ is an uncountable discrete group, so it is not ω-narrow.

The proof of the following fact is very similar to the proof of Theorem 3.3 and hence is omitted.

Theorem 3.4. *An ω-narrow topological group X is strongly realcompact if and only if X_ω is Raĭkov complete.*

It was proved in [8, Theorem 3.16] that an $\mathbb{R}$-factorizable topological group G is strongly realcompact if and only if G is a realcompact space. Since $\mathbb{R}$-factorizable groups are ω-narrow and precompact groups are $\mathbb{R}$-factorizable [1, Corollary 8.1.17], Theorem 3.4 implies the following corollaries.

Corollary 3.5. *An $\mathbb{R}$-factorizable group G is realcompact if and only if the P-modification of G is Raĭkov complete.*

Corollary 3.6. *For every precompact topological group G, the following conditions are equivalent:*
(i) *G is realcompact.*
(ii) *The P-modification of G is Raĭkov complete.*

4 Quotients of strongly realcompact and strongly Dieudonné complete groups

It is shown in [1, Corollary 7.6.19] that any topological group is a quotient of a zero-dimensional topological group with countable pseudocharacter. Since every ω-balanced topological group of countable pseudocharacter is strongly Dieudonné complete [8, Theorem 3.4], we see that the properties of being strongly Dieudonné complete or strongly realcompact are not invariant under taking quotient groups.

It turns out that quotients with respect to compact subgroups inherit both properties. According to [8, Theorem 3.17], if H is a compact invariant subgroup of a topological group G, then G is strongly realcompact (strongly Dieudonné complete) if and only if G/H is strongly realcompact (strongly Dieudonné complete).

Our aim in this section is to extend this result to Čech-complete subgroups H. This requires several auxiliary facts.

Our first lemma is evident and its proof is omitted.

Lemma 4.1. *If H is an arbitrary subgroup of a topological group G, then the identity mapping of H_ω onto the corresponding subgroup of G_ω is a topological isomorphism.*

The next result is a key fact for the proof of Lemma 4.3.

Lemma 4.2. *Let H be an invariant completely metrizable subgroup of a topological group G with identity e. Let also $\{V_n : n \in \omega\}$ be a local base for H at e and $\{U_n : n \in \omega\}$ a sequence of open symmetric neighborhoods of e in G such that $U_{n+1}^2 \subseteq U_n$ and $U_n \cap H \subseteq V_n$ for each $n \in \omega$. Then the subgroup $P = \bigcap_{n\in\omega} U_n$ of G satisfies the equality $PH = \bigcap_{n\in\omega} U_n H$.*

Proof. It follows from the properties of the sequence $\{U_n : n \in \omega\}$ that $P = \bigcap_{n\in\omega} U_n$ is a subgroup of G. It is also clear that $PH \subset \bigcap_{n\in\omega} U_n H$. So it suffices to verify the inverse inclusion.

Take an arbitrary element $x \in \bigcap_{n\in\omega} U_n H$. For every $n \in \omega$, choose elements $y_n \in U_n$ and $h_n \in H$ such that $x = y_n h_n$. We claim that $\{h_n : n \in \omega\}$ is a Cauchy sequence in H. Indeed, let V be an open neighborhood of e in H. There exists $N \in \omega$ such that $V_N \subseteq V$. Take $m, n \in \omega$ with $m > N$ and $n > N$. We can assume without loss of generality that $n \le m$. It follows from $y_m h_m = x = y_n h_n$ that

$$y_n^{-1} y_m = h_n h_m^{-1} \in H \cap U_n^{-1} U_m = H \cap U_n U_m \subseteq H \cap U_n^2 \subseteq H \cap U_{n-1} \subseteq V_N.$$

Since H is invariant in G, a similar argument shows that $y_m y_n^{-1} \in V_N$ provided that $m, n > N$. This proves our claim.

