Topol. Algebra Appl. 2016; 4:18–30 DE GRUYTER OPEN

Research Article Open Access

Boris G. Averbukh*

On unitary extensions and unitary completions of topological monoids

DOI 10.1515/taa-2016-0003
Received November 3, 2014; accepted August 18, 2016

Abstract: The concept of a unitary Cauchy net in an arbitrary Hausdorff topological monoid generalizes the concept of a fundamental sequence of reals. We construct extensions of this monoid where all its unitary Cauchy nets converge.

Keywords: topological monoid, Cauchy space, completion

MSC: 22A15, 54D35, 54E15

Introduction

In analysis, one often uses sequences of points of normed linear spaces possessing the property that all differences of their far enough members lie in any preassigned neighborhood of zero. In paper [1], we have defined nets in an arbitrary Hausdorff topological monoid $\mathcal{X}$ which have a similar property. We called them unitary Cauchy nets or shortly C-nets and the corresponding filters C-filters and proved that the underlying set X of this monoid endowed with the set of all C-filters is a Cauchy space.

Our main purpose is to construct extensions of a given monoid where all its C-nets converge. We call them unitary. Thus, we intend to transfer the classical notions of a completeness and of a completion into the theory of topological monoids.

In paper [2], we studied the Wyler completion $\tilde{X}$ of the aforesaid Cauchy space. This Hausdorff topological space is a unitary extension of $\mathcal{X}$ and has a universal property over other its unitary extensions. However, the topology of $\tilde{X}$ is mostly rather fine. In this paper, we construct more suitable unitary extensions of $\mathcal{X}$. They are obtained from $\tilde{X}$ by means of topological operations and look simpler than compactifications.

A given Hausdorff topological monoid is said to be unitarily complete if all its C-filters converge. It was proved in [1] that a monoid on a T_3-underlying space possesses this property if its identity has a neighborhood whose closure is compact. For a given Hausdorff topological monoid $\mathcal{X}$, a couple $(f, \mathcal{Y})$ consisting of a unitarily complete monoid $\mathcal{Y}$ and of an algebraic and topological embedding $f\colon \mathcal{X} \to \mathcal{Y}$ is called its unitary completion if $\mathcal{Y}$ properly contains no unitarily complete submonoid containing $f(\mathcal{X})$. We find conditions of the existence of such a completion.

Not every topological monoid has it. Therefore, we also consider unitary extensions up to a topological space. In the most important case, such space Y is an underlying space of a Hausdorff topological monoid, too, and f is an algebraic and topological embedding of $\mathcal{X}$. For commutative monoids, we find a necessary and sufficient condition of the existence of such an extension. The proof of the sufficiency is constructive, and this construction generalizes the classical one: for $\mathcal{X} = (Q_0^+, +)$ with the usual topology, it leads to $Y = (R_0^+, +)$ with the usual topology.

***Corresponding Author: Boris G. Averbukh:** Moscow State Forestry University, department of mathematics, Moscow, Russian Federation, E-mail: averbuch@gmx.de

A unitary extension is said to be precise if the inverse image on X of the topology of Y coincides with the initial topology of $\mathcal{X}$. Not any monoid has such an extension. We give its construction for monoids satisfying some additional requirements.

1 *C*-filters and strict *C*-filters

In order to make this paper more self-contained, we begin from the reminding of the main concepts and statements of the previous papers [1] and [2] of this series. Moreover, we define the notion of a strict C-filter which we use later.

A) Let $\mathcal{X} = (X, m, \tau)$ be a Hausdorff topological monoid with an identity 1. Here, X denotes its underlying set, m its multiplication and τ its topology. In the following, we always shorten $m(a,\ b)$ to ab.

A net $S = \{x_\alpha\}_{\alpha \in A}$ in X is called *a C-net* (a left, a right C-net) if, for each neighborhood U of 1, there exists $\alpha_0 \in A$ and, for each $\alpha \geq \alpha_0$, there exists $\alpha_0' \in A$ such that $x_{\alpha'} \in \overline{Ux_\alpha U}$ ($x_{\alpha'} \in \overline{x_\alpha U}$, $x_{\alpha'} \in \overline{Ux_\alpha}$) for all $\alpha' \geq \alpha_0'$. The line on top denotes the topological closure. Throwing away this line, we obtain a definition of a *strict C-net* (of a left, of a right strict C-net).

The filters corresponding to C-nets (strict C-nets) are said to be *C-filters (strict C-filters)*. Their direct definition is: a filter $\mathcal{F}$ on X (i.e. in the power set $P(X)$) is called a C-filter (a left, a right C-filter) on $\mathcal{X}$ if the set $M_U = \{x \in X\colon\ \overline{UxU} \in \mathcal{F}\}$ (respectively, $L_U = \{x \in X\colon\ \overline{xU} \in \mathcal{F}\}$, $R_U = \{x \in X\colon\ \overline{Ux} \in \mathcal{F}\}$) belongs to $\mathcal{F}$ for every neighborhood U of 1. To obtain a definition of a strict C-filter (of a left and of a right strict C-filter), it is necessary to use similar sets $M_U^s = \{x \in X\colon\ UxU \in \mathcal{F}\}$, L_U^s and R_U^s.

The concepts of a C-filter and of a strict C-filter coincide if there is a neighborhood of 1 with a compact closure. Indeed, in this case, the closure of any sufficiently small neighborhood of 1 is compact, and it is straightforward that $\overline{UxU} = \overline{U}x\overline{U}$ for any such a neighborhood U of 1. Moreover, U is T_3, and there exists a neighborhoods V of 1 such that $\overline{V} \subset U$. Then $\overline{VxV} = \overline{V}x\overline{V} \subset UxU$ for any $x \in X$. Therefore, $M_V(\mathcal{F}) \subset M_U^s(\mathcal{F})$ for an arbitrary C-filter $\mathcal{F}$ and such neighborhoods U and V of 1. Hence, $M_U^s(\mathcal{F}) \in \mathcal{F}$ for any U, and $\mathcal{F}$ is a strict C-filter. The converse is evident.

It is evident, each left (right) C-net is a C-net, and each left (right) C- filter is a C-filter. If $\mathcal{X}$ is commutative, then these concepts coincide. If $\mathcal{X}$ is a topological group, then right (left, two-sided) C-filters are Cauchy filters of the right (left, Rölke) uniformity on X, and any C-filter is a strict one. As a rule, *we only consider two-sided C-filters and strict C-filters in the following*. For any point $x \in X$, the symbol $\dot{x}$ denotes the strict C-filter consisting of all subsets of X containing this point x.

For C-filters $\mathcal{F}_1$, $\mathcal{F}_2$, we write $\mathcal{F}_1 \geq \mathcal{F}_2$ if $M_U(\mathcal{F}_2) \in \mathcal{F}_1$ for any neighborhood U of 1 . The corresponding condition for C-nets $S_1 = \{x_\alpha\}_{\alpha \in A}$ and $S_2 = \{y_\beta\}_{\beta \in B}$ means: for each neighborhood U of 1, there exists $\alpha_0 \in A$ such that, for each $\alpha \geq \alpha_0$, there exists $\beta_0 \in B$ such that the inclusion $y_\beta \in \overline{Ux_\alpha U}$ holds for any $\beta \geq \beta_0$. We set $\mathcal{F}_1 \approx \mathcal{F}_2$ if both $\mathcal{F}_1 \geq \mathcal{F}_2$ and $\mathcal{F}_2 \geq \mathcal{F}_1$ are true. It is proved in [1] (see Proposition 1.7) that $\geq$ is a quasi-order relation, and $\approx$ is an equivalence on the set Σ of all C- filters. By Proposition 1.9 from [1], if $\mathcal{F}_1$, $\mathcal{F}_2$ are C-filters with $\mathcal{F}_1 \geq \mathcal{F}_2$ and $\mathcal{F}_1$ has a cluster point x_0, then $\mathcal{F}_2$ can not have any cluster points different from x_0.

The intersection of any family of equivalent to each other C-filters is a C-filter from the same equivalence class. If such a family is an equivalence class, then this intersection is said to be least with respect to inclusion or shortly $\subset$-least C-filter. For the class of a C-filter $\mathcal{F}$, we denote it by $\mathcal{F}_{lst}$.

Example 1.1. Let $\mathcal{X}$ be $(R_0^+,\ +)$ with the topology of the Sorgenfrey line (= the right half-open topology whose a base at each point x consists of intervals $[x;\ x+\epsilon[$ with $\epsilon > 0$). The set of $\subset$-least C-filters is formed by all ultrafilters $\dot{x}$, $x \in R_0^+$, and filters $\mathcal{F}_x$, $x > 0$, with bases consisting of intervals $[x-\epsilon;\ x[$. These filters $\mathcal{F}_x$ are strict, diverge, and $\mathcal{F}_x \geq \dot{x}$ for any x.

By Theorems 2.5 and 3.2 from [1], the set X endowed with the family Σ is a Cauchy space (see [5], [6], [7], [8], [9], and there is a larger list of papers devoted to Cauchy spaces and their completions in [1] and [2]) whose convergence structure defines a $T_{3\frac{1}{2}}$-topology. This topology is said to be unitary. It is proved in Theorem 3.2 from [1] that, for any $x \in X$, the filter $\dot{x}_{\mathrm{lst}}$ is the neighborhood filter of x in the unitary topology. The monoid

$\mathcal{X}$ is said to be *unitarily separable* if its initial topology τ is coarser than the unitary one or coincides with it. In this case, for any $x \in X$, the filter $\dot{x}_{\text{lst}}$ converges to x in the topology τ.

