Research Article

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Entropy on noncompact sets

<https://doi.org/10.1515/taa-2019-0003>

Received December 20, 2018; accepted July 31, 2019

Abstract: In this paper we review and explore the notion of topological entropy for continuous maps defined on non compact topological spaces which need not be metrizable. We survey the different notions, analyze their relationship and study their properties. Some questions remain open along the paper.

Keywords: Topological entropy, non compact spaces.

MSC: 37B40, 37A35, 54C70.

1 Introduction

Topological entropy [1] is one of the most accepted tools to characterize complicated or chaotic behavior in discrete dynamical systems. It is commonly accepted that positive topological entropy implies the existence of dynamics that we identify as unpredictable. From the theoretical point of view is very useful, e.g. it is invariant by conjugacy. On the other hand, although it is quite difficult to compute in an effective way, it gives you a very valuable information when you are able to compute it when, for instance, one is analyzing models depending on one or several parameters because it gives you an analytical proof on then existence of chaotic behavior, or more precisely, for what parameter values it is possible to find chaotic motions.

However, the above paragraph holds when the phase space is compact, and usually metric. In this paper we are interested in analyzing the notion of topological entropy when the topological space need not be neither compact nor metrizable. As we will point out, there exist several possible entropy notions which are not equivalent, some well-known properties in the compact case are not true when we remove the compactness, and extremely dynamically simple maps may exhibit positive entropy.

The aim of this paper is to survey what notions of topological entropy can be found in the literature, trying to show their advantages and disadvantages, and exploring the link among them. As we are going to work with several definitions of topological entropy, to distinguish them, we denote it with the classical letter h plus a subscript with the initials of the authors who introduced or inspired the notion. For instance, we denote by $h_{AKM}(f)$ the Adler, Konheim and McAndrew's definition of topological entropy for maps $f : X \rightarrow X$ defined on a compact space X [1]. In case some author introduces more than one definition, we may add some characteristic to distinguish them. For instance, Hofer's definition on compactifications will be denoted by $h_H^*(f)$ [10].

Now, let $f : X \rightarrow X$ be a continuous map on a non necessarily compact space. Fixed $x \in X$, we denote by $\text{Orb}_f(x)$ the forward orbit of x under f , that is, the sequence $f^n(x)$ for $n \in \mathbb{N}$. The set of accumulation points of $\text{Orb}_f(x)$ is the ω -limit set $\omega_f(x)$. This ω -limit set is closed and strictly invariant by f , and then we can define the following subsets of X by

$$K_f = \{x \in X : \omega_f(x) \text{ is compact and non empty}\},$$

$$C_f = \{x \in X : \omega_f(x) \text{ is non empty and non compact}\},$$

$$E_f = \{x \in X : \omega_f(x) \text{ is empty}\}.$$

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Since $\omega_f(x)$ is strictly invariant by f , we have that K_f and C_f are invariant by f . It is easy to see that E_f is invariant as well. The above sets allow us to consider the following decomposition of X by $X = K_f \cup C_f \cup E_f$. As we will show, the several definitions that we will consider can be invisible for them. Although K_f need not be compact, the restriction of the map to K_f gives us the dynamics of f which is intrinsically compact, while $f|_{C_f}$ and $f|_{E_f}$ is related to the dynamics of f that are essentially non compact.

Now, let us start by introducing the notions we are going to deal with. We start by recalling the well-known notion of entropy for compact spaces.

2 The compact case

Before introducing the non compact case, we recall briefly the well-known Adler, Konheim and McAndrew's definition of topological entropy for a continuous map $f : X \rightarrow X$ on a compact topological Hausdorff space X is stated from its open covers [1]. Namely, if $\mathcal{A} = \{A_i\}$ is an open cover of X , and $\mathcal{N}(\mathcal{A})$ is the minimum number of elements from $\mathcal{A}$ to cover X , then the topological entropy of f relative to $\mathcal{A}$ is given by the non negative number

$$h(f, \mathcal{A}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{N} \left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A}) \right),$$

where $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A}) = \{\cap_{i=0}^{n-1} f^{-i}(A_{j_i}) : A_{j_i} \in \mathcal{A}\}$, $f^{-i} = (f^i)^{-1}$, and $f^i = f \circ f^{i-1}$, $i \geq 1$, f^0 is the identity on X . The limit exists due to the subadditivity properties of $\frac{1}{n} \log \mathcal{N} \left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A}) \right)$ given by the fact that

$$\mathcal{N} \left(\bigvee_{i=0}^{k+n-1} f^{-i}(\mathcal{A}) \right) \leq \mathcal{N} \left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A}) \right) \mathcal{N} \left(\bigvee_{i=0}^{k-1} f^{-i}(\mathcal{A}) \right).$$

The topological entropy of f is then

$$h_{AKM}(f) = \sup\{h(f, \mathcal{A}) : \mathcal{A} \text{ is an open cover of } X\}.$$

It is a simple observation that, due to the compactness of X , the open cover $\mathcal{A}$ can be taken finite. The basic properties of $h_{AKM}(f)$ are introduced below.

Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be continuous maps on the topological spaces X and Y , respectively and let $\varphi : X \rightarrow Y$ be continuous and such that $\varphi \circ f = g \circ \varphi$. The product map $f \times g : X \times Y \rightarrow X \times Y$ is given by $(f \times g)(x, y) = (f(x), g(y))$ for all $(x, y) \in X \times Y$. The basic properties of topological entropy on compact topological spaces are the following:

(TE1) If φ is surjective, then $h_{AKM}(f) \geq h_{AKM}(g)$. If φ is injective, then $h_{AKM}(f) \leq h_{AKM}(g)$. If φ the map φ is an homeomorphism, then $h_{AKM}(f) = h_{AKM}(g)$ (see [1] or [2, Chapter 4]).

(TE2) Power formula. For $n \in \mathbb{N}$, $h_{AKM}(f^n) = nh_{AKM}(f)$. Additionally, if f is an homeomorphism then $h_{AKM}(f) = h_{AKM}(f^{-1})$. [1].

(TE3) If $Y \subset X$ is compact and invariant by f , that is, $f(Y) \subseteq Y$, then $h_{AKM}(f|_Y) \leq h_{AKM}(f)$. In addition, if $X = \bigcup_{i=1}^k X_i$ and $f(X_i) \subseteq X_i$, then $h_{AKM}(f) = \max\{h_{AKM}(f|_{X_i}) : i = 1, 2, \dots, k\}$ (see [1] or [2, Chapter 4]).

(TE4) Product formula. $h_{AKM}(f \times g) = h_{AKM}(f) + h_{AKM}(g)$ [8].

In addition, compact subsets Y of X with the property that $h_{AKM}(f|_Y) = h_{AKM}(f)$ have a special interest. In particular, the nonwandering set $\Omega(f)$ given by the points $x \in X$ for which any open neighborhood U , with $x \in U$, satisfies that $f^n(U) \cap U \neq \emptyset$ for some $n \in \mathbb{N}$, is the smallest dynamical subset such that $h_{AKM}(f|_{\Omega(f)}) = h_{AKM}(f)$ [3] or [2, Chapter 4].

In this paper, we will try to study in detail when all the above properties (TE1)–(TE4) are satisfied in the non compact case, but in some cases we are unable to establish that the property is fulfilled. We will point out when some property is not fulfilled and make comments on the validity of the above properties when it is known.

3 Hofer's approaches

3.1 Hofer's definition #1

If the space X is not compact, then its open covers need not have finite refinements. Hence, in [10], the topological entropy is introduced in an obvious way; just consider finite open covers of X , and then

$$h_H(f) = \sup\{h(f, \mathcal{A}) : \mathcal{A} \text{ finite open cover of } X\}.$$

It is also mentioned in [10] there that (TE1) and (TE2) are true for $h_H(f)$. Clearly, for the first part of (TE2) the proof is analogous to that of [1], based in the equality

$$\bigvee_{i=0}^{kn-1} f^{-i}(\mathcal{A}) = \bigvee_{i=0}^{k-1} f^{-ni}(\bigvee_{i=0}^{n-1} \mathcal{A})$$

for the inequality

$$h_H(f^n) \geq n \cdot h_H(f)$$

and since $\bigvee_{i=0}^{n-1} \mathcal{A}$ is finer than $\mathcal{A}$ we have that

$$h(f^n, \mathcal{A}) \leq h(f^n, \bigvee_{i=0}^{n-1} \mathcal{A}) = n \cdot h(f, \mathcal{A}),$$

which gives us the second inequality

$$h_H(f^n) \leq n \cdot h_H(f).$$

This property was proved in [5], jointly with the formula $h_H(f) = h_H(f^{-1})$ for homeomorphisms. For property (TE1), again in [5] was proved that if φ is injective and $\varphi(X)$ is closed in Y , then $h_H(f) \leq h_H(g)$. Additionally it is proved that if φ is surjective, then $h_H(f) \geq h_H(g)$, implying the equality if f and g are conjugate. It is unclear if property (TE3) holds in general. In [5] it is proved that if Y is closed in X , then