As the group H is completely metrizable, it follows from [1, Theorem 4.3.7] that H is Raĭkov complete. Hence the sequence $\{h_n : n \in \omega\}$ converges in H, i.e. there exists $h_\star \in H$ such that $h_n \to h_\star$ for $n \to \infty$. Therefore, the elements $y_n = x h_n^{-1}$ converge to $x h_\star^{-1}$. Since $y_k \in U_n$ whenever $k \ge n$, we conclude that $x h_\star^{-1} \in \overline{U_n}$ for each $n \in \omega$, whence it follows that $x h_\star^{-1} \in \bigcap_{n\in\omega} \overline{U_n} = \bigcap_{n\in\omega} U_n = P$. We have thus proved that $x \in P h_\star \subset PH$, which implies the required equality. □

The following lemma prepares ground for the proof of Theorem 4.5.

Lemma 4.3. *Let H be an invariant completely metrizable subgroup of a topological group G. Then the identity mapping of G_ω/H_ω onto $(G/H)_\omega$ is a topological isomorphism.*

Proof. Denote by φ the identity mapping of G_ω/H_ω onto $(G/H)_\omega$. It is clear that φ is continuous, so it suffices to verify that φ is open. Let $\pi\colon G \to G/H$ be the quotient homomorphism. The same homomorphism considered as a mapping of G_ω onto G_ω/H_ω will be denoted by π_ω. Clearly π and π_ω are continuous open homomorphisms. Then $p = \varphi \circ \pi_\omega$ is a continuous homomorphism of G_ω onto $(G/H)_\omega$. The conclusion of the lemma will follow if we prove that p is open since the latter means that φ is open as well. To this end it suffices to present a local base $\mathcal{B}$ at the identity e of the group G_ω such that $p(W)$ is open in $(G/H)_\omega$ for each $W \in \mathcal{B}$.

Let $\{V_n : n \in \omega\}$ be a local base at the identity e in H. Take an arbitrary open set Q in G_ω containing e. Then there exists a sequence $\{O_n : n \in \omega\}$ of open neighborhoods of e in G such that $\bigcap_{n\in\omega} O_n \subset Q$. It is easy to define by induction a sequence $\{U_n : n \in \omega\}$ of symmetric open neighborhoods of e in G such that $U_{n+1}^2 \subset U_n \subset O_n$ and $U_n \cap H \subset V_n$ for each $n \in \omega$. Then $P = \bigcap_{n\in\omega} U_n \subset Q$ and Lemma 4.2 implies that $PH = \bigcap_{n\in\omega} U_n H$. The set $R = \bigcap_{n\in\omega} \pi(U_n)$ is an open neighborhood of the identity in $(G/H)_\omega$. We have that $p^{-1}(R) = \bigcap_{n\in\omega} p^{-1}(\pi(U_n)) = \bigcap_{n\in\omega} U_n H = P$, whence it follows that $R = p(P)$. Hence $p(P)$ is open in $(G/H)_\omega$. We have thus proved that for every open neighborhood Q of e in G_ω, the image $p(Q)$ contains an open neighborhood of the identity in $(G/H)_\omega$. Therefore the homomorphism p is open and φ is a topological isomorphism, as claimed. □

Combining Lemma 4.3 and several results from [7, 8] we prove the following theorem:

Theorem 4.4. *Let G be an ω-balanced (ω-narrow) group and H a completely metrizable invariant subgroup of G. Then G is strongly Dieudonné complete (strongly realcompact) if and only if G/H is strongly Dieudonné complete (strongly realcompact).*

Proof. It suffices to prove the theorem in the case of strong Dieudonné completeness, the argument in the case of strong realcompactness is almost the same. So we assume that that the group G is ω-balanced. Then the quotient group G/H is also ω-balanced. According to [8, Theorem 3.3] we have to verify that if one of the groups G or G/H is G_δ-closed in its Raĭkov completion, then so is the other.

Let $\varphi\colon \varrho G \to (\varrho G)/H$ be the quotient homomorphism. If G/H is G_δ-closed in $(\varrho G)/H$ then $G = \varphi^{-1}(G/H)$ is G_δ-closed in ϱG and, by [8, Theorem 3.3], G is strongly Dieudonné complete.