Main statements on properties of strict C-filters slightly strengthen the corresponding propositions proved in the papers [1] and [2] for arbitrary ones. As a rule, in order to obtain their proofs, it is only necessary to add the word "strict" into their wordings and to delete the line on top in their proofs. Therefore, we only point out properties of strict C-filters, different from the corresponding properties of all C-filters. In particular, the statement of Proposition 1.9 from [1] can be strengthened in the following way: *if $\mathcal{F}_1, \mathcal{F}_2$ are strict C-filters such that $\mathcal{F}_1 \geq^s \mathcal{F}_2$ (i.e. $M_U^s(\mathcal{F}_2) \in \mathcal{F}_1$), and $\mathcal{F}_1$ clusters to a point x_0, then $\mathcal{F}_2$ converges to x_0.*

Strict C-filters $\mathcal{F}_1, \mathcal{F}_2$ are said to be *s-equivalent* if both the inequalities $\mathcal{F}_1 \geq^s \mathcal{F}_2$ and $\mathcal{F}_2 \geq^s \mathcal{F}_1$ take place. Applying Corollary 1.10 from [1] to strict C-filters, one may omit the assumption that (X, τ) is a T_3-space. It means that each strict C-filter either converges or does not have any cluster points and s-equivalent strict C-filters have equal limits (see also Example 2.4). For any strict C-filter $\mathcal{F}$, there exists a corresponding to it $\subset$-least strict C-filter $\mathcal{F}^s_{\text{lst}}$. It is equal to the intersection of all s-equivalent to $\mathcal{F}$ strict C-filters and can differ from $\mathcal{F}_{\text{lst}}$.

Properties of arbitrary C-filters proved at the beginning of the second section of the paper [2] can be transferred to strict ones as follows: *the images of strict C-filters under continuous identity preserving homomorphisms between Hausdorff monoids and under translations in a given monoid are strict C-filters,* and, *for a commutative Hausdorff monoid, the product of any couple of its strict C-filters ($\mathcal{F}_1\mathcal{F}_2$ is the filter generated by all sets of the form F_1F_2 with $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$) is a strict C-filter, too.* But statement (ii) of Proposition 2.1 of the paper [2] can be false for strict C-filters.

The underlying set X endowed with the family Σ^s of strict C-filters is a Cauchy space, too, and its convergence structure defines a $T_{3\frac{1}{2}}$-topology. It is finer than the topology which is defined by the family Σ, or coincides with it. Moreover, it is always finer than the initial one or coincides with it.

B) The Wyler completion $\tilde{X}$ of the Cauchy space (X, Σ) is a Hausdorff topological space (see Theorem 1.9 from [2]) whose topology is said to be natural. The set $\tilde{X}$ consists of equivalence classes of C-filters or (that is equivalent) of $\subset$-least C-filters. Its point corresponding to such a filter $\mathcal{F}$ is denoted by $[\mathcal{F}]$. There exists a canonical topological embedding i of X endowed with the unitary topology into $\tilde{X}$, so that $i(x) = [\dot{x}_{lst}]$. The subspace $i(X)$ is dense and open, and $\tilde{X} \setminus i(X)$ is discrete. It is proved in [2] that any filter $i(\mathcal{F})$ with $\mathcal{F} \in \Sigma$ converges in $\tilde{X}$ to the point $[\mathcal{F}_{lst}]$, and that is why any point of $\tilde{X}$ is the limit of such a filter.

It is also proved there that translations of the space $\tilde{X}$ by elements of X which are induced by translations of C-filters, are continuous maps. If $\mathcal{X}$ is commutative, then $\tilde{X}$ is an abstract monoid whose multiplication corresponds to the multiplication of C-filters, and i is an algebraic embedding.

The above construction of the Wyler completion of the Cauchy space (X, Σ) can be transferred to the space (X, Σ^s). Theorem 1.9 from [2] describing (together with its lemmas) the natural topology of $\tilde{X}$, remains true for the obtained space $\tilde{X}^s$.

As the next example shows, the above-mentioned does not mean that strict C-filters are more convenient object of study than the non-strict ones.

Example 1.2. Let $\mathcal{X}$ be a submonoid of $(R_0^+, +)$ with the topology of the Sorgenfrey line (see Example 1.1) arising if we exclude all irrationals between 0 and 1. Denote by U its neighborhood of 0 lying in the interval $[0; 1[$ and by $\mathcal{F}$ a strict C-filter on $\mathcal{X}$ differing from ultrafilters of the form $\dot{x}$. Suppose that its limit a in the usual topology of R_0^+ is bigger than 1. Then $x_1 - x_2$ is rational for any $x_1,\ x_2 \in M_U^s(\mathcal{F})$ since otherwise $(x_1 + U) \cap (x_2 + U) = \emptyset$. It means that $M_U^s(\mathcal{F})$ lies wholly in some coset from $(R_0^+, +)/(Q_0^+, +)$. This coset does not depend on U, and we denote it by $r(\mathcal{F})$. Strict C-filters $\mathcal{F}_1,\ \mathcal{F}_2$ of such a kind are equivalent in Σ^s if and only if they are equivalent in Σ and, moreover, $r(\mathcal{F}_1) = r(\mathcal{F}_2)$. For any such coset, there exists a strict C-filter $\mathcal{F}$ with the corresponding value of $r(\mathcal{F})$. Hence, each equivalence class from Σ differing from classes of ultrafilters of the form $\dot{x}$ consists of continuum of equivalence classes from Σ^s and, moreover, of C-filters which are not strict ones. For example, the $\subset$-least C-filter whose base consists of intervals of the form $]a - \epsilon,\ a[$, is not a strict one. It shows that the corresponding point from $\tilde{X} \setminus i(X)$ splits into a continuum of points from $\tilde{X}^s$.

In the following, we are going to develop both these theories (of C-filters and of strict C-filters) in parallel. In each formulation, to obtain the corresponding statement relating to strict C-filters, it is necessary to include in the text separate words in brackets. Otherwise, these words should be omitted.

C) Let Y be a topological space and $f\colon\ X \to Y$ a map of the underlying set X of the considered monoid $\mathcal{X}$. This map is said to be *unitarily quasi-extending* if $\lim f(\mathcal{F})$ consists of an only point for each C-filter $\mathcal{F}$ on $\mathcal{X}$ and each point of Y is the limit of such a filter. For example, the map i above is a unitarily quasi-extending one. By Theorem 2.12 from [2], for any such map f, there exists a unique continuous map $\tilde{f}\colon\ \tilde{X} \to Y$ such that $f = \tilde{f} \circ i$, i.e. i has the universal property over other unitarily quasi-extending maps. That is why $\tilde{X}$ is said to be *the finest unitary extension* of $\mathcal{X}$. Such a map $\tilde{f}$ is surjective. Replacing C-filters by strict C-filters, we obtain a definition of *a weak unitarily quasi-extending map* and the formulation of the corresponding universal property of *the weak finest unitary extension* $\tilde{X}^s$.

Like the unitary topology on X, the natural topology of $\tilde{X}$ is mostly rather fine. For many monoids (see, in particular, Example 1.12 from [2]), it is finer than the usually used topology of their extensions having the above properties of images of C-filters. On the other hand, all more suitable extensions of X with these properties can be obtained from $\tilde{X}$ by means of topological operations. This is the subject of this paper.

2 The concept of a unitary extension

In this section, we consider a definition, some properties and constructions of (weak) unitary extensions.

A) Definition 2.1. Let $\mathcal{X}$ be a Hausdorff topological monoid, Y a topological space, and $f\colon\ X \to Y$ a (weak) unitarily quasi-extending map. The couple (f, Y) is called a (weak) unitary extension of $\mathcal{X}$ and f a (weak) unitarily extending map if f is an injection of the set X.

For an arbitrary unitary extension $(f,\ Y)$ of $\mathcal{X}$, let Y' be the subset of Y consisting of all points of the form $\lim f(\mathcal{F})$ where $\mathcal{F}$ is a strict C-filter on $\mathcal{X}$. It is evident that $Y' \supset f(X)$ since all $\dot{x}$ with $x \in X$ are strict C-filters and $f(x) = \lim f(\dot{x})$ because of $f(x) \in f(F)$ for all $F \in \dot{x}$. Therefore, $(f,\ Y')$ is a weak unitary extension of $\mathcal{X}$.

For a given (weak) unitary extension $(f,\ Y)$ of $\mathcal{X}$, denote by τ_f the topology on X induced by the topology of Y. This topology is said to be *extensible* by means of f. For example, if Y is $\tilde{X}$ endowed with the natural topology, then the extensible by means of i topology on X is unitary one. It is evident, f is a homeomorphic embedding of the space (X, τ_f) into Y.

The (weak) unitary extension (f, Y) is said to be *precise* if τ_f coincides with the initial topology τ of X. It is said to be *homomorphic* if Y is the underlying space of a Hausdorff topological monoid $\mathcal{Y}$ and f is a continuous identity preserving homomorphism of $\mathcal{X}$ into $\mathcal{Y}$. If $\mathcal{X}$ is commutative, then this $\mathcal{Y}$ is also commutative for any (weak) homomorphic unitary extension of $\mathcal{X}$ since $f(X)$ is dense in Y. For any (weak) precise homomorphic unitary extension $(f,\ Y)$, the map f is an algebraic and topological embedding.

Proposition 2.2. Any (weak) unitary extension $(f,\ Y)$ of $\mathcal{X}$ possesses the next properties:

(i) $(X,\ \tau_f)$ is a T_1-space.

(ii) The topology τ_f is coarser than the unitary one or coincides with it.

(iii) Any (strict) C-filter on $\mathcal{X}$ either converges in the topology τ_f to an only point which is its unique cluster point, or does not have any cluster points.

(iv) $\lim f(\mathcal{F}_1) = \lim f(\mathcal{F}_2)$ for any (s-)equivalent (strict) C-filters $\mathcal{F}_1$, $\mathcal{F}_2$.

Proof. It is a simple corollary of Proposition 2.12 from [2]. □

The last property is also true if $(f,\ Y)$ is a weak homomorphic unitary extension and strict C-filters $\mathcal{F}_1$, $\mathcal{F}_2$ are equivalent but not necessarily s-equivalent. Indeed, $\lim f(\mathcal{F}_1) = \lim f(\mathcal{F}_2)$ by Corollary 1.10 from [1] since strict C-filters $f(\mathcal{F}_1)$, $f(\mathcal{F}_2)$ on $\mathcal{Y}$ are equivalent and converge.

Not each topological monoid has a precise unitary extension since it follows from statement (ii) of Proposition 2.2 that *monoids having such extensions are unitarily separable.* That is why we study not only precise unitary extensions.

It is evident, each submonoid of a monoid having a (weak) precise unitary extension, has it, too. Show that the operation of a finite direct product also preserves this property.

