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# Topological Rough Groups

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**Abstract:** The concept of topological group is a simple combination of the concepts of abstract group and topological space. The purpose of this paper is to combine the concepts of topological space and rough groups; called topological rough groups on an approximation space.

**Keywords:** Rough sets, rough group,rough subgroup, topological group, topological rough group

**MSC:** 03E99, 20A05, 22A99

## 1 Introduction

Rough sets were introduced by Z. Pawlak [18] in 1982 as an powerful mathematical tool for uncertain data while modeling the problems in computer science, medical science, data analysis and many other diverse fields [19, 21, 26]. In recent years, the rough sets has been combined with some mathematical theories such as algebra and topology [1, 8–10, 13, 15, 17, 20, 24, 25, 30, 31]. Algebraic structures of rough sets have been studied by many authors, for example, Bonikowaski [4], Iwinski [7], and Pomykala and Pomykala [22]. In 1994, Biswas and Nanda [3] introduced the notion of rough group and rough subgroups that their notion depends on the upper approximation and does not depend on the lower approximation. Miao et al. [16] improve definitions of rough group and rough subgroup, and prove their new properties. On the other hand, Kuroki and Wang [11] presented some properties of the lower and upper approximations with respect to the normal subgroups in 1996. In addition, some properties of the lower and the upper approximations with respect to the normal subgroups were studied in [5, 14, 27–29]. Also, Kuroki [12], introduced the notion of a rough ideal in a semigroup. Bağırmaz and Özcan [2], studied the notion of rough semigroup on approximation spaces. Davvaz [6], introduced the notion of rough subring with respect to an ideal of a ring.

The main purpose of this paper is to introduce topological rough groups, which extends the notion of a topological group to include the algebraic structures of rough groups in [3] and [16]. In section 2, some basic notions of rough sets, rough groups and topological groups are given. In section 3, new definition of topological rough group is introduced and its properties are given ; and some examples for topological rough groups are given. In section 4, topological rough ( normal ) subgroup are defined and some properties of them are proved. Finally, our conclusions are presented.

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## 2 Preliminaries

In this section, we are going to list some definitions and results about rough sets, rough groups and topological groups used in this paper.

**Definition 1.** *[18]Let $U$ be a finite non-empty set called universe and $R$ be a family equivalence relation on $U$. The pair $(U, R)$ is called an approximation space.*

**Definition 2.** *[18]Let $U$ be a universe and $R$ be an equivalence relation on $U$. We denote the equivalence class of object $x$ in $R$ by $[x]_R$.*

**Definition 3.** *[18]Let $(U, R)$ be an approximation space and $X$ be a subset of $U$.*
*The sets*
(1) $\overline{X} = \{x \mid [x]_R \cap X \neq \emptyset\}$ ,
(2) $\underline{X} = \{x \mid [x]_R \subseteq X\}$ ,
*are called upper approximation and lower approximation of $X$ in $(U, R)$, respectively.*

Let $X, Y \subset U$, where $U$ is a universe. Then, the approximations have the following properties:
(1) $\underline{X} \subset X \subset \overline{X}$,
(2) $\underline{\emptyset} = \overline{\emptyset} = \emptyset$, $\underline{U} = \overline{U} = U$,
(3) $\underline{X} \cap \underline{Y} = \underline{X \cap Y}$,
(4) $\overline{X} \cap \overline{Y} \subset \overline{X \cap Y}$,
(5) $\underline{X} \cup \underline{Y} \subset \underline{X \cup Y}$,
(6) $\overline{X} \cup \overline{Y} = \overline{X \cup Y}$,
(7) $X \subset Y$ if and only if $\underline{X} \subset \underline{Y}$, $\overline{X} \subset \overline{Y}$.
Let $(U, R)$ be an approximation space and $(*)$ be a binary operation defined on $U$. In the present paper onwards, we shall write $xy$ instead of $x * y$, $\forall\, x, y \in U$. If $X$ and $Y$ are two subsets of a group $G$, we denote by $XY$ the subset composed of all the elements of the form $xy$, where $x \in X$, $y \in Y$. We denote by $X^{-1}$ the subset composed of all the elements of the form $x^{-1}$, where $x \in X$. If $(X, T)$ is a topological space and $x \in X$, we write $\mathcal{N}(x)$ for the set of neighborhood of $x$.