$$h_H(f) \geq h_H(f|_Y)$$

and that if $X = \bigcup_{i=1}^k Y_i$ with all the sets Y_i closed in X then we get the equality

$$h_H(f) = \max\{h_H(f|_{X_i}) : i = 1, 2, \dots, k\}.$$

Finally, it is unclear whether the property (TE4) remains true for Hofer's definition #1. It is clear that if $\mathcal{A}$ is a finite open cover of X and $\mathcal{B}$ is a finite open cover of Y , then the inequality

$$\mathcal{N}(\mathcal{A} \times \mathcal{B}) \leq \mathcal{N}(\mathcal{A}) \cdot \mathcal{N}(\mathcal{B}), \quad (1)$$

and hence we have that

$$h_H(f \times g) \leq h_H(f) + h_H(g),$$

but it is unclear whether this inequality can be replaced by an equality.

3.2 Hofer's definition #2

Also in [10] is given a second definition of topological entropy for non compact spaces. We assume that X is a completely regular space (Tychonoff) and given the continuous map f , we consider $f^* : X^* \rightarrow X^*$, the unique continuous extension of f to the Stone–Čech compactification X^* of X . Then

$$h_H^*(f) = h_{AKM}(f^*).$$

Both definitions are equivalent when the space X is normal (see [10]). In addition, in [6], it was proved that the above result is false in general when X is not normal, disproving a conjecture from [10]. Additionally, in [6], it is proved that for completely regular spaces:

$$h_H^*(f) = \sup\{h(f, \mathcal{A}) : \mathcal{A} \text{ finite cozero cover of } X\},$$

where $\mathcal{A}$ is a cozero cover of X if any $A \in \mathcal{A}$ is of the form $A = \phi^{-1}((0, 1])$, $\phi : X \rightarrow [0, 1]$ continuous. In addition the inequality $h_H^*(f) \leq h_H(f)$ was proved when the space is completely regular.

Concerning the related properties (TE1)–(TE4), the injective case in (TE1), as well as the inequality $h_H^*(f|_Y) \leq h_H^*(f)$ for invariant subsets Y of X , are not true in general for $h_H^*(f)$ (see [6]). It is proved in [5] that the surjectivity in (TE1) holds when $\varphi(X)$ is dense in Y and when φ is a homeomorphism then we get the equality $h_H^*(f) = h_H^*(g)$. Property (TE2) follows since $\bigvee_{i=0}^{n-1} \mathcal{A}$ is a finite cozero cover of X whenever $\mathcal{A}$ has this property and it is also proved in [5]. It is unclear whether property (TE4) holds, although it is clear that the inequality $h_H^*(f \times g) \leq h_H^*(f) + h_H^*(g)$ holds. Anyway, from now on we are mainly interested in normal spaces where Hofer's definitions are equivalent.

4 Definition based on extensions

4.1 Infimum of extensions

In what follows we will assume that all the spaces are normal. The Hofer's approach makes use of compactifications of a normal topological space and the extensions of continuous maps. However, extending the map implies to embed the space into a compact space, which roughly speaking implies to add points to the dynamics of f , of course changing the topology. The following result from [9] shows that Hofer topological entropy is not a good tool to analyze a class of maps with essentially non compact dynamics (see [9]).

Theorem 1. *Let $f : X \rightarrow X$ be a continuous map such that $E_f \neq \emptyset$. Then $h_H(f) = +\infty$.*

Moreover, the following example from [10] shows that other possibilities should be taken into account in order to define topological entropy on noncompact spaces.

Example 2. *We consider the map $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(n) = n + 1$ for all $n \in \mathbb{Z}$ holds that $h_H^*(f) = h_H(f) = \infty$. Note that the dynamics of f is quite simple, although E_f is the whole space. In addition, clearly f can be compactified by just one point and then, the extended map f^* has zero topological entropy.*

This is then another problem in the compactification ideas, that is, it may not be unique and therefore, different compactifications may give different values of entropy, as the above example shows. Then, a possible solution to this problem is to consider

$$h^*(f) = \inf \{ h_{AKM}(f^*) : f^* \text{ is a extension of } f \}.$$

With this definition it is easy to see that the topological entropy of the map $f(n) = n + 1$ defined above is zero.