Proposition 2.3. Let $(f_1,\ Y_1), \dots, (f_n,\ Y_n)$ be (weak) precise unitary extensions of Hausdorff topological monoids $\mathcal{X}_1, \dots, \mathcal{X}_n$. Then $(f_1 \times \cdots \times f_n,\ Y_1 \times \cdots \times Y_n)$ is a (weak) precise unitary extension of $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$.

Proof. Denote by $X,\ X_1, \dots,\ X_n$ the underlying spaces of $\mathcal{X}$, respectively, $\mathcal{X}_1, \dots,\ \mathcal{X}_n$ and by p_i the i-th projection of X, $i = \overline{1,\ n}$.

First, we prove that $\mathcal{F} = \mathcal{F}_1 \times \dots \times \mathcal{F}_n$ is a $\subset$-least C-filter on $\mathcal{X}$ for any $\subset$-least C-filters $\mathcal{F}_1, \dots,\ \mathcal{F}_n$ on $\mathcal{X}_1, \dots,\ \mathcal{X}_n$, respectively. Note that $\mathcal{F}$ is a C-filter on $\mathcal{X}$ by Proposition 2.4 from [2]. If $\mathcal{F}'$ is a C-filter on $\mathcal{X}$ such that $\mathcal{F}' \approx \mathcal{F}$ and $\mathcal{F}' \subset \mathcal{F}$, then, by Proposition 2.1 from [2], $p_i(\mathcal{F}') \subset \mathcal{F}_i$ and $p_i(\mathcal{F}') \approx \mathcal{F}_i$. It implies $p_i(\mathcal{F}') = \mathcal{F}_i$ for $i = \overline{1,\ n}$ since all $\mathcal{F}_i$ are $\subset$-least filters. For arbitrary members $F_1 \in \mathcal{F}_1, \dots,\ F_n \in \mathcal{F}_n$, it follows from $F_i \in p_i(\mathcal{F}')$ that $p_i^{-1}(F_i) \in \mathcal{F}'$, $i = \overline{1,\ n}$. Therefore, $F_1 \times \dots \times F_n = p_1^{-1}(F_1) \cap \dots \cap p_n^{-1}(F_n) \in \mathcal{F}'$. It means $\mathcal{F}' \supset \mathcal{F}_1 \times \dots \times \mathcal{F}_n$ and $\mathcal{F}' = \mathcal{F}_1 \times \dots \times \mathcal{F}_n$.

Let now $\mathcal{G}$ be an arbitrary C-filter on $\mathcal{X}$ and $\mathcal{F}_i$, $i = \overline{1,\ n}$, $\subset$-least C-filter on $\mathcal{X}_i$ which is equivalent to $p_i(\mathcal{G})$. Then $\mathcal{F}_1 \times \dots \times \mathcal{F}_n \approx p_1(\mathcal{G}) \times \dots \times p_n(\mathcal{G}) \approx \mathcal{G}$ by the same Propositions from [2], and, therefore, $\mathcal{G} \supset \mathcal{F}_1 \times \dots \times \mathcal{F}_n$ and $(f_1 \times \dots \times f_n)(\mathcal{G}) \supset (f_1 \times \dots \times f_n)(\mathcal{F}_1 \times \dots \times \mathcal{F}_n) = f_1(\mathcal{F}_1) \times \dots \times f_n(\mathcal{F}_n)$. The filter $f_i(\mathcal{F}_i)$, $i = \overline{1,\ n}$, converges to an only point of Y_i which we denote by y_i. It implies $(f_1 \times \dots \times f_n)(\mathcal{G})$ converges to $y = (y_1, \dots,\ y_n) \in Y_1 \times \dots \times Y_n$. If $y' = (y_1', \dots,\ y_n')$ is a cluster point of this filter, then y_i' is a cluster point of $f_i(p_i(\mathcal{G}))$. But this filter also converges to y_i and does not have any other cluster points by Proposition 2.2, i.e. $y_i = y_i'$ for all i. □

Example 2.4. Let $\mathcal{X}$ be the submonoid of $(C, +)$ consisting of $z = 0$ and of all z with $0 \le \arg z \le \pi$. Define its topology so that a base of neighborhoods of each $z_0 \in X$ consists of all subsets $U_\epsilon(z_0)$ given by means of inequalities $|z - z_0| < \epsilon$ with $\epsilon > 0$ and, for $z \ne z_0$, $0 \le \arg(z - z_0) < \pi$. We obtain a topological monoid on a Hausdorff but non-T_3 underlying space. It was considered in Examples 2.9 and 2.15 from [1]. This monoid does not have precise unitary extensions since it is not unitary separable. For any z, the filter $\dot{z}$ and the filter corresponding to the sequence $\{z - 1/n\}_{n \in N}$ are equivalent but not-s-equivalent strict C-filters. The first of them converges, and the second one does not have any cluster points.

Change now the topology so that a base at each point z_0 consists of subsets given by means of inequalities $|z - z_0| < \epsilon$, $0 \le \arg(z - z_0) \le \pi$ (instead of $< \pi$). Endowed with this topology, $\mathcal{X}$ is topologically isomorphic to the product of $(R,\ +)$ with the usual topology and $(R_0^+,\ +)$ with the topology of the Sorgenfrey line (see Example 1.1). These monoids have precise unitary extensions since all C-filters of the first one converge and, for the second one, its precise unitary extension is a slightly modified space "two arrows" (see [4] and Example 2.8 below). Therefore, $\mathcal{X}$ with the changed topology has a precise unitary extension by Proposition 2.3.

B) A Hausdorff monoid can have many (weak) unitary extensions. Here, we define their *equivalence* and *an order relation on the set of its classes*. Similar definitions for completions of Cauchy spaces can be found in [9]. Then we prove that any subset of this set has a unique least upper bound.

Let $(f_1,\ Y_1)$, $(f_2,\ Y_2)$ be (weak) unitary extensions of $\mathcal{X}$. We write $(f_1,\ Y_1) \ge (f_2,\ Y_2)$ if there exists a continuous map $h\colon\ Y_1 \to Y_2$ such that $h \circ f_1 = f_2$. In this case, the extensible topology τ_{f_1} is finer than the extensible topology τ_{f_2}, or these topologies coincide. By quoted above Theorem 2.13 from [2], $(i,\ \tilde{X}) \ge (f,\ Y)$ $((i^s,\ \tilde{X}^s) \ge (f,\ Y))$ for each (weak) unitary extension $(f,\ Y)$ of $\mathcal{X}$.

Such a map h *is unique*. Indeed, for each point $y \in Y_1$, there exists a (strict) C-filter $\mathcal{F}$ on $\mathcal{X}$ such that $y = \lim f_1(\mathcal{F})$, and so $h(y) = \lim f_2(\mathcal{F})$. The map h *is surjective* since, for each point $y \in Y_2$, there exists a (strict) C-filter $\mathcal{F}$ on $\mathcal{X}$ such that $y = \lim f_2(\mathcal{F})$, and it implies that $y = h(\lim f_1(\mathcal{F}))$.

Let now Y_1, Y_2 be underlying spaces of Hausdorff topological monoids $\mathcal{Y}_1$, $\mathcal{Y}_2$ such that $(f_1,\ Y_1)$, $(f_2,\ Y_2)$ are (weak) homomorphic unitary extensions of $\mathcal{X}$. The relation $(f_1,\ Y_1) \ge (f_2,\ Y_2)$ implies that the corresponding map h is an identity preserving continuous homomorphism of $\mathcal{Y}_1$ onto $\mathcal{Y}_2$. Indeed, for any $y_1,\ y_2 \in Y_1$, if $\mathcal{F}_1, \mathcal{F}_2$ are (strict) C-filters on $\mathcal{X}$ such that $y_i = \lim f_1(\mathcal{F}_i)$, $i = 1,\ 2$, then $h(y_i) = \lim(h \circ f_1)(\mathcal{F}_i) = \lim f_2(\mathcal{F}_i)$, $i = 1,\ 2$, and $y_1 y_2 = \lim f_1(\mathcal{F}_1\mathcal{F}_2)$. Therefore, $h(y_1 y_2) = \lim(h \circ f_1)(\mathcal{F}_1\mathcal{F}_2) = \lim f_2(\mathcal{F}_1\mathcal{F}_2) = \lim f_2(\mathcal{F}_1) \lim f_2(\mathcal{F}_2) = h(y_1)h(y_2)$. We didn't assume here that $\mathcal{F}_1\mathcal{F}_2$ is a C-filter.

(Weak) unitary extensions $(f_1,\ Y_1)$, $(f_2,\ Y_2)$ are said to be *equivalent* if both the inequalities $(f_1,\ Y_1) \ge (f_2,\ Y_2)$ and $(f_2,\ Y_2) \ge (f_1,\ Y_1)$ hold. In this case, the corresponding maps $h_1\colon\ Y_1 \to Y_2$ and $h_2\colon\ Y_2 \to Y_1$ are canonical reciprocal homeomorphisms. Therefore, equivalent (weak) unitary extensions induce the same extensible topology.

In the case of equivalent (weak) homomorphic unitary extensions of a commutative $\mathcal{X}$, the map h is an isomorphism of monoids $\mathcal{Y}_1$ and $\mathcal{Y}_2$. If a (weak) homomorphic unitary extension (f_1, Y_1) of $\mathcal{X}$ and its (weak) unitary extension (f_2, Y_2) are equivalent, then the second one is homomorphic, too. Indeed, for arbitrary

$y_1, y_2 \in Y_2$, the formula $y_1 \star y_2 = h_1(h_2^{-1}(y_1)h_2^{-1}(y_2))$ defines a multiplication $\star$ in Y_2 having all necessary properties.

The relation ≥ induces an order on the set of equivalence classes. The greatest element of this ordered set is the class of the (weak) finest unitary extension.

Theorem 2.5. Any non-empty subset of the ordered set of equivalence classes of (weak) unitary extensions has a unique least upper bound. Its extensible topology is the least upper bound of the extensible topologies corresponding to classes from this subset. In particular, for each extensible topology τ_0 there exists a greatest class of (weak) unitary extensions with this extensible topology.

If $\mathcal{X}$ is commutative and the considered subset consists of classes of (weak) homomorphic unitary extensions, then its least upper bound is again a class of such extensions.