**Definition 4.** *[23] A topological group is a group $(G, *)$ together with a topology on G that satisfies the following two properties:*
(1) *the mapping $f : G \times G \to G$ defined by $f(x, y) = xy$ is continuous when $G \times G$ is endowed with the product topology;*
(2) *the inverse mapping $g : G \to G$ defined by $g(x) = x^{-1}$ is continuous.*

We remark that item (1) is equivalent to the statement that, whenever $W \subseteq G$ is open, and $W \in \mathcal{N}(x_1x_2)$, then there exist open sets $V_1$ and $V_2$ such that $V_1 \in \mathcal{N}(x_1)$; $V_2 \in \mathcal{N}(x_2)$ and $V_1V_2 = \{x_1x_2 : x_1 \in V_1; x_2 \in V_2 \subseteq W\} \subseteq W$. Also, item (2) is equivalent to showing that whenever $V \subseteq G$ is open, then $V^{-1} = \left\{x^{-1} \mid x \in V\right\} \in \mathcal{N}\left(x^{-1}\right)$is open.

Let $G$ be a topological group and let $H$ be a subgroup of $G$. Then $H$ becomes a topological group when endowed with the topology induced by $G$.

**Definition 5.** *[3] Let $S = (U, R)$ be an approximation space and $(*)$ be a binary operation defined on $U$. A subset $G$ of universe $U$ is called a rough group if the following properties are satisfied:*
(1) $\forall x, y \in G$, $xy \in \overline{G}$,
(2) *Associativity property holds in $\overline{G}$,*
(3) $\exists e \in \overline{G}$ *such that* $\forall x \in G$, $xe = ex = x$, *where $e$ is called the rough identity element of rough group $G$.*
(4) $\forall x \in G$, $\exists y \in G$ *such that $xy = yx = e$, where $y$ is called the rough inverse element of $x$ in $G$, we denote it by $x^{-1}$.*

**Definition 6.** *[3] A non-empty subset H of rough group G is called its rough subgroup, if it is a rough group itself with respect to operation in G.*

**Remark 7.** *[3] There is only one guaranteed trivial rough subgroup of rough group G, i.e. G itself. A necessary and sufficient condition for $\{e\}$ to be a trivial rough subgroup of rough group G is $e \in G$.*

**Definition 8.** *[3] A necessary and sufficient condition for a subset H of a rough group G to be a rough subgroup is that:*
(1) $\forall x, y \in H, xy \in \overline{H}$,
(2) $\forall x \in H, x^{-1} \in H$.

**Definition 9.** *[16] A rough subgroup N of rough group G is called a rough normal subgroup, if $\forall a \in G$, $a * N = N * a$.*

# 3 Topological rough group

In this section, we are going to give a definition of topological rough group and its related properties.

**Definition 10.** *A topological rough group is a rough group $(G, *)$ together with a topology $T$ on $\overline{G}$ satisfying the following two properties:*
*(a) the mapping $f : G \times G \to \overline{G}$ defined by $f(x, y) = xy$ is continuous with respect to product topology on $G \times G$ and the topology $T_G$ on G induced by $T$,*
*(b) the inverse mapping $g : G \to G$ defined by $g(x) = x^{-1}$ is continuous with respect to the topology $T_G$ on G induced by $T$.*

We remark that item (1) is equivalent to the statement that, whenever $W \subseteq \overline{G}$ is open and $W \in \mathcal{N}(x_1x_2)$, then there exist open sets $V_1 \subseteq G$ and $V_2 \subseteq G$ such that $V_1 \in \mathcal{N}(x_1)$; $V_2 \in \mathcal{N}(x_2)$ and $V_1V_2 = \{x_1x_2 : x_1 \in V_1; x_2 \in V_2 \subseteq W\} \subseteq W$. Also, item (2) is equivalent to showing that whenever $V \subseteq G$ is open, then $V^{-1} = \left\{x^{-1} \mid x \in V\right\} \in \mathcal{N}\left(x^{-1}\right)$ is open.