Now assume that G is G_δ-closed in ϱG. It follows from Lemma 3.2 that $\varrho(G_\omega) = (\varrho_\omega G)_\omega = G_\omega$, so the group G_ω is Raĭkov complete. Since H is metrizable, the group H_ω is discrete and hence Čech-complete. By [7, Theorem 11.18], the quotient group G_ω/H_ω is also Raĭkov complete. Hence Lemma 4.3 implies that $(G/H)_\omega \cong G_\omega/H_\omega$ is Raĭkov complete and, therefore, the group G/H is strongly Dieudonné complete by Theorem 3.3. The latter means that G/H is G_δ-closed in $\varrho(G/H) \cong (\varrho G)/H$. □

Theorem 4.5. *Let X be an ω-balanced (ω-narrow) group and H a Čech-complete invariant subgroup of X. Then X is strongly Dieudonné complete (strongly realcompact) if and only if so is X/H.*

Proof. As in the proof of Theorem 4.4 we consider only the case of strong Dieudonné completeness. Let φ be the canonical homomorphism of ϱX onto $(\varrho X)/H$. Since H is Čech-complete, the quotient group $(\varrho X)/H$ is Raĭkov complete [7, Theorem 11.18]. So $(\varrho X)/H$ is the Raĭkov completion of X/H. If X/H is strongly Dieudonné complete, then it is G_δ-closed in $(\varrho X)/H$ by [8, Theorem 3.3] and hence $X = \varphi^{-1}(X/H)$ is G_δ-closed in ϱX. So the same theorem implies that X is strongly Dieudonné complete.

Conversely, assume that X is G_δ-closed in ϱX. Since the group H is Čech-complete, it follows from [1, Corollary 4.3.5] that there exists a compact subset C of H with a countable neighborhood base in H which contains the identity element e of H. Let $\{O_n : n \in \omega\}$ be a family of open neighborhoods of e in X such that $\{O_n \cap H : n \in \omega\}$ is a base for C in H. Then $C = H \cap \bigcap_{n\in\omega} O_n$. Since X is ω-balanced, every neighborhood of the identity e in X contains a closed invariant subgroup of type G_δ in X (see [1, Theorem 3.4.18]). Therefore, for every $n \in \omega$, there exists a closed invariant subgroup P_n of type G_δ in X satisfying $P_n \subset O_n$. Then $P = \bigcap_{n\in\omega} P_n$ is a closed invariant subgroup of type G_δ in X with $P \cap H \subset C$.

It is clear that $P \cap H$ is a closed G_δ-set in H and in C. Since both C and its closed subset $P \cap H$ are compact, the set $P \cap H$ has a countable neighborhood base in C. By the transitivity of character in Hausdorff spaces we conclude that the compact subgroup $N = P \cap H$ of H has a countable neighborhood base in H. Since H and P are invariant subgroups of X, so is the subgroup N. Hence the quotient group H/N is metrizable by [1, Lemma 4.3.19].

The quotient homomorphism of H onto H/N is a perfect mapping, while perfect mappings preserve Čech-completeness according to [2, Theorem 3.9.10]. Hence the group H/N is Čech-complete. Since the space H/N is metrizable, we conclude that it is completely metrizable.

The quotient group X/N is strongly Dieudonné complete by [8, Theorem 3.17], and H/N is an invariant completely metrizable subgroup of X/N. Hence Theorem 4.4 implies that the group $X/H \cong (X/N)/(H/N)$ is strongly Dieudonné complete. □

Suppose that $\eta = \{G_\alpha : \alpha \in A\}$ is a family of topological groups and $\Pi\eta = \prod_{\alpha\in A} G_\alpha$ is the topological product of the family η. Then the Σ-product of η, denoted by $\Sigma\Pi\eta$, is the subgroup of $\Pi\eta$ consisting of all points $g \in \Pi\eta$ such that $|\{\alpha \in A : \pi_\alpha(g) \neq e_\alpha\}| \leq \omega$, where $\pi_\alpha\colon \Pi\eta \to G_\alpha$ is the natural projection of $\Pi\eta$ onto the factor G_α and e_α is the neutral element of G_α, for every $\alpha \in A$. Similarly, the σ-product of η, denoted by $\sigma\Pi\eta$, is the subgroup of $\Pi\eta$ consisting of all points $g \in \Pi\eta$ such that $|\{\alpha \in A : \pi_\alpha(g) \neq e_\alpha\}| < \omega$. It is easy to see that both $\Sigma\Pi\eta$ and $\sigma\Pi\eta$ are dense subgroups of $\Pi\eta$. A description of properties of these subgroups can be found in [1, Section 1.6].