Proof. This proof resembles the proof of the existence of the Bohr compactification from [3]. A construction of the greatest (weak) unitary extension with a given extensible topology will be given in the following paper of this series.

Our first step is similar to Lemma 2.43 from [3]. In order to show that equivalence classes form a set, we will find a set consisting of (weak) unitary extensions of $\mathcal{X}$ and containing members from all equivalence classes. Let A be an arbitrary set with $|A| = 2^{2^{|X|}}$. Denote by S the set of all (weak) unitary extensions (f, Y) of $\mathcal{X}$ such that Y is a subset of A endowed with some topology. A straightforward cardinality argument shows that S is really a set. If (f', Y') is an arbitrary (weak) unitary extension of $\mathcal{X}$, then there exists an injection j of the set Y' into A. Set $Y = j(Y')$, $f = j \circ f'$ and define a topology on Y as the quotient topology of the given topology of Y'. It is evident that (f, Y) is a (weak) unitary extension which is equivalent to (f', Y').

Let now B be a set consisting of (weak) unitary extensions of $\mathcal{X}$ and containing only members from all equivalence classes belonging to the considered subset of classes. For any $(f, Y) = b \in B$, denote $Y = Y_b$, $f = f_b$ and consider the space $\hat{Y} = \prod_{b \in B} Y_b$. Let $f = \Delta_{b \in B} f_b$ denote the diagonal map of X into $\hat{Y}$. Every f_b is injective, and it implies that f is injective, too.

The projection p_b of $\hat{Y}$ onto Y_b defines a canonical bijection of $f(X)$ onto $f_b(X)$. A subbase of the topology on X induced by the product topology of $\hat{Y}$ consists of sets of the form $f^{-1}(p_b^{-1}(V))$ where b runs B and V runs open subsets of Y_b. A subbase of the least upper bound of extensible topologies on X induced by all extensions (f_b, Y_b) consists of sets of the form $f_b^{-1}(V)$ with the same b and V, i.e. these topologies coincide since $f_b = p_b \circ f$.

For an arbitrary (strict) C-filter $\mathcal{F}$ on $\mathcal{X}$ in its initial topology, denote $x_b = \lim f_b(\mathcal{F})$. Then $x = \{x_b\}_{b \in B} = \lim f(\mathcal{F})$. Set now

$$Y = \{y \in \hat{Y} : \text{ there exists a } C\text{–filter } \mathcal{F} \text{ on } \mathcal{X} \text{ such that } y = \lim f(\mathcal{F})\}.$$

Then (f, Y) is a (weak) unitary extension of $\mathcal{X}$, and $f_b = p_b \circ f$ implies $(f, Y) \geq (f_b, Y_b)$ for any $b \in B$.

Let now (f', Y') be a (weak) unitary extension of $\mathcal{X}$ such that $(f', Y') \geq (f_b, Y_b)$ for any $b \in B$, and h_b is the corresponding continuous map of Y' into Y_b with $h_b \circ f' = f_b$. Denote by h the diagonal map $\Delta_{b \in B} h_b$ of Y' into $\hat{Y}$. Then $h(Y') \subset Y$. Indeed, for any $y' \in Y'$, there exists a (strict) C-filter $\mathcal{F}$ on $\mathcal{X}$ such that $y' = \lim f'(\mathcal{F})$. Therefore, $h(y') = \lim f(\mathcal{F}) \in Y$ since $h\ circf' = f$. The last equality means that $(f', Y') \geq (f, Y)$, and it completes the first part of the proof.

If $\mathcal{X}$ is commutative, then $\hat{Y}$ is the underlying space of the Hausdorff commutative topological monoid $\hat{\mathcal{Y}} = \prod_{b \in B} \mathcal{Y}_b$ and f is a continuous identity preserving homomorphism of $\mathcal{X}$ into this monoid. Denote by $\star$ the standard multiplication in $\hat{\mathcal{Y}}$. Since the product of (strict) C-filters on a commutative Hausdorff monoid is a (strict) C-filter, too, and a continuous identity preserving homomorphism between such monoids takes a (strict) C-filter into a (strict) C-filter, then, for any (strict) C-filters $\mathcal{F}_1, \mathcal{F}_2$ on $\mathcal{X}$, there exists and belongs to Y $\lim f(\mathcal{F}_1) \star \lim f(\mathcal{F}_2) = \lim f(\mathcal{F}_1\mathcal{F}_2)$. Therefore, Y is closed under the multiplication $\star$, and (f, Y) is a (weak) homomorphic unitary extension of $\mathcal{X}$. □

Corollary 2.6. If the set of (weak) unitary extensions of $\mathcal{X}$ with a given extensible topology is non-empty, then there exists its element possessing the universal property over other such extensions. If $\mathcal{X}$ is commutative, then a similar statement is true for (weak) homomorphic unitary extensions.

C) Here, we give a construction of a (weak) precise unitary extension of a monoid which satisfies some additional assumptions. This construction resembles the Wallman extension.

Assume that the initial topology τ of $\mathcal{X}$ is T_3 and all divergent in this topology $\subset$-least (strict) C-filters have bases consisting of closed sets. By Corollary 1.10 from [1], such filters do not cluster. For any of them, the intersection of its closed sets is empty.

Let Y be the set of divergent $\subset$-least (strict) C-filters. Endow the set $X \cup Y$ with the topology whose a base consists of subsets of the form

$$V^{\star} = j(V) \cup \{\mathcal{F} \in Y : \ V \in \mathcal{F}\}$$

where j denotes the canonical embedding $X \to X \cup Y$ and V runs open subsets of (X, τ). It is easy to check that $(V_1 \cap V_2)^{\star} = V_1^{\star} \cap V_2^{\star}$ for any V_1, V_2, and the axioms of a base are satisfied.

Denote the set $X \cup Y$ endowed with this topology by υX. Then j is a topological embedding of (X, τ) into υX. To avoid misunderstandings, we will use the symbol $[\mathcal{F}]$ for the point of υX corresponding to a considered filter $\mathcal{F} \in Y$.

Example 2.7. Let $\mathcal{X} = (Q, +)$ with the usual topology. For any commutative topological group, C-filters are Cauchy filters of the standard uniformity, and $\subset$- least C-filters are minimal ones. They have bases consisting of closed sets. Assign to each point $j(x)$, $x \in Q$, its image under the canonical embedding $Q \to R$ and to each point $[\mathcal{F}] \in Y$ the limit of the image of the filter $\mathcal{F}$ under this embedding. In the latter case, it is an irrational, and the image of $\mathcal{F}$ is generated by intersections of neighborhoods of this irrational with Q. We obtain a standard one-to-one correspondence between the sets υQ and R. Identify them. For any interval $]a, b[\subset R$, if $V =]a, b[\cap Q$, then $V^{\star} =]a, b[$. Therefore, the above topology on υQ coincides with the usual one on R.

Let now $\mathcal{X} = (Q_0^+, +)$ with the usual topology. As above, identify Y with positive irrationals and υX with R_0^+. If b is the irrational corresponding to $[\mathcal{F}] \in Y$, then the image of $\mathcal{F}$ in R_0^+ is generated by intersections of the form $V_\epsilon =]b - \epsilon, b[\cap Q$ with $\epsilon > 0$. Therefore, the above base of υX at an irrational b is generated by semi-intervals $V_\epsilon^{\star} =]b - \epsilon, b]$. This topology is coarser than the topology on R_0^+ from Example 1.12 (ii) of paper [2] but finer than the usual one. These topologies give the same standard extensible topology on Q_0^+.

Return now to the description of the topology of υX. For each closed $Z \subset X$, denote $Z_\star = \upsilon X \setminus (X \setminus Z)^{\star}$. This closed subset of υX consists of $j(Z)$ and of all point $[\mathcal{F}] \in Y$ such that $Z \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. Each closed subset of υX is an intersection of subsets of the form $Z_\star$. The equality $\overline{j(H)} = (\overline{H})_\star$ holds for every subset $H \subset X$.

Proposition 2.8. The map j is a dense embedding.

Proof. It is evident, $j(V) = V^{\star} \cap j(X)$ for every open $V \subset X$. □

Proposition 2.9. υX is a T_1- space. It is Hausdorff if and only if, for any different divergent $\subset$-least (strict) C-filters $\mathcal{F}_1, \mathcal{F}_2$ on $\mathcal{X}$, there exist open $V_1 \in \mathcal{F}_1$, $V_2 \in \mathcal{F}_2$ with the empty intersection. This requirement is satisfied if the initial topology τ is T_4 or if all divergent $\subset$-least (strict) C-filters have bases consisting of open sets.

Proof. Each point of the form $j(x)$ is closed in υX since $\{x\}_\star = j(x)$. Show that any singleton $\{[\mathcal{F}]\} \subset Y$ is closed in υX. It is evident, $[\mathcal{F}] \in Z_\star$ for any closed $Z \in \mathcal{F}$. If $y \in \upsilon X$ belongs to the closure of this singleton, then $y \in Z_\star$ for each closed $Z \in \mathcal{F}$, too. Since it is impossible for points from $j(X)$, then $y \in Y$. Denote by $\mathcal{F}'$ the $\subset$-least C-filter corresponding to y. The intersection $Z \cap H$ is non-empty for any closed $Z \in \mathcal{F}$, $H \in \mathcal{F}'$. Hence, any members $Z \in \mathcal{F}$, $H \in \mathcal{F}'$ have non-empty intersections. Therefore, there exists a filter $\hat{\mathcal{F}}$ on X such that $\mathcal{F}, \mathcal{F}' \subset \hat{\mathcal{F}}$. By Proposition 2.2 from [1], $\hat{\mathcal{F}}$ is a C-filter and $\mathcal{F} \approx \mathcal{F}' \approx \hat{\mathcal{F}}$. Hence, $\mathcal{F} = \mathcal{F}'$ because of the uniqueness of a $\subset$-least C-filter in its equivalence class.