Additionally, it is clear that $h^*(f) \leq h_H^*(f) = h_H(f)$. However, it is unclear whether properties (TE1)–(TE4) hold. For (TE2) it is clear that $(f^*)^2$ is an extension of f^2 for a suitable compactification of X and hence it is easy to see that $h^*(f^n) \leq nh^*(f)$. For (TE4), note that $f^* \times g^*$ are extensions on suitable compactifications of $f \times g$ and therefore $h^*(f \times g) \leq h^*(f) + h^*(g)$.

4.2 Friedland's like approach

Friedland's approach was made for totally bounded metric spaces and makes use of embeddings. We adopt a similar approach when X is not metric as follows. Let $f : X \rightarrow X$ be continuous and assume that X^* is a compactification of X . Then the product space $(X^*)^{\mathbb{N}}$ is also compact. Consider the shift map $\sigma : X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$, given by $\sigma(x_n) = (x_{n+1})$, which is continuous. Let X^f be the subset of $X^{\mathbb{N}}$ given by

$$X^f = \{(x_i)_{i=0}^{\infty} : x_i = f(x_{i-1}) \text{ with } i = 1, 2, \dots\}.$$

Then $\sigma(X^f) \subset X^f$ and Friedland topological entropy [7] is defined as

$$h_F(f) = h_{AKM}(\sigma|_{\text{Cl}(X^f)}).$$

It should be emphasized that Friedland's approach was for invariant subsets of compact metric spaces, when there is no ambiguity in the selection of the compactification. Clearly, we can define as above

$$h_F^*(f) = \inf\{h_F(f) : X^* \text{ is a compactification of } X\}.$$

In the compact case, that is, when X is compact, Goodwyn proved in [8] that $h_{AKM}(f) = h_{AKM}(\sigma|_{X^f})$, and therefore Friedland's approach makes sense. It is unclear how this definition is related with the previous ones as well as the validity of properties (TE1)–(TE4).

5 Compact subsets definition

5.1 Non invariant case definition

There is a possibility of defining topological entropy by following the definition based on Misiurewicz's conditional topological entropy [12]. Let $K \subset X$ be a compact subspace of X . For an open cover $\mathcal{A}$ of X , we define the number

$$\mathcal{N}\left(K, \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A})\right)$$

as the smallest cardinality of a subcover of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A})$ such that the union of its elements contains K . Then

$$h(f, K, \mathcal{A}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{N}\left(K, \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A})\right)$$

exists, notice that subadditivity is not guaranteed, and then we can take

$$h(f, K) = \sup\{h(f, K, \mathcal{A}) : \mathcal{A} \text{ open cover of } X\}.$$

Clearly if $K_1 \subseteq K_2$ are compact subsets on X , then $h(f, K_1) \leq h(f, K_2)$. Then we can define the topological entropy as

$$h_M(f) = \sup\{h(f, K) : K \in \mathcal{K}(X)\},$$

where $\mathcal{K}(X)$ denotes the family of compact subsets of X .

Concerning property (TE1), we can state the following result.

Proposition 3. *Let assumptions of (TE1) hold for noncompact sets. Then*

1. *If φ is injective and $\varphi(X)$ is closed in Y , then $h_M(f) \leq h_M(g)$.*
2. *If φ is surjective and proper, then $h_M(f) \geq h_M(g)$.*
3. *If φ is a homeomorphism, then $h_M(f) = h_M(g)$.*

Proof. We prove 1. Let K be a compact subset of X . Since $\varphi(X)$ is closed in Y , for any open cover $\mathcal{A}$ there is an open cover $\mathcal{B}$ of Y such that $\varphi^{-1}(\mathcal{B}) = \mathcal{A}$. This idea is taken from [5] and the open cover $\mathcal{B}$ is given by sets B such that $\varphi^{-1}(B) = A \in \mathcal{A}$ jointly with the set $Y \setminus \varphi(X)$, which is open in Y . Then

$$\mathcal{N}\left(K, \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A})\right) \leq \mathcal{N}\left(\varphi(K), \bigvee_{i=0}^{n-1} g^{-i}(\mathcal{B})\right),$$

which gives us the inequality $h_M(f, K) \leq h_M(g)$, which obviously implies 1.