The following example shows that the properties of being strongly realcompact or strongly Dieudonné complete are not three spaces properties.

Example 4.6. There exists a pseudocompact non-compact topological Abelian group G containing a closed separable metrizable subgroup H such that the quotient group G/H is compact. Hence G fails to be strongly realcompact.

Proof. Let X be the group $\mathbb{Z}_2^{\mathfrak{c}}$, where $\mathfrak{c} = 2^\omega$, and let Y be the group $\mathbb{Z}_2^\omega$. Denote by $[\mathfrak{c}]^{\leq\omega}$ the family of all non-empty countable subsets of $\mathfrak{c}$. It is clear that $|[\mathfrak{c}]^{\leq\omega}| = \mathfrak{c}$. For every $A \in [\mathfrak{c}]^{\leq\omega}$ and $u \in \mathbb{Z}_2^A$, let $C(A,u) = \pi_A^{-1}(u)$, where $\pi_A\colon \mathbb{Z}_2^{\mathfrak{c}} \to \mathbb{Z}_2^A$ is the natural projection. Observe that the family $\mathcal{E} = \{C(A,u) : A \in [\mathfrak{c}]^{\leq\omega},\ u \in \mathbb{Z}_2^A\}$ is of cardinality $\mathfrak{c}$. Let $\mathcal{F}$ be the product $\mathcal{E} \times Y$. It is clear that any point of Y is a G_δ-set and that $|\mathcal{F}| = \mathfrak{c}$. Let $\{(E_\alpha, y_\alpha) : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{F}$. By recursion on $\alpha < \mathfrak{c}$ we define a subset $\{x_\alpha : \alpha < \mathfrak{c}\}$ of X such that the following conditions hold for each $\alpha < \mathfrak{c}$:
(i) $x_\alpha \in E_\alpha$;
(ii) $x_\alpha \notin \langle\{x_\beta : \beta < \alpha\}\rangle$.

Condition (ii) is possible because $|E_\alpha| = 2^{\mathfrak{c}}$. For any $y \in Y$, the cardinality of the set $\{\alpha : y_\alpha = y\}$ is equal to $\mathfrak{c}$. Let $\tilde{X} = \{x_\alpha : \alpha < \mathfrak{c}\}$. Define a mapping $f : \tilde{X} \to Y$ by $f(x_\alpha) = y_\alpha$ for each $\alpha < \mathfrak{c}$. The set $\tilde{X}$ is linearly independent by (ii), so we can extend f to a homomorphism g of $\langle\tilde{X}\rangle$ to Y. Since every subgroup of the Boolean group X is a direct summand in X, g extends to a homomorphism $\varphi\colon X \to Y$.

Let $P = \{(x, \varphi(x)) : x \in X\}$. Observe that P is a G_δ-dense subgroup of $X \times Y$. Indeed, let U be a non-empty G_δ-subset of $X \times Y$. Then there exist $y \in Y$, $A \in [\mathfrak{c}]^{\leq\omega}$, and $u \in \mathbb{Z}_2^\omega$ such that $C(A,u)\times\{y\} \subset U$. There exists $\alpha < \mathfrak{c}$ such that $(E_\alpha, y_\alpha) = (C(A,u), y)$. Then (i) implies that $(x_\alpha, y_\alpha) \in U$ and, by our definition of φ, $\varphi(x_\alpha) = y_\alpha$, whence $(x_\alpha, y_\alpha) \in U \cap P$.