Prove now the second statement. Let $y_1 \in j(X)$ and $y_2 = [\mathcal{F}] \in Y$ be points of υX. Since τ is T_3 and $\mathcal{F}$ does not cluster to $j^{-1}(y_1)$, there exist a neighborhood V_1 of $j^{-1}(y_1)$ and a set $H \in \mathcal{F}$ such that $\overline{V}_1 \cap H = \emptyset$. Set $V_2 = X \setminus \overline{V}_1 \in \mathcal{F}$. Then $V_1^{\star}$ and $V_2^{\star}$ are neighborhoods of y_1 and y_2 with the empty intersection. Observe, moreover, that for any $V_1^{\star}, V_2^{\star}$ belonging to the base above, the set $V_1^{\star} \cap V_2^{\star}$ is empty iff $V_1 \cap V_2$ is empty. It implies immediately that y_1, y_2 have also neighborhoods with the empty intersection in both the cases $y_1, y_2 \in j(X)$ and $y_1, y_2 \in Y$.

Consider the last statement. As above, any different $\mathcal{F}_1, \mathcal{F}_2 \in Y$ have members F_1 and F_2, respectively, with the empty intersection. If these filters have bases consisting of open sets, then we may assume that F_1

and F_2 are open. Anyway, we may assume that these members are closed. If τ is T_4, then there exist open V_1 and V_2 such that $V_1 \supset F_1$, $V_2 \supset F_2$ and $V_1 \cap V_2 = \emptyset$. It completes the proof. □

Example 2.10. Let $\mathcal{X}$ be $(R_0^+, +)$ with the topology of the Sorgenfrey line (see Example 1.1). It is an example of a monoid which does not have unitary completions (see Example 3.1 below). Assign to each point $x \in X = R_0^+$ the corresponding point $(x, 0)$ of the real plane. The set Y is formed by filters $\mathcal{F}_x$ (see Example 1.1) with $x \in R^+$. Assign to $\mathcal{F}_x$ the point $(x, 1)$ of this plane. Thus, the set υX is the pair of half-lines one of which is open.

If V runs the base $\{[x_0, x_0+\epsilon[\}_{\epsilon>0}$ at the point $x_0 \in X$, then $V^\star$ runs the family of sets $V_\epsilon^\star = \{(x, 0) : x_0 \le x < x_0+\epsilon\} \cup \{(x, 1) : x_0 < x \le x_0+\epsilon\}$. The set $V_\epsilon^\star \setminus \{(x_0+\epsilon, 1)\}$ is open in υX, too, and such sets form a base of the space υX at the point $(x_0, 0)$. If V runs the base $\{]x_0-\epsilon, x_0[\}_{\epsilon>0}$ of the filter of open sets from $\mathcal{F}_{x_0}$, then $V^\star$ runs the family of sets $\{(x, 0) : x_0-\epsilon < x < x_0\} \cup \{(x, 1) : x_0-\epsilon < x \le x_0\}$. It is a base of the space υX at the point $(x_0, 1)$. Hence, the space υX is homeomorphic to the space "two arrows" (see [4]) without one point since one usually uses the interval $[0; 1[$ as the first arrow, and then the second arrow is the interval $]0; 1]$ instead of R^+ in our case. This space was defined by P. S. Alexandrov and P. S. Uryson as a container of the limits of all fundamental sequences in R_0^+ endowed with the topology of the Sorgenfrey line.

As it was already mentioned, the addition in $\mathcal{X}$ defines an addition of C-filters (see [2], section 2A) and generates a commutative and associative addition in υX. It can be given by the formulas $(a, 0)+(b, 0) = (a+b, 0)$, $(a, 0)+(b, 1) = (a, 1)+(b, 0) = (a+b, 1)$, $(a, 1)+(b, 1) = (a+b, 1)$. This addition is not continuous. Indeed, let $\{a_n\}$ be an increasing sequence of reals convergent to some $a > 0$ and $\{b_n\}$ a decreasing sequence of reals convergent to some $b > 0$. Then the sequence $\{(a_n, 0)\}$ from υX converges to $(a, 1)$, and the sequence $\{(b_n, 0)\}$ converges to $(b, 0)$. However, if the sequence $\{a_n+b_n\}$ decreases, then the sequence $\{(a_n+b_n, 0)\}$ converges to $(a+b, 0) \ne (a, 1)+(b, 0)$.

Now, we are going to show that the couple $(j, \upsilon X)$ is a (weak) precise unitary extension of the monoid $\mathcal{X}$. We need to prove that the limit of any filter of the form $j(\mathcal{F})$ where $\mathcal{F}$ is a (strict) C-filter on $\mathcal{X}$, consists of an only point. For an arbitrary $x \in X$, it is straightforward to show that $j(\mathcal{F})$ clusters to $j(x)$ iff $\mathcal{F}$ clusters to x. It is only possible if $\mathcal{F}$ converges to x. Similarly, $x = \lim \mathcal{F}$ is equivalent to $j(x) \in \lim j(\mathcal{F})$.

Proposition 2.11. As above, let $\mathcal{F}_{\mathrm{lst}}$ denote the $\subset$-least (strict) C-filter corresponding to a given (strict) C-filter $\mathcal{F}$. Then:

(i) if $\mathcal{F}$ converges to a point x, then $\lim j(\mathcal{F}) = j(x)$,

(ii) if $\mathcal{F}$ diverges, then $\lim j(\mathcal{F}) = [\mathcal{F}_{\mathrm{lst}}]$.

Proof. (i) Since $\mathcal{F}$ converges to x and does not have other cluster points, then $j(x) \in \lim j(\mathcal{F})$ and $j(\mathcal{F})$ cannot have other cluster points in $j(X)$. Show that $j(\mathcal{F})$ cannot converge to any point in Y. Let $[\mathcal{G}]$ be such a point and $V \in \mathcal{G}$ an arbitrary open set. Then $V^\star$ belongs to $j(\mathcal{F})$. Hence, V belongs to $\mathcal{F}$ since a base of $j(\mathcal{F})$ lies in $j(X)$ and $V^\star \cap j(X) = j(V)$. That is why, it intersects any member of $\mathcal{F}$. However, x is not a cluster point of $\mathcal{G}$, and there exist a neighborhood U of x and a set $G \in \mathcal{G}$ such that $G \cap \overline{U} = \emptyset$. Therefore, $V = X \setminus \overline{U}$ is a member of $\mathcal{G}$ which does not intersect U.

ii) In this case, $[\mathcal{F}_{\mathrm{lst}}] \in Y$. Let $V^\star$ be its arbitrary neighborhood from the base above. Then $V \in \mathcal{F}_{\mathrm{lst}}$ implies $V \in \mathcal{F}$ and $j(V) \in j(\mathcal{F})$. Therefore, $V^\star \in j(\mathcal{F})$ and $[\mathcal{F}_{\mathrm{lst}}] \in \lim j(\mathcal{F})$.

Let now $y \in \upsilon X$ be an arbitrary cluster point of $j(\mathcal{F})$. Then $y \in Y$ since $j(\mathcal{F})$ does not have any cluster point in $j(X)$ by Corollary 1.10 from [1]. Denote $y = [\mathcal{G}]$. Any closed set from $j(\mathcal{F})$ contains y. Let $Z \in \mathcal{F}_{\mathrm{lst}}$ be closed in X. $Z \in \mathcal{F}$ implies $j(Z) \in j(\mathcal{F})$. Then $Z_\star$ is a closed set from $j(\mathcal{F})$, and $Z_\star$ contains y. Therefore, the intersection $Z \cap H$ is non-empty for any closed $Z \in \mathcal{F}_{\mathrm{lst}}$, $H \in \mathcal{G}$. It means that $\mathcal{F}_{\mathrm{lst}} = \mathcal{G}$, and the proof is complete. □

Theorem 2.12. Let $\mathcal{X}$ be a topological monoid on a T_3-underlying space such that all its divergent $\subset$-least (strict) C-filters have bases consisting of closed sets. Then the couple $(j, \upsilon X)$ is a (weak) precise unitary extension of $\mathcal{X}$.

Remark. Examples show that our assumptions cannot be omitted.

Proof. The statement follows from Propositions 2.8 and 2.11. □

We prove now that the inclusion $X \to \upsilon X$ is universal over continuous (weak) unitarily quasi-extending maps of X into T_3-topological spaces.

Proposition 2.13. For any continuous (weak) unitarily quasi-extending map f of X into a T_3-topological space K, there exists a unique continuous map $\hat{f}: \upsilon X \to K$ such that $\hat{f} \circ j = f$. In particular, each continuous homomorphism of $\mathcal{X}$ into a locally compact monoid can be uniquely continuously extended to υX.

Proof. For an arbitrary point $y = [\mathcal{F}] \in Y$, $f(\mathcal{F})$ is a convergent filter on K, and we set $\hat{f}(y) = \lim f(\mathcal{F})$. If $y = j(x)$, then we set $\hat{f}(y) = f(x)$. It is the only possible continuous extension of f. Prove that it is really continuous. For any $y \in \upsilon X$, let U be an arbitrary neighborhood of the point $\hat{f}(y)$ and V its neighborhood such that $\overline{V} \subset U$. The open subset $W = f^{-1}(V)^{\star}$ is a neighborhood of y. Indeed, if $y = j(x)$, then $f(x) \in V$ implies $y \in f^{-1}(V)^{\star}$. For $y = [\mathcal{F}] \in Y$, $\hat{f}(y) \in V$ implies $V \in f(\mathcal{F})$ and $f^{-1}(V) \in \mathcal{F}$. Therefore, $y \in f^{-1}(V)^{\star}$ holds again. Show now that $\hat{f}(W) \subset U$. For any point $y' \in W \cap j(X)$, the inclusion $\hat{f}(y') \in U$ follows from $j^{-1}(y') \in f^{-1}(V)$. If $y' = [\mathcal{F}'] \in Y$, then $y' \in f^{-1}(V)^{\star}$ implies $f^{-1}(V) \in \mathcal{F}'$ and $V \in f(\mathcal{F}')$. Hence, $\hat{f}(y') = \lim f(\mathcal{F}') \in \overline{V} \subset U$. □

3 The concept of a unitary completion

A) Here, we consider (weak) precise homomorphic unitary extensions. Not each unitarily separable monoid $\mathcal{X}$ has them. Indeed, let $(f, \mathcal{Y})$ be such an extension of $\mathcal{X}$. By Proposition 2.1 from [2], the filter $f(\mathcal{F})$ is a C-filter on $\mathcal{Y}$ for any C-filter $\mathcal{F}$ on $\mathcal{X}$. Moreover, the condition $\mathcal{F}_1 \geq \mathcal{F}_2$ implies $f(\mathcal{F}_1) \geq f(\mathcal{F}_2)$ for such filters. By Proposition 1.9 from [1], it follows from this inequality that $\lim f(\mathcal{F}_1) = \lim f(\mathcal{F}_2)$. However, it is impossible if only one of filters $\mathcal{F}_1$, $\mathcal{F}_2$ converges in X, since, in this case, only one of the points $\lim f(\mathcal{F}_1)$, $\lim f(\mathcal{F}_2)$ belongs to $f(X)$.