**Example 11.** *Let $U = \left\{\overline{0}, \overline{1}, \overline{2}\right\}$ be a set of surplus class with respect to module 3 and $(*)$ be the plus of surplus class. A classification of U is $U/R = \{E_1, E_2\}$, where $E_1 = \left\{\overline{0}, \overline{1}\right\}, E_2 = \left\{\overline{2}\right\}$. Let $G = \left\{\overline{1}, \overline{2}\right\}$, then $\overline{G} = \{E_1, E_2\} = U$.*

*From Definition 5, since the conditions*
***(1)*** $\forall x, y \in G, xy \in \overline{G}$,
***(2)*** *Association property holds in $\overline{G}$,*
***(3)*** $\overline{1} * \overline{2} = \overline{0} \in \overline{G}$,
***(4)*** $\overline{1}^{-1} = \overline{2} \in G, \overline{2}^{-1} = \overline{1} \in G$,
*hold, G is a rough group.*

*Let $T = \left\{\emptyset, \overline{G}, \left\{\overline{1}\right\}, \left\{\overline{2}\right\}, \left\{\overline{1}, \overline{2}\right\}\right\}$ is a topology on $\overline{G}$, then $T_G = \left\{\emptyset, G, \left\{\overline{1}\right\}, \left\{\overline{2}\right\}\right\}$ is a relative topology on G.*

*From Definition 10, since the conditions*
*(a) $\overline{1} * \overline{1} = \overline{2}$, for all $W \in \mathcal{N}\left(\overline{2}\right) \in T$, then there exists open set*
*$U = \left\{\overline{1}\right\} \in \mathcal{N}\left(\overline{1}\right) \in T_G$ such that $U * U \subseteq W$,*
*$\overline{2} * \overline{2} = \overline{1}$, for all $W \in \mathcal{N}\left(\overline{1}\right) \in T$, then there exists open set*
*$U = \left\{\overline{2}\right\} \in \mathcal{N}\left(\overline{2}\right) \in T_G$ such that $U * U \subseteq W$,*
*$\overline{1} * \overline{2} = \overline{0}$, for all $W \in \mathcal{N}\left(\overline{0}\right) \in T$, then there exist open sets*
*$U = \left\{\overline{1}\right\} \in \mathcal{N}\left(\overline{1}\right) \in T_G$ and $V = \left\{\overline{2}\right\} \in \mathcal{N}\left(\overline{2}\right) \in T_G$ such that $U * V \subseteq W$,*
*(b) $\overline{1}^{-1} = \overline{2}$, for all $W \in \mathcal{N}\left(\overline{2}\right) \in T_G$, then there exists open set*

$U = \{\overline{1}\} \in \mathcal{N}(\overline{1}) \in T_G$ *such that* $U^{-1} = \{\overline{2}\} \subseteq W$,
$\overline{2}^{-1} = \overline{1}$, *for all* $W \in \mathcal{N}(\overline{1}) \in T_G$, *then there exists open set*
$U = \{\overline{2}\} \in \mathcal{N}(\overline{2}) \in T_G$ *such that* $U^{-1} = \{\overline{1}\} \subseteq W$,
*hold, G is a topological rough group.*

**Example 12.** *Let U be the set of all permutation of* $S_4$ *and* $(*)$ *be the multiplication operation of permutation. A classification of U is* $U/R = \{E_1, E_2, E_3, E_4\}$, *where*

$E_1 = \{(1), (12), (13), (14), (23), (24), (34)\}$,
$E_2 = \{(123), (132), (124), (142), (134), (143), (234), (243)\}$,
$E_3 = \{(1234), (1243), (1324), (1342), (1423), (1432)\}$,
$E_4 = \{(12)(34), (13)(24), (14)(23)\}$.
*Let* $G = \{(12), (123), (132)\}$, *then* $\overline{G} = E_1 \cup E_2$.
*From Definition 5, since the conditions*
*(1)* $\forall x, y \in G, xy \in \overline{G}$,
*(2) Association property holds in* $\overline{G}$,
*(3)* $(12) * (12) = (1) \in \overline{G}$,
*(4)* $(12)^{-1} = (12) \in G, (123)^{-1} = (132) \in G$,
*hold, G is a rough group.*

*Let* $T = \{\emptyset, \overline{G}, \{(12)\}, \{(1), (123), (132)\}, \{(1), (12), (123), (132)\}\}$ *is a topology on* $\overline{G}$, *then* $T_G = \{\emptyset, G, \{(12)\}, \{(123), (132)\}\}$ *is a relative topology on G.*

*From Definition 10, since the conditions*

$(a)$. $(12) * (12) = (1)$, *for all* $W \in \mathcal{N}((1)) \in T$, *then there exists open set* $U \in \mathcal{N}((12)) \in T_G$ *such that* $U.U \subseteq W$,