Let us put $Z = \sigma\mathbb{Z}_2^\omega$ and $H = \{e\}\times Z$, where e is the identity element of X. Then H is a separable metrizable subgroup of $X \times Y$. We define a subgroup G of $X \times Y$ as the sum $G = P + H$. Let p be the restriction to G of the natural projection of $X \times Y$ onto the first factor X. Observe that $G \cap (\{e\} \times Y) = H$, so H is a closed subgroup of G. In particular G is a proper subgroup of $X \times Y$. Since H is dense in $\{e\} \times Y$, p is an open homomorphism of G onto X (see [3, Lemma 1.3]). Therefore, the quotient group G/H is topologically isomorphic to the compact group X.

As P is a G_δ-dense subgroup of the compact group $X \times Y$, we see that P is pseudocompact. Since $P \subset G \subset X \times Y$, the group G is also pseudocompact. Finally, a pseudocompact space is realcompact iff it is compact, so the group G cannot be either realcompact or strongly realcompact. □

5 Quotients and *P*-modifications

It was shown in Lemma 4.3 that for any invariant completely metrizable subgroup H of a topological group G, the quotient groups $(G/H)_\omega$ and G_ω/H_ω are naturally isomorphic as topological groups. In this sections we are going to extend this result to the case when H is Čech-complete.

We start with considering the case of a compact subgroup H of G.

Lemma 5.1. *If K is a compact invariant subgroup of a topological group G, then the identity isomorphism of G_ω/K_ω onto $(G/K)_\omega$ is a topological isomorphism.*

Proof. As in the proof of Lemma 4.3 it suffices to show the canonical homomorphism of G_ω onto $(G/K)_\omega$, say, p is open. Clearly p coincides pointwise with the quotient homomorphism $\pi\colon G \to G/K$.

Let Q be a G_δ-set in G containing the identity e in G, say, $Q = \bigcap_{n\in\omega} O_n$, where O_n is open in G for each $n \in \omega$. We define by induction a sequence $\{U_n : n \in \omega\}$ of symmetric open neighborhoods of e in G such that $U_{n+1}^2 \subset U_n \subset O_n$ for each $n \in \omega$. Then $C = \bigcap_{n\in\omega} U_n$ is a closed subgroup of G satisfying $C \subset Q$. We claim that $p(C) = \bigcap_{n\in\omega} p(U_n)$. Indeed, the inclusion $p(C) \subset \bigcap_{n\in\omega} p(U_n)$ is evident. Conversely, take an arbitrary element $y \in p(\bigcap_{n\in\omega} U_n)$. Then $p^{-1}(y) \cap U_n \neq \emptyset$ for each $n \in \omega$. Since K is compact, so is the fiber $p^{-1}(y)$.

Clearly $\overline{U_{n+1}} \subseteq U_n$ for each $n \in \omega$, so $p^{-1}(y) \cap \bigcap_{n\in\omega} \overline{U_n} \neq \emptyset$ or, equivalently, $p^{-1}(y) \cap \bigcap_{n\in\omega} U_n \neq \emptyset$. We conclude that $y \in \bigcap_{n\in\omega} p(U_n)$, whence our claim follows.

It follows from $\bigcap_{n\in\omega} p(U_n) = p(C) \subset p(Q)$ that the image $p(Q)$ contains the open neighborhood $\bigcap_{n\in\omega} p(U_n)$ of the identity in $(G/H)_\omega$, so the homomorphism p is open. This proves the lemma. □

Theorem 5.2. *Let H be an invariant Čech-complete subgroup of an ω-balanced topological group G. Then the identity mapping of G_ω/H_ω onto $(G/H)_\omega$ is a topological isomorphism.*

Proof. The argument that follows is close to the proof of Theorem 4.5. Since the group H is Čech-complete, we can apply [1, Corollary 4.3.5] to find a compact subset C of H with a countable neighborhood base in H which contains the identity element e of H. Let $\{O_n : n \in \omega\}$ be a family of open neighborhoods of e in G such that $\{O_n \cap H : n \in \omega\}$ is a base for C in H. Then $C = H \cap \bigcap_{n\in\omega} O_n$. Since G is ω-balanced, every neighborhood of the identity e in G contains a closed invariant subgroup of type G_δ in G (see [1, Theorem 3.4.18]). Therefore, for every $n \in \omega$, there exists a closed invariant subgroup P_n of type G_δ in G satisfying $P_n \subset O_n$. Then $P = \bigcap_{n\in\omega} P_n$ is a closed invariant subgroup of type G_δ in G with $P \cap H \subset C$.