Example 3.1. Let $\mathcal{X}$ be $(R_0^+, +)$ with the topology of the Sorgenfrey line. By Example 1.1, there exist couples $\mathcal{F}_1$, $\mathcal{F}_2$ with the above property, and that is why $\mathcal{X}$ does not have any precise homomorphic unitary extensions. However, we have already mentioned that it has a precise one.

We assume now that *the considered Hausdorff topological monoid $\mathcal{X}$ is commutative and unitarily separable*, and find conditions of the existence of its precise homomorphic unitary extension.

Let $(f, \mathcal{Y})$ be such an extension. $\mathcal{Y}$ is commutative since $f(X)$ is dense in its underlying space Y. For each C-filter $\mathcal{F}$ on $\mathcal{X}$, denote by $\mathcal{O}_{\mathcal{F}}$ the family of preimages in X of all neighborhoods of the point $\lim f(\mathcal{F})$. By Proposition 2.2, the filters $f(\mathcal{F}_1)$, $f(\mathcal{F}_2)$ have equal limits if $\mathcal{F}_1$, $\mathcal{F}_2$ are equivalent C-filters on $\mathcal{X}$. Therefore, such families coincide for equivalent C-filters, and, instead of $\{\mathcal{F}\}$, we can use the set of $\subset$-least C-filters or (that is equivalent) the set of points of the finest unitary extension $\tilde{X}$ as the set of indexes of the family $\{\mathcal{O}_{\mathcal{F}}\}$ of families $\mathcal{O}_{\mathcal{F}}$.

It is evident that these families possess the following properties:

(i) For each $\subset$-least C-filter $\mathcal{F}$, the family $\mathcal{O}_{\mathcal{F}}$ is a filter in the set of open subsets of X, and $\mathcal{O}_{\mathcal{F}} \subset \mathcal{F}$.

(ii) For each convergent in (X, τ) $\subset$-least C-filter $\mathcal{F}$ (in particular, for each filter of the form $\dot{x}_{\text{lst}}$), $\mathcal{O}_{\mathcal{F}}$ is the family of neighborhoods of the point $\lim \mathcal{F}$ ($= x$ for $\mathcal{F} = \dot{x}_{\text{lst}}$) in this space.

(iii) For any $\subset$-least C-filters $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ with $\mathcal{F} \approx \mathcal{F}_1\mathcal{F}_2$ and for an arbitrary $W \in \mathcal{O}_{\mathcal{F}}$, there exist $V_1 \in \mathcal{O}_{\mathcal{F}_1}$, $V_2 \in \mathcal{O}_{\mathcal{F}_2}$ such that $V_1 \in \mathcal{O}_{\mathcal{G}_1}$, $V_2 \in \mathcal{O}_{\mathcal{G}_2}$ imply $W \in \mathcal{O}_{\mathcal{G}}$ for any $\subset$-least C-filters $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ with $\mathcal{G} \approx \mathcal{G}_1\mathcal{G}_2$.

This property follows from the continuity of the multiplication in $\mathcal{Y}$ and involves the next one:

(iii') For any $\subset$-least C-filter $\mathcal{F}$ and for an arbitrary $W \in \mathcal{O}_{\mathcal{F}}$, there exist $V \in \mathcal{O}_{\mathcal{F}}$ and a neighborhood U of 1 such that $VU \subset W$.

(iv) For any $\subset$-least C-filters $\mathcal{F}_1, \mathcal{F}_2$, $\mathcal{O}_{\mathcal{F}_1} \neq \mathcal{O}_{\mathcal{F}_2}$ implies that there exist $V_1 \in \mathcal{O}_{\mathcal{F}_1}$ and $V_2 \in \mathcal{O}_{\mathcal{F}_2}$ such that $V_1 \cap V_2 = \emptyset$.

It follows from our assumption that $\mathcal{Y}$ is Hausdorff.

To formulate property (v), remind that, in the considered commutative case, $\tilde{X}$ is an abstract commutative monoid whose multiplication • corresponds to the multiplication of C-filters (for details, see Proposition 2.9 from [2]).

(v) The relation $E = \{([\mathcal{F}_1], [\mathcal{F}_2]) \in \tilde{X} \times \tilde{X} : \mathcal{O}_{\mathcal{F}_1} = \mathcal{O}_{\mathcal{F}_2}\}$ is a congruence in $(\tilde{X}, \bullet)$.

It follows from the same assumption.

Theorem 3.2. Let $\mathcal{X}$ be a commutative unitarily separable Hausdorff topological monoid. It has a precise homomorphic unitary extension if and only if there exists a family of non-empty families $\{\mathcal{O}_{\mathcal{F}}\}_{[\mathcal{F}] \in \tilde{X}}$ possessing properties (i) - (v) above.

Proof. The *necessity* has already been proved.

For the proof of the *sufficiency*, we give a construction of this extension. Let such a family $\{\mathcal{O}_{\mathcal{F}}\}$ exist for any $\subset$-least C-filter $\mathcal{F}$. As the first step, we introduce another topology on $\tilde{X}$. For any open in the initial topology τ on X set V, denote

$$V^{\star} = \{[\mathcal{F}] \in \tilde{X}: \ V \in \mathcal{O}_{\mathcal{F}}\}.$$

It is easy to verify that $(V_1 \cap V_2)^{\star} = V_1^{\star} \cap V_2^{\star}$. Therefore, such sets $V^{\star}$ form a base of a topology on $\tilde{X}$. We denote it by τ_c.

As it follows from property (iii) of the family $\{\mathcal{O}_{\mathcal{F}}\}$, the multiplication $\bullet$ on $\tilde{X}$ is continuous in the topology τ_c. By condition (ii), for any $x \in X$, the inclusions $i(x) = [\dot{x}_{\text{lst}}] \in V^{\star}$ and $x \in V$ follow from each other, i.e. the preimage of τ_c under i coincides with the initial topology τ of $\mathcal{X}$. It will follow from this fact that the extension which we construct, is precise.

Show now that the topology τ_c is coarser than the natural topology or coincides with it. It involves, in particular, that i is a dense embedding of X into $\tilde{X}$ endowed with the topology τ_c. Let $\tilde{x} \in \tilde{X}$ be an arbitrary point and $V^{\star}$ its neighborhood in the topology τ_c from the base above. First, assume that there exists $x \in X$ such that $\tilde{x} = i(x)$. Then $\tilde{x} \in i(V) \subset V^{\star}$. Since $\mathcal{X}$ is unitarily separable, its initial topology τ is coarser than the unitary topology on X (see Theorem 3.2 from [1]). Therefore, the set V is open in the unitary topology, too. The map i is a homeomorphism of X endowed with the unitary topology onto an open subspace $i(X)$ of $\tilde{X}$ endowed with the natural topology (see Theorem 1.9 from [2]). Hence, $i(V)$ is a neighborhood of $\tilde{x}$ in the natural topology lying in $V^{\star}$. Consider the second case. If $\tilde{x} = [\mathcal{F}] \in \tilde{X} \setminus i(X)$, then $V \in \mathcal{F}$. By Lemma 1.10 from [2], the set $\{[\mathcal{F}]\} \cup i(V)$ is a neighborhood of the point $[\mathcal{F}]$ in the natural topology on $\tilde{X}$. It is evident that this neighborhood lies in $V^{\star}$.

Denote by Y the quotient space $(\tilde{X}, \ \tau_c)/E$ and by p the corresponding quotient map. All the sets $V^{\star}$ are saturated by the relation E. That is why the map p is open and closed, and sets $p(V^{\star})$ with $V^{\star} \ \in \tau_c$ form a base of the quotient topology τ_Y on Y. This topology is Hausdorff by property (iv). By property (v), the multiplication $\bullet$ induces a multiplication in Y which we denote by $\star$. It is easy to verify that it is continuous in the topology τ_Y, and $\mathcal{Y} = (Y, \star, \tau_Y)$ is a commutative Hausdorff topological monoid.

Define now a map $f\colon \ X \to Y$ as the composition $p \circ i$. It follows from Proposition 2.9 from [2] that f is algebraically an identity preserving homomorphism of (X, m) into $(Y, \star)$. Moreover, f is injective since, for any $x, y \in X$, $i(x)Ei(y)$ is only possible if $x = y$. It is evident that $f^{-1}(p(V^{\star})) = i^{-1}(V^{\star}) = V$ for any open $V \subset X$. Therefore, f is a dense algebraic and topological embedding of $\mathcal{X}$ into $\mathcal{Y}$.

For an arbitrary C-filter $\mathcal{F}$ on $\mathcal{X}$, the filter $i(\mathcal{F})$ converges to the point $[\mathcal{F}_{\text{lst}}]$ in the natural topology on $\tilde{X}$ and, hence, in the topology τ_c. Therefore, the filter $f(\mathcal{F})$ converges to $p([\mathcal{F}_{lst}])$ in the topology τ_Y. It also involves that each point from Y is the limit of a filter of the form $f(\mathcal{F})$ where $\mathcal{F}$ is a C-filter on $\mathcal{X}$. It completes the proof. □

Remark 3.3. In the following, we will also use the next property of the topology τ_Y: if the topology τ has a denumerable base at the point $1_{\mathcal{X}}$, then there exists a denumerable base of the topology τ_Y at the point $1_{\mathcal{Y}}$. Indeed, for any open $V_1, \ V_2 \subset X$, $V_1 \subset V_2$ implies $V_1^{\star} \subset V_2^{\star}$. Therefore, if the point $1_{\mathcal{X}}$ possesses a denumerable base of neighborhoods in the topology τ, then there exists a denumerable base of neighborhoods of the point $i(1_{\mathcal{X}})$ in the topology τ_c. The images of elements of this base under p form the required base at the point $1_{\mathcal{Y}}$.