$(12) * (123) = (23)$, *for all* $W \in \mathcal{N}((23)) \in T$, *then there exist open sets* $U \in \mathcal{N}((12)) \in T_G$ *and* $V \in \mathcal{N}((123)) \in T_G$ *such that* $U.V \subseteq W$,

$(12) * (132) = (13)$, *for all* $\forall W \in \mathcal{N}((13)) \in T$, *then there exist open sets* $U \in \mathcal{N}((12)) \in T_G$ *and* $V \in \mathcal{N}((132)) \in T_G$ *such that* $U.V \subseteq W$,

$(123) * (132) = (1)$, *for all* $\forall W \in \mathcal{N}((1)) \in T$, *then there exist open set* $U \in \mathcal{N}((123)) \in T_G$ *and* $V \in \mathcal{N}((132)) \in T_G$ *such that* $U.V \subseteq W$,

$(b)$ $(12)^{-1} = (12)$, *for all* $W \in \mathcal{N}((12)) \in T_G$, *then there exist open set* $U \in \mathcal{N}((12)) \in T_G$ *such that* $U^{-1} \subseteq W$,

$(123)^{-1} = (132)$, *for all* $W \in \mathcal{N}((132)) \in T_G$, *then there exist open set* $U \in \mathcal{N}((123)) \in T_G$ *such that* $U^{-1} \subseteq W$,

*hold, G is a topological rough group.*

**Proposition 13.** *Let G be a topological rough group and fix* $a \in G$. *Then*
$(a)$ *The map* $L_a : G \to \overline{G}$ *defined by* $L_a(x) = ax$ *is one-to-one and continuous, for every* $x \in G$;
$(b)$ *The map* $R_a : G \to \overline{G}$ *defined by* $R_a(x) = xa$ *is one-to-one and continuous, for every* $x \in G$;
$(c)$ *The map* $f : G \to G$ *defined by* $f(x) = x^{-1}$ *is a homeomorphism, for every* $x \in G$.

*Proof.* $(a)$ For every $x_1, x_2 \in G$, then $L_a(x_1) = L_a(x_2) \Rightarrow ax_1 = ax_2$. Since $a \in G$ then $a^{-1} \in G \subseteq \overline{G}$. Thus $a^{-1}(ax_1) = a^{-1}(ax_2) \Rightarrow x_1 = x_2$. Hence $L_a$ is one-to-one.

For every $x \in G$ then $L_a(x) = ax$. Let $W \in \mathcal{N}(ax)$. Then, from Definition 10 item $(a)$ $\exists U \in \mathcal{N}(a) \in T_G$ and $\exists V \in \mathcal{N}(x) \in T_G$ such that $UV \subseteq W$. Since, $aV \subseteq UV \subseteq W$, then $L_a(V) = aV \subseteq W$. Therefore $L_a$ is continuous on $x$. Since $x$ be any element of $G$, then $L_a$ is continuous on $G$.

$(b)$ Similarly proof item $(1)$.

$(c)$ Firs, the map $f : G \to G$ defined by $f(x) = x^{-1}$ is one-to-one and onto. Then, from Definition 10 item $(b)$ $f$ is continuous. The continuity of the inverse mapping $f^{-1}(x) = x^{-1}$ can be proved in the same way. □

**Remark 14.** *The maps* $L_a$ *and* $R_a$, *defined in previous proposition, are not onto. Furthermore, the maps* $L_a$ *and* $R_a$ *are not open. Let* $G = \{\overline{1}, \overline{2}\}$, $T = \{\emptyset, \overline{G}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{1}, \overline{2}\}\}$ *and* $T_G = \{\emptyset, G, \{\overline{1}\}, \{\overline{2}\}\}$ *from the Example*

*11, while fix $\overline{1} \in G$ and $V = \{\overline{2}\} \in T_G$. Thus $V \in T_G$ is open, but then $L_{\overline{1}}(\{\overline{2}\}) = \overline{1} \oplus \{\overline{2}\} = \{\overline{0}\}$ is not open in $T$.*

**Proposition 15.** *Let $G$ be a topological rough group. Then*

$$G = G^{-1}.$$

*Proof.* By definition rough group and topological rough group are clear. □

**Proposition 16.** *Let $G$ be a topological rough group and $V \subseteq G$. Then*

$$V \text{ is open (closed)} \Leftrightarrow V^{-1} \text{ is open (closed)}.$$

*Proof.* Let $V \subseteq G$ is open (closed). Since by Proposition 13 item $(c)$ the map $f : G \to G$ defined by $f(x) = x^{-1}$ is a homeomorphism, then $f(V) = V^{-1}$ is open (closed). □