It is clear that $P\cap H$ is a closed G_δ-set in H and in C. Since both C and its closed subset $P\cap H$ are compact, the set $P \cap H$ has a countable neighborhood base in C. By the transitivity of character in Hausdorff spaces we conclude that the compact subgroup $K = P \cap H$ of H has a countable neighborhood base in H. Since the subgroups H and P of G are invariant in G, so is the subgroup K. Hence the quotient group H/K is metrizable [1, Lemma 4.3.19]. The quotient homomorphism of H onto H/K is a perfect mapping, while perfect mappings preserve Čech-completeness according to [2, Theorem 3.9.10]. Hence the group H/K is Čech-complete. Since the space H/K is metrizable, we conclude that it is completely metrizable.

The final step in our argument is to apply the Second Isomorphism Theorem for topological groups which implies that $G/H \cong (G/K)/(H/K)$. Since the subgroup K of G is compact, the identity mapping of G_ω/K_ω onto $(G/K)_\omega$ is a topological isomorphism, by Lemma 5.1. Similarly, the groups $(H/K)_\omega$ and H_ω/K_ω are topologically isomorphic. Let $H^\star = H/K$ and $G^\star = G/K$. Using this notation we have that $G/H \cong G^\star/H^\star$, $H^\star_\omega \cong H_\omega/K_\omega$ and $G^\star_\omega \cong G_\omega/K_\omega$. We now apply Lemmas 4.3 and 5.1 along with the Second Isomorphism Theorem to conclude that

$$(G/H)_\omega \cong (G^\star/H^\star)_\omega \cong (G_\omega/K_\omega)/(H_\omega/K_\omega) \cong G_\omega/H_\omega.$$

Thus the groups $(G/H)_\omega$ and G_ω/H_ω are topologically isomorphic. □

In Lemmas 4.3 and 5.1 the corresponding subgroups H and K of a topological group G are Raĭkov complete. It is also known that every Čech-complete group is Raĭkov complete [1, Theorem 4.3.7]. Hence the subgroup H of G in Theorem 5.2 is Raĭkov complete as well. However, the natural equivalence of the groups $(G/H)_\omega$ and G_ω/H_ω does not imply the Raĭkov completeness of H, even if H is separable and metrizable:

Proposition 5.3. *If H and K are arbitrary topological groups and $G = H \times K$, then the identity mapping of G_ω/H_ω onto $(G/H)_\omega$ is a topological isomorphism.*

Proof. It is clear that $G/H \cong K$, so $(G/H)_\omega \cong K_\omega$. Since $G_\omega \cong H_\omega \times K_\omega$, we see that $G_\omega/H_\omega \cong (H_\omega \times K_\omega)/H_\omega \cong K_\omega$. Therefore the groups G_ω/H_ω and $(G/H)_\omega$ are topologically isomorphic. □

Under certain circumstances, the Raĭkov completeness of a closed invariant subgroup H of a topological group G becomes a necessary condition for the validity of the equivalence $G_\omega/H_\omega \cong (G/H)_\omega$. In the following proposition we show that this is the case when the group G is locally pseudocompact.

Proposition 5.4. *Let H be a closed metrizable invariant subgroup of a locally pseudocompact topological group G. Then the identity mapping of G_ω/H_ω onto $(G/H)_\omega$ is a topological isomorphism if and only if the group H is locally compact.*

Proof. First we note that a locally compact metrizable group is completely metrizable. So the sufficiency part of the propostion follows from Lemma 4.3, even without the assumption that G is locally pseudocompact.