Example 3.4. Show that, in classical cases, the construction above leads to classical results.

(i) Let $\mathcal{X} = (Q_0^+, \ +)$ with the usual topology. Its $\subset$-least C-filters are either trivial ultrafilters or correspond to classes of equivalent (in the usual sense) increasing fundamental sequences. For any convergent C-filter $\mathcal{F}$, denote by $\mathcal{O}_{\mathcal{F}}$ the family of neighborhoods of its limit point. If $\mathcal{F}$ does not converge, then $\mathcal{O}_{\mathcal{F}}$ is the filter of open sets which is generated by intervals belonging to $\mathcal{F}$ and having rational endpoints. We need the last condition to guarantee that $\mathcal{O}_{\mathcal{F}}$ does not contain intervals of the form $]x - \epsilon; \ x[$ if $\mathcal{F}$ defines an irrational x. Otherwise, it would not have property (iii') above. For this choice of $\mathcal{O}_{\mathcal{F}}$, $\mathcal{Y}$ is algebraically and topologically isomorphic to $(R_0^+, \ +)$ with the usual topology.

(ii) Let $\mathcal{X}$ be an abelian topological group. Then C-filters on $\mathcal{X}$ are Cauchy filters of its uniformity, and their classes corresponds to points of the completion Y of X endowed with this uniformity (see [4]). For any C-filter

$\mathcal{F}$, if its image in Y converges to $y \in Y$, then denote by $\mathcal{O}_{\mathcal{F}}$ the family of preimages in X of neighborhoods of y in Y. One can prove that $\mathcal{O}_{\mathcal{F}}$ consists of all open member U from $\mathcal{F}$ such that $U \supset FV$ for some $F \in \mathcal{F}$ and some neighborhood V of $1_{\mathcal{X}}$. Then our construction applied to this family $\mathcal{O}_{\mathcal{F}}$ gives the same Y endowed with a structure of a topological monoid.

Consider now weak precise homomorphic unitary extensions. Let $(f, \mathcal{Y})$ be such an extension of a commutative Hausdorff monoid $\mathcal{X}$. As above, for each strict C-filter $\mathcal{F}$ on $\mathcal{X}$, denote by $\mathcal{O}_{\mathcal{F}}$ the family of preimages in X of all neighborhoods of the point $\lim f(\mathcal{F})$. As it was mentioned above, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent in Σ, then $\mathcal{O}_{\mathcal{F}_1} = \mathcal{O}_{\mathcal{F}_2}$. Therefore, we can assume that the family of families $\mathcal{O}_{\mathcal{F}}$ is indexed by elements of the subset $\tilde{X}^w$ from $\tilde{X}$ consisting of classes of C-filters containing strict ones. It is possible that this subset is proper, but it is always a submonoid of $(\tilde{X}, \bullet)$ containing $i(X)$ since the product of strict C-filters is again a strict C-filter. Thus, for each $\tilde{x} \in \tilde{X}^w$, $\mathcal{O}_{\tilde{x}} = \mathcal{O}_{\mathcal{F}}$ for an arbitrary strict C-filter $\mathcal{F}$ belonging to the equivalence class from Σ corresponding to $\tilde{x}$. Properties of the family $\{\mathcal{O}_{\tilde{x}}\}_{\tilde{x} \in \tilde{X}^w}$ are similar to properties (i) - (v) above, and we omit their wordings.

A commutative Hausdorff topological monoid $\mathcal{X}$ has a weak precise homomorphic unitary extension if and only if there exists a family of non-empty families $\{\mathcal{O}_{\tilde{x}}\}_{\tilde{x} \in \tilde{X}^w}$ with these properties.

The proof is similar to the proof of Theorem 3.2 above. The only difference is that it is necessary to use the space $\tilde{X}^w$ instead of the space $\tilde{X}$ for the construction of this extension. The required monoid $\mathcal{Y}^w$ is the quotient-monoid $(\tilde{X}^w, \bullet)/E^s$ where E^s is the congruence $\{(\tilde{x}_1, \tilde{x}_2) \in \tilde{X}^w \times \tilde{X}^w : \mathcal{O}_{\tilde{x}_1} = \mathcal{O}_{\tilde{x}_2}\}$.

B) We define the concept of a (weak) unitary completion here. First of all, remind that a Hausdorff topological monoid is said to be *(weakly) unitarily complete* if all its (strict) C-filters converge (see Definition 1.11 from [1]).

Definition 3.5. Let $\mathcal{X}$ be a Hausdorff monoid, $\mathcal{Y}$ a (weakly) unitarily complete monoid, and $f: \mathcal{X} \to \mathcal{Y}$ an algebraic and topological embedding. The couple $(f, \mathcal{Y})$ is called *a (weak) unitary completion* of $\mathcal{X}$ if $\mathcal{Y}$ properly contains no (weakly) unitarily complete submonoid containing $f(\mathcal{X})$.

Any closed submonoid of a (weakly) unitarily complete monoid is (weakly) unitarily complete, since its embedding takes any C-filter into a C-filter and any strict C-filter into a strict one. Therefore, the map f above is dense.

Moreover, each filter of the form $f(\mathcal{F})$ where $\mathcal{F}$ is a (strict) C-filter on $\mathcal{X}$, converges, and it means that the map f and the set of all points of the form $\lim f(\mathcal{F})$ where $\mathcal{F}$ runs (strict) C-filters on $\mathcal{X}$, form a (weak) precise unitary extension of $\mathcal{X}$; it is homomorphic if $\mathcal{X}$ is commutative. It implies, in particular, that, by Proposition 2.2 or Corollary 1.10 from [1], filters $f(\mathcal{F}_1)$, $f(\mathcal{F}_2)$ have equal limits if (strict) C-filters $\mathcal{F}_1$, $\mathcal{F}_2$ are equivalent in Σ.

It is of interest that though the monoids $(R_0^+, +)$ with the usual topology and with the topology of the Sorgenfrey line (see Example 1.1) have the same finest unitary extension (see Example 1.12 from [2]), but the first of them is unitarily complete while the second one does not have any unitary completions (see Example 3.1 above).

Proposition 3.6. If there exists an algebraic and topological embedding of a given Hausdorff topological monoid into a (weakly) unitarily complete one (in particular, into a locally compact one, see Proposition 1.12 from [1]), then there exists its (weak) unitary completion.

Proof. Let $\mathcal{K}$ be a unitarily complete monoid and f an algebraic and topological embedding of a given monoid $\mathcal{X}$ into $\mathcal{K}$. It is evident, there exists the least with respect to inclusion (weakly) unitarily complete submonoid of $\mathcal{K}$ containing $f(\mathcal{X})$. The couple consisting of f and of this submonoid is a (weak) unitary completion of $\mathcal{X}$. □

Any submonoid of a monoid having a (weak) unitary completion, and the product of a family of such monoids have (weak) unitary completions. For the product, it follows from the fact that the product of (weakly) unitarily complete monoids is (weakly) unitarily complete since images of any its (strict) C-filter by projections onto factors are (strict) C-filters on these factors and converge.

C) We turn now to conditions of the existence of a (weak) unitary completion of a commutative monoid.

Theorem 3.7. Let a commutative unitarily separable Hausdorff monoid $\mathcal{X}$ possess a denumerable base of neighborhoods of its identity $1_{\mathcal{X}}$ and a family $\{\mathcal{O}_{\tilde{x}}\}_{\tilde{x} \in \tilde{X}}$ of filters of open subsets satisfying conditions (i) - (v) from section 3.A). Suppose that the underlying space Y of the precise homomorphic unitary extension $(f, \mathcal{Y})$

constructed for these $\mathcal{X}$ and $\{\mathcal{O}_{\tilde{x}}\}_{\tilde{x}\in\tilde{X}}$ in the proof of Theorem 3.2, is a T_3-space. Then all strict C-filters on $\mathcal{Y}$ converge.

Proof. In this proof, it is more convenient to consider strict C-nets in $\mathcal{Y}$ instead of strict C-filters. Since $\mathcal{X}$ and $\mathcal{Y}$ are commutative, we will use the formula $x_{\alpha'} \in x_\alpha U$ in the definition of a strict C-net (see section 1.A)).

Let $T = \{y_\alpha\}_{\alpha\in A}$ be such a net in Y. To prove its convergence, we will construct a C-net $S = \{x_\beta\}_{\beta\in B}$ in X such that $f(S) \geq T$. The net $f(S)$ converges since (f, Y) is a unitary extension of $\mathcal{X}$, and, therefore, T converges by Corollary 1.10 from [1]. To find S, we will use C-filters $\mathcal{F}_\alpha$ on $\mathcal{X}$ such that $\lim f(\mathcal{F}_\alpha) = y_\alpha$ for each $\alpha \in A$. It was proved in Theorem 3.2 that they exist. The chosen filters will form a sequence where each member can be obtained from its preceding one by means of a suitable translation (see Proposition 2.3 from [2]). But it will be sometimes necessary to use translations by elements from $Y \setminus f(X)$, and we start with a definition of such a translation.