**Proposition 17.** *Let $G$ be a topological rough group and $W \subseteq \overline{G}$ be an open subset with $e \in W$. Then there exists an open set $V$, with $e \in V$ such that $V = V^{-1}$ and $VV \subseteq W$.*

*Proof.* Since $f : G \times G \to \overline{G}$ is continuous, $f^{-1}(W)$ is open in $G \times G$ and $ee = e \in W \in T$. Hence, there exist open sets $V_1, V_2 \in T_G$ with $e \in V_1, e \in V_2$ such that $V_1V_2 \subseteq W$. By the above proposition $V_1^{-1}, V_2^{-1}$ are also open, hence, $V = V_1 \cap V_2 \cap V_1^{-1} \cap V_2^{-1}$ is also open, $e \in V$, $V = V^{-1}$ and $VV \subseteq V_1V_2 \subseteq W$. □

**Proposition 18.** *Let $G$ be a topological rough group. Let $G$ ,$\{e\}$ be open subsets of $\overline{G}$. Then;*
$(a)$ *If $e \in G$, then $\{e\}$ is open in space $G$,*
$(b)$ *If $e \notin G$, then there exists an open subset $V \subseteq G$ such that $V = V^{-1}$.*

*Proof.* $(a)$ Let $e \in G$. Since $f : G \times G \to \overline{G}$ is continuous, $f^{-1}(\{e\})$ is open in $G \times G$ and $ee = e \in \{e\} \in T$. Then, there exist open sets $V_1, V_2 \in T_G$ with $e \in V_1, e \in V_2$ such that $V_1V_2 \subseteq \{e\}$. Thus, if $U \neq \{e\}$ then $U.U \nsubseteq \{e\}$ ,and so $V_1 = V_2 = \{e\}$.

$(b)$ Let $e \notin G$. Since $G$ is a topological rough group and $\{e\} \subset \overline{G}$ is an open subset, then from Definition 10 item $(a)$ $f^{-1}(\{e\})$ is open in $G \times G$. So, for all $a, b \in G$, $ab = e$, then there exist open sets $V_1 \in \mathcal{N}(a)$ and $V_2 \in \mathcal{N}(b)$ such that $V_1V_2 \subseteq \{e\}$. Thus, if $V_1V_2 \neq \emptyset$,then $V_1V_2 = \{e\}$. Hence $V_1 = \{a\}$, $V_2 = \{b\}$ and $V_1 = V_2^{-1}$. □

The following two propositions it gives the relationship between the topological rough groups and topological groups.

**Proposition 19.** *Let $G$ be a topological rough group. If $G = \overline{G}$, then $G$ is a topological group.*

*Proof.* Let $G$ be a topological rough group and $G = \overline{G}$.

(1) $\forall x, y \in G$, $xy \in \overline{G} = G$.
(2) $\forall x, y, z \in G$, $(xy)z = x(yz)$, association property holds in $\overline{G} = G$.
(3) $e \in G$.
(4) $\forall x \in G$, $x^{-1} \in G$.
Thus $G$ is a group.
Since $G$ is a topological rough group and $G = \overline{G}$, we have the maps

$$f : G \times G \to G, (x, y) \mapsto x.y$$
$$g : G \to G, x \mapsto x^{-1}$$

are continuous. Hence $G$ is a topological group. □

**Proposition 20.** *Let $(G, T_1)$ be a topological group and $(\overline{G}, T)$ be a topological space. Then $G$ is a topological rough group if and only if the topology $T_1$ and the topology $T_G$ on $G$ induced by $T$ same topologies.*

*Proof.* From Definition 10 and Definition 4 are clear. □

## 3.1 Topological rough subgroups and topological rough normal subgroups

In this section we introduce the concept of topological rough subgroups and topological rough normal subgroups. We consider the relative topology on a rough subgroup.