Hence we assume that the groups G_ω/H_ω and $(G/H)_\omega$ are topologically isomorphic, where the subgroup H of G is metrizable. The Raĭkov completion of G, ϱG, is a locally compact group. Since the group ϱH is topologically isomorphic to the closure of H in ϱG and this closure is an invariant subgroup of ϱG, we see that the canonical embedding of G/H to $\varrho G/\varrho H$ is a topological isomorphism from G/H onto its image (see [1, Theorem 1.5.16]). Notice that the group ϱH is metrizable and locally compact, so it is completely metrizable. Hence Lemmas 4.1 and 4.3 imply that $(G/H)_\omega$ is a topological subgroup of the group $(\varrho G/\varrho H)_\omega \cong (\varrho G)_\omega/(\varrho H)_\omega$. Since the groups G_ω/H_ω and $(G/H)_\omega$ are topologically isomorphic, we see that the natural embedding of G_ω/H_ω to $(\varrho G)_\omega/(\varrho H)_\omega$ is a topological isomorphism from G_ω/H_ω onto its image or, equivalently, the restriction to G_ω of the quotient homomorphism $\pi_\omega\colon (\varrho G)_\omega \to (\varrho G)_\omega/(\varrho H)_\omega$, say, φ is an open mapping of G_ω onto the subgroup $\pi_\omega(G_\omega)$ of $(\varrho G)_\omega/(\varrho H)_\omega$ and $\pi_\omega(G_\omega) \cong G_\omega/H_\omega$.

Finally, the locally pseudocompact group G meets every non-empty G_δ-set in ϱG (see [1, Problem 3.7.J]), i.e. G_ω is a dense subgroup of $(\varrho G)_\omega$. Since the homomorphism φ is open, [1, Theorem 1.5.16] implies that $H_\omega = (\varrho H)_\omega \cap G_\omega$ is dense in $(\varrho H)_\omega$. Notice that the group $(\varrho H)_\omega$ is discrete since ϱH is metrizable. Hence $H_\omega = (\varrho H)_\omega$, i.e. $H = \varrho H$. In other words, the group H is Raĭkov complete. Hence $H = \varrho H$ is a closed subgroup of the locally compact group ϱG, whence the local compactness of H follows. □

Let us note that every precompact Abelian group is topologically isomorphic to a closed subgroup of a pseudocompact (connected) Abelian group [9]. Hence the following corollary describes quite a common situation.

Corollary 5.5. *Let H be a closed metrizable invariant subgroup of a pseudocompact topological group G. Then the identity mapping of G_ω/H_ω onto $(G/H)_\omega$ is a topological isomorphism if and only if the group H is compact.*

Proof. According to Proposition 5.4, the groups G_ω/H_ω and $(G/H)_\omega$ are topologically isomorphic iff H is locally compact. The group H is precompact as a subgroup of the pseudocompact group G. The required conclusion now follows from the fact that precompact locally compact topological groups are compact. □

The corollary below extends Lemma 5.1 to a slightly more general case.

Corollary 5.6. *Let K be a closed invariant pseudocompact subgroup of a topological group G. Then the identity mapping of G_ω/K_ω onto $(G/K)_\omega$ is a topological isomorphism.*

Proof. Let ϱG be the Raĭkov completion of the group G. Then the closure of K in ϱG is a compact group topologically isomorphic to ϱK. Denote by π the quotient homomorphism of ϱG onto $\varrho G/\varrho K$ and let $G^\star = \pi^{-1}\pi(G)$. Then $G \subset G^\star \subset \varrho G$. Since K is pseudocompact it meets every non-empty G_δ-set in ϱK. Hence G meets every non-empty G_δ-set in $G^\star = G\cdot\varrho K$. Since K is dense in ϱK, the restriction of π to G is a continuous open homomorphism of G onto $\pi(G) \cong G/K$ and, similarly, $G^\star/\varrho K \cong \pi(G) \cong G/K$. The group ϱK is compact, so Lemma 5.1 implies that

$$(G/K)_\omega \cong (G^\star/\varrho K)_\omega \cong (G^\star)_\omega/(\varrho K)_\omega. \tag{1}$$

We know that G is G_δ-dense in $G^\star$ and K is G_δ-dense in ϱK, i.e. G_ω is dense in $(G^\star)_\omega$ and K_ω is dense in $(\varrho K)_\omega$. Hence $(G^\star)_\omega/(\varrho K)_\omega \cong G_\omega/K_\omega$ by virtue of [1, Theorem 1.5.16]. The latter fact and (1) together imply that $(G/K)_\omega \cong G_\omega/K_\omega$. □

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