Let $\mathcal{F}$ be a C-filter on $\mathcal{X}$, $u \in Y$ and $R = \{u_\gamma\}_{\gamma\in G}$ a C-net in $\mathcal{X}$ such that $u = \lim f(R)$. Denote by $\mathcal{F}_\gamma$ the image of $\mathcal{F}$ by the translation $x \to xu_\gamma$ of X. It is a C-filter by Proposition 2.3 from [2]. If $\lim f(\mathcal{F}) = y$, then $\lim f(\mathcal{F}_\gamma) = yf(u_\gamma)$. Set now

$$\mathcal{F}_R = \{F \subset X | \,\exists\, \gamma_0 \in G : \ F \in \mathcal{F}_\gamma \ \forall\, \gamma \geq \gamma_0\}.$$

It is evident that $\mathcal{F}_R$ is a filter. To show that it is a C-filter, find the set $M_V(\mathcal{F}_R)$ for an arbitrary neighborhood V of the identity $1_{\mathcal{X}}$. Observe that the following properties are equivalent for each $x \in X$: (i) $x \in M_V(\mathcal{F}_R)$, (ii) $\overline{xV} \in \mathcal{F}_R$, (iii) there exists $\gamma_0 \in G$ such that $\overline{xV} \in \mathcal{F}_\gamma$ for all $\gamma \geq \gamma_0$, (iv) there exists $\gamma_0 \in G$ such that $x \in M_V(\mathcal{F}_\gamma)$ for all $\gamma \geq \gamma_0$, (v) there exists $\gamma_0 \in G$ such that $x \in \bigcap_{\gamma\geq\gamma_0} M_V(\mathcal{F}_\gamma)$, (vi) $x \in \bigcup_{\gamma_0\in G}\bigcap_{\gamma\geq\gamma_0} M_V(\mathcal{F}_\gamma)$. Hence,

$$M_V(\mathcal{F}_R) = \bigcup_{\gamma_0\in G}\bigcap_{\gamma\geq\gamma_0} M_V(\mathcal{F}_\gamma).$$

Verify now that $M_V(\mathcal{F}_R) \in \mathcal{F}_R$, i.e. $\mathcal{F}_R$ is a C-filter. Let W be a neighborhood of $1_{\mathcal{X}}$ such that $W^2 \subset V$. There exists $\gamma_0 \in G$ such that, for each $\gamma \geq \gamma_0$, there exists γ_0' such that $u_{\gamma'} \in \overline{u_\gamma W}$ for any $\gamma' \geq \gamma_0'$. If $\gamma \geq \gamma_0$ and $x \in M_W(\mathcal{F}_\gamma)$, then $\overline{xW} \in \mathcal{F}_\gamma$, and there exists $F \in \mathcal{F}$ such that $\overline{xW} \supset u_\gamma F$. Therefore, $u_\gamma WF \subset \overline{xV}$. It implies $\overline{u_{\gamma'}F} \subset \overline{xV}$, $u_{\gamma'}F \subset \overline{xV}$ and $\overline{xV} \in \mathcal{F}_{\gamma'}$, i.e. $x \in M_V(\mathcal{F}_{\gamma'})$ for any $\gamma' \geq \gamma_0'$. Thus, $M_W(\mathcal{F}_\gamma) \subset M_V(\mathcal{F}_{\gamma'})$ holds for such γ and γ'. That is why $M_W(\mathcal{F}_\gamma) \subset \bigcap_{\gamma'\geq\gamma_0'} M_V(\mathcal{F}_{\gamma'}) \subset M_V(\mathcal{F}_R)$ for all $\gamma \geq \gamma_0$, and it follows now from $M_W(\mathcal{F}_\gamma) \in \mathcal{F}_\gamma$ that $M_V(\mathcal{F}_R) \in \mathcal{F}_R$.

Show that $\lim f(\mathcal{F}_R) = yu$ where $y = \lim f(\mathcal{F})$. Let U be any neighborhood of yu. Then $\lim f(\mathcal{F}_\gamma) = yf(u_\gamma)$ lies eventually in U, and $f^{-1}(U)$ belongs eventually to $\mathcal{F}_\gamma$. Therefore, $f^{-1}(U) \in \mathcal{F}_R$ and $U \in f(\mathcal{F}_R)$.

In the following, we use also the next property of $\mathcal{F}_R$. Let U be a neighborhood of the identity $1_{\mathcal{Y}}$ of $\mathcal{Y}$ containing u, $V = f^{-1}(U)$ and W a neighborhood of $1_{\mathcal{X}}$ such that $V^2 \subset W$. Then $M_V(\mathcal{F}) \subset M_W(\mathcal{F}_R)$. Indeed, u_γ lies eventually in V, and $x \in M_V(\mathcal{F})$ implies $\overline{xV} \in \mathcal{F}$, $u_\gamma\overline{xV} \in \mathcal{F}_\gamma$, and $\overline{xW} \in \mathcal{F}_\gamma$. Therefore, $\overline{xW} \in \mathcal{F}_R$ and $x \in M_W(\mathcal{F}_R)$.

We choose now the sequence of filters. As the first step, fix a base $\{U_n\}_{n\in N}$ of the topology τ_Y at the point $1_{\mathcal{Y}}$ so that $U_{n+1}^2 \subset U_n$ for all n. Such a base exists by Remark 3.3 above. Denote $V_n = f^{-1}(U_n)$. The family $\{V_n\}_{n\in N}$ forms a base at the point $1_{\mathcal{X}}$, and $U_n \cap f(X) = f(V_n)$.

Since T is a strict C-net on $\mathcal{Y}$, then, for each $n \in N$, there exists $\alpha_0(n) \in A$ such that for any $\alpha \geq \alpha_0(n)$ there exists $\alpha_0'(n, \alpha)$ such that $y_{\alpha'} \in y_\alpha U_n$ for all $\alpha' \geq \alpha_0'(n, \alpha)$. Choose a subsequence $\{\alpha_n\}_{n\in N}$ of A as follows. Fix an arbitrary $\alpha_1 \geq \alpha_0(1)$ and if α_n, $n \in N$, $\alpha_n \geq \alpha_0(n)$, is already chosen, take as α_{n+1} any element $\alpha \in A$ with $\alpha \geq \alpha_n$, $\alpha \geq \alpha_0'(n, \alpha_n)$, and $\alpha \geq \alpha_0(n+1)$. Then $y_{\alpha_k} \in y_{\alpha_n}U_n$ for all natural n, k with $k \geq n$. In particular, for any $n \in N$, there exists $u_n \in U_n$ such that $y_{\alpha_{n+1}} = y_{\alpha_n}u_n$.

Take now an arbitrary C-filter $\mathcal{F}_1$ on $\mathcal{X}$ with $\lim f(\mathcal{F}_1) = y_{\alpha_1}$. If the filter $\mathcal{F}_n$, $n \in N$, is already chosen, denote by $\mathcal{F}_{n+1}$ the result of its translation to u_n, i.e. the filter $(\mathcal{F}_n)_{R_n}$ for some C-net R_n such that $\lim f(R_n) = u_n$. By the first part of the proof, $\mathcal{F}_n$ is a C-filter on $\mathcal{X}$ and $\lim f(\mathcal{F}_n) = y_{\alpha_n}$ for all $n \in N$. Moreover, it follows from $u_n \in U_n$, $V_n = f^{-1}(U_n)$, and $V_n^2 \subset V_{n-1}$ that $M_{V_n}(\mathcal{F}_n) \subset M_{V_{n-1}}(\mathcal{F}_{n+1})$ for all $n \geq 2$.

We can now construct the net S. It is a sequence. As x_1, we take an arbitrary element from $M_{V_2}(\mathcal{F}_2)$. Let x_n, $n \geq 1$, be chosen so that it belongs to $M_{V_{n+1}}(\mathcal{F}_{n+1}) \subset M_{V_n}(\mathcal{F}_{n+2})$. Then x_{n+1} is an arbitrary element from $\overline{x_nV_n} \cap M_{V_{n+2}}(\mathcal{F}_{n+2})$. The intersection of these sets is non-empty since they belong to $\mathcal{F}_{n+2}$. Verify that S is a C-net. Indeed, for any n, $k \in N$, we have $x_{n+k} \in \overline{x_nV_n \cdot \cdots \cdot V_{n+k-1}} \subset \overline{x_nV_{n-1}}$.

It remains to show that $f(S) \geq T$. It follows from the definition of S that $\overline{x_n V_{n+1}} \in \mathcal{F}_{n+1}$. Therefore, $\overline{f(x_n)U_{n+1}} \in f(\mathcal{F}_{n+1})$ and $y_{\alpha_{n+1}} \in \overline{f(x_n)U_{n+1}}$. Then $y_\alpha \in \overline{f(x_n)U_n}$ if $\alpha \geq \alpha_0'(n+1, \alpha_{n+1})$, since $y_\alpha \in y_{\alpha_{n+1}} U_{n+1}$ for these α. It completes the proof. □

For strict C-filters, the next statement is true.

Theorem 3.8. Suppose that a commutative unitarily separable Hausdorff monoid $\mathcal{X}$ possesses a denumerable base of neighborhoods of its identity and there exists such as above family $\{\mathcal{O}_{\tilde{x}}\}_{\tilde{x} \in \tilde{X}^w}$. If the underlying space of its constructed above weak precise homomorphic unitary extension $\mathcal{Y}^w$ is a T_3 - space, then $\mathcal{Y}^w$ is a weak unitary completion of $\mathcal{X}$.

Proof. The previous arguments can be repeated almost unchanged. □

In the next paper of this series, as an application of the results of this paper to known problems, we prove a criterion of the existence of a dense embedding of a monothetic monoid into a topological group.

References

[1] B. G. Averbukh, On unitary Cauchy filters on topological monoids, Topological Algebra and its Applications **1** (2013), 46-59.
[2] B. G. Averbukh, On finest unitary extensions of topological monoids, Topological Algebra and its Applications (appears).
[3] J. H. Carruth, J. A. Hildebrant, R. J. Koch, The theory of topological semigroups. Pure and Applied Mathematics, Marcel Dekker, Inc., 1983, vi + 244 pp., ISBN 0-8-8247-1795-3.
[4] R. Engelking, General topology. Rev. and compl. ed., Sigma Series in Pure Mathematics, 6., Berlin: Heldermann Verlag, 1989, viii + 529 pp., ISBN 3-88538-006-4, Zbl 0684.54001.
[5] R. Fric, D. C. Kent, Completion functors for Cauchy spaces, Int. J. Math. & Math. Sci. **2**, No. 4 (1979), 589-604. MR 80#54042. Zbl 428.54018.
[6] Lowen-Colebunders E., Function Classes of Cauchy Continuous Maps, Pure and Applied Mathematics, Marcel Dekker Inc., New York, 1989.
[7] Ramaley J.F., Wyler O., Cauchy spaces, http://repository.cmu.edu/math/97, 1968.
[8] N. Rath, Completions of Filter Semigroups, Acta Math. Hungar. **107** (1-2) (2005), 45-54.
[9] E. E. Reed, Completions of Uniform Convergence Spaces, Math. Ann. **194** (1971), 83-108. MR 45#1109. Zbl 217.19603.