**Definition 21.** *Let $G$ be a topological rough group and let $H$ be a subgroup of $G$. Then, $H$ is called a topological rough subgroup of $G$ if*

(1) *the mapping $f_H : H \times H \to \overline{H}$ defined by $f_H(x, y) = xy$ is continuous where $\overline{H}$ carries the topology induced by $\overline{G}$.*

(2) *the inverse mapping $g_H : H \to H$ defined by $g_H(x) = x^{-1}$ is continuous.*

**Proposition 22.** *Let $G$ be a topological rough group. Then, every rough subgroup $H$ of $G$ with relative topology is a topological rough subgroup.*

*Proof.* Since the restriction of mapping $f : G \times G \to \overline{G}$, $f_H : H \times H \to \overline{H}$, and inverse to $H$ are continuous the result holds. □

**Proposition 23.** *Let $G$ be a topological rough group. If $H$ is a topological rough subgroup of $G$, then the topological closure of $H$, $H^c$, is a topological rough subgroup of $G$.*

*Proof.* Let $H$ is a topological subgroup of $G$. Suppose that $x, y \in H^c$. We then prove that $xy \in \overline{H^c}$ and $x^{-1} \in H^c$. Let $W$ be a neighborhood of the element $xy$. Then there exist neighborhoods $U$ and $V$ of the elements $x$ and $y$ such that $UV \subseteq W$. Since $x \in H^c$ and $y \in H^c$ there exist elements $a$ and $b$ of $H$ such that

$$a \in U \cap H \Rightarrow a \in U \cap \overline{H} \text{ and } b \in V \cap H \Rightarrow b \in V \cap \overline{H}.$$

Thus we have

$$ab \in UV \text{ and } ab \in \overline{H}.$$

This implies that

$$ab \in UV \cap \overline{H}.$$

Therefore $UV \cap \overline{H} \neq \varnothing$ and so $W \cap \overline{H} \neq \varnothing$. Hence $xy \in \overline{H^c}$.

Let $x \in H^c$. Let $W$ be a neighborhood of the element $x^{-1}$. Then there exist neighborhoods $U$ of the element $x$ such that $U^{-1} \subseteq W$, where $U^{-1} = \{x^{-1} \mid x \in U\}$. Since $x \in H^c$ there exist elements $a$ of $H$ such that $a \in U$. Thus $a \in U \cap H$. This implies that $a^{-1} \in U^{-1} \cap H$ and so $U^{-1} \cap H \neq \varnothing$. Hence $x^{-1} \in H^c$.

This prove $H^c$ is a topological rough subgroup of $G$. □

**Remark 24.** *It is known that if $H$ is a group then for any $p \in H, p.H = H$. But if $H$ is a rough group since composition operation defined in $\overline{H}$, for any $p \in H$, $p.H \neq H$. Therefore in a topological group if a $H$ subgroup is open also it is closed but this is not valid for a topological rough group. Since in a topological rough group as it is stated in Remark 14 $p.H$, may not be open neighborhood of $p$.*

**Proposition 25.** *Let $H_1$ and $H_2$ be two topological rough subgroups of the topological rough group $G$. A sufficient condition for intersection of two topological rough subgroups of a topological rough group to be a topological rough subgroup is $\overline{H_1} \cap \overline{H_2} = \overline{H_1 \cap H_2}$.*

*Proof.* Suppose $H_1$ and $H_2$ are two topological rough subgroups of $G$. It is obvious that $H_1 \cap H_2 \subset G$. Consider $x, y \in H_1 \cap H_2$. Because $H_1$ and $H_2$ are rough subgroups, we have $xy \in \overline{H_1}$, $xy \in \overline{H_2}$, and $x^{-1} \in H_1, x^{-1} \in H_2$, $i.e. xy \in \overline{H_1} \cap \overline{H_2}$ and $x^{-1} \in H_1 \cap H_2$. Assuming $\overline{H_1} \cap \overline{H_2} = \overline{H_1 \cap H_2}$, we have $xy \in \overline{H_1 \cap H_2}$ and $x^{-1} \in H_1 \cap H_2$. Thus $H_1 \cap H_2$ is a rough subgroup of $G$. Hence with Proposition 22, $H_1 \cap H_2$ is a topological subgroup of $G$. □

**Proposition 26.** *Let $K$ be a topological rough subgroup of the topological rough group $G$ and $H$ be any nonempty subsets of $G$. If $H$ be a topological rough subgroup of $K$, then also $H$ be a topological rough subgroup of $G$.*

*Proof.* Let. $H \subset G$ and $H$ be a topological rough subgroup of $K$. Thus, since $\forall x, y \in H$ we have $xy \in \overline{H}$ and so $\forall z \in H$ we have $z^{-1} \in H$. Hence $H$ is a topological rough subgroup of $G$.

Let us show that $H$ is a topological rough subgroup. Since $G$ is a topological rough group and $K$ is a topological rough subgroup we can take topological spaces $(\overline{G}, T)$ and $(\overline{K}, T_{\overline{K}})$ . On the other hand since $H$ is a topological rough subgroup of $K$, topological space $\left(\overline{H}, T_{\overline{K}|_{\overline{H}}}\right)$ exists. Hence the topologies $T_{\overline{H}}$ and $T_{\overline{K}|_{\overline{H}}}$on $\overline{H}$ induced by $T$ are same then $H$ is topological rough subgroup of $G$. □

**Proposition 27.** *Let $H_1$ and $H_2$ are two topological rough subgroup of the topological rough group $G$ and let $\overline{H}_1\overline{H}_2 = \overline{H_1H_2}$. Then $H_1H_2$ be a topological rough subgroup of the topological rough group $G$ if and only if $H_1H_2 = H_2H_1$*

*Note that $H_1H_2 = \{h_1h_2 \mid h_1 \in H_1, h_2 \in H_2\}$ .*

*Proof.* ($\Rightarrow$) Let $H_1H_2$ be a topological rough subgroup of $G$. We then prove that $H_1H_2 = H_2H_1$.

Suppose that

$$h_2h_1 \in H_2H_1 \Rightarrow (h_2h_1)^{-1} = h_1^{-1}h_2{}^{-1} \in H_1H_2.$$

Since $H_1H_2$ is a rough subgroup then

$$\left(h_1^{-1}h_2{}^{-1}\right)^{-1} = h_2h_1 \in H_1H_2.$$

Thus $H_2H_1 \subseteq H_1H_2$.

Again suppose that $x \in H_1H_2$. Since $H_1H_2$ is a rough subgroup, $x^{-1} = h_1h_2 \in H_1H_2$, where $h_1 \in H_1$ and $h_2 \in H_2$. Thus $x = h_2^{-1}h_1{}^{-1} \in H_2H_1$ and so $H_1H_2 \subseteq H_2H_1$. Hence $H_1H_2 = H_2H_1$.

($\Leftarrow$) Let $H_1H_2 = H_2H_1$.We then prove that $H_1H_2$ be a topological rough of $G$. Suppose that $h_1h_2, g_1g_2 \in H_1H_2$. Then

$$(h_1h_2)(g_1g_2) = h_1(h_2g_1)g_2 = h_1(g_1h_2)g_2 = (h_1g_1)(h_2g_2) \in \overline{H}_1\overline{H}_2.$$

Since $\overline{H}_1\overline{H}_2 = \overline{H_1H_2}$ we have $(h_1h_2)(g_1g_2) \in \overline{H_1H_2}$.

Suppose that $h_1h_2 \in H_1H_2$. Then

$$(h_1h_2)^{-1} = h_2^{-1}h_1{}^{-1} \in H_2H_1.$$

Since $H_1H_2 = H_2H_1$ we have $(h_1h_2)^{-1} \in H_1H_2$. □

**Definition 28.** *Let $G$ be a topological rough group and let $N$ be a normal subgroup of $G$. Then, $N$ is called a topological rough normal subgroup of $G$ if $\forall a \in G$, $aN = Na$.*

**Proposition 29.** *Let $H_1$ and $H_2$ are two topological rough normal subgroups of the topological rough group $G$ and let $\overline{H}_1\overline{H}_2 = \overline{H_1H_2}$. Then $H_1H_2$ be a topological rough normal subgroup of the topological rough group $G$*

*Proof.* We already know by Proposition 27 that $H_1H_2$ is a topological rough subgroup of $G$. It suffices to show that $H_1H_2$ is normal. Let $a$ be any elements of $G$. Since $a(H_1H_2) = (aH_1)H_2 = (H_1a)H_2 = H_1(aH_2) = H_1(H_2a) = (H_1H_2)a$, then from Definition 28 that $H_1H_2$ is a topological rough normal subgroup of $G$. □

# 4 Conclusion

In this paper, we deal with one of the newest argument from rough set theory namely topological rough groups. We have studied the concept of topological rough groups. Then, we present the concept of topological rough subgroups. By considering the relative topology on rough subgroups, we prove some properties of them.

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