Now, we prove 2. Let $K \subset Y$ be compact and let $\mathcal{A}$ be an open cover of Y . Then $\varphi^{-1}(K) \in \mathcal{K}(X)$ and $\mathcal{B} = \varphi^{-1}(\mathcal{A})$ is an open cover of X . Then

$$\mathcal{N}\left(K, \bigvee_{i=0}^{n-1} g^{-i}(\mathcal{B})\right) \leq \mathcal{N}\left(\varphi^{-1}(K), \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A})\right),$$

which gives us the inequality $h_M(g, K) \leq h_M(f)$, which obviously implies 2. Then, 3 follows directly from 1 and 2. $\square$

Now, we consider property (TE2), stating the following.

Proposition 4. *Let assumptions of (TE2) hold for noncompact sets. Then $h_M(f^n) = n \cdot h_M(f)$.*

Proof. We take ideas from [2]. Since

$$\bigvee_{i=0}^{kn-1} f^{-i}(\mathcal{A}) = \bigvee_{i=0}^{k-1} f^{-ni}(\bigvee_{i=0}^{n-1} \mathcal{A}),$$

we obtain that for any compact subset K of X

$$\mathcal{N}\left(K, \bigvee_{i=0}^{kn-1} f^{-i}(\mathcal{A})\right) = \mathcal{N}\left(K, \bigvee_{i=0}^{k-1} f^{-ni}(\bigvee_{i=0}^{n-1} \mathcal{A})\right),$$

and hence

$$n \cdot h(f, K, \mathcal{A}) = h(f^n, K, \bigvee_{i=0}^{n-1} \mathcal{A}) \leq h_M(f),$$

and so

$$n \cdot h_M(f) \leq h_M(f^n).$$

For the converse inequality, we simply realize that

$$h(f^n, K, \mathcal{A}) \leq h(f^n, K, \bigvee_{i=0}^{n-1} \mathcal{A}) = n \cdot h(f, K, \mathcal{A}) \leq n \cdot h_M(f)$$

and obtain

$$n \cdot h_M(f) \geq h_M(f^n),$$

which concludes the proof. $\square$

Note that it remains open the property $h_M(f) = h_M(f^{-1})$ for f homeomorphism. Regarding property (TE3) we show the following.

Proposition 5. *Let $X = \bigcup_{i=1}^k Y_i$, for Y_i closed invariant subsets of X . Then $h_M(f) = \max\{h_M(f|_{Y_i}) : i = 1, 2, \dots, k\}$.*

Proof. Again, we take ideas from [2]. Since $h_M(f|_{Y_i}) \leq h_M(f)$, we obtain that $h_M(f) \leq \max\{h_M(f|_{Y_i}) : i = 1, \dots, k\}$. To prove the converse inequality, let K be a compact subset of X and note that $K_i = K \cap Y_i$ is compact for $i = 1, \dots, k$. Let $\mathcal{A}$ be an open cover of X and consider $\mathcal{A}_i = \mathcal{A} \cap Y_i = \{A \cap Y_i : A \in \mathcal{A}\}$ open covers of Y_i . If $\mathcal{B}_1, \dots, \mathcal{B}_k$ are subcovers of $\mathcal{A}_1, \dots, \mathcal{A}_k$ such that $\#\mathcal{B}_i = \mathcal{N}(K_i, \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{A}_i))$, note that each element $B = B_1 \cup \dots \cup B_k$, $B_i \in \mathcal{B}_i$, $i = 1, \dots, k$, is contained in $\bigvee_{j=0}^{n-1} f^{-j}(\mathcal{A})$ and their union contains K . Hence

$$\mathcal{N}(K, \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{A})) \leq \sum_{i=1}^k \mathcal{N}(K_i, \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{A}_i)),$$

and applying Lemma 4.1.9 from [2] we obtain that $h(f, K, \mathcal{A}) \leq \max\{h_M(f|_{Y_i}) : i = 1, \dots, k\}$, which gives us

$$h_M(f) \leq \max\{h_M(f|_{Y_i}) : i = 1, \dots, k\},$$

which concludes the proof. $\square$

Concerning the property (TE4), it is clear from (1) that $h_M(f \times g) \leq h_M(f) + h_M(g)$, but it is unclear whether the converse inequality holds.

5.2 Invariant compact subsets definition

We can consider a subfamily within the compact subsets of X as follows. Let $\mathcal{K}(X, f)$ be the family of compact subsets K of X which are invariant by f , that is, $f(K) \subseteq K$. The restricted map $f|_K$ is the properly defined in a compact space, and therefore we can compute its topological entropy $h(f|_K)$. Then, the topological entropy of f is defined by

$$h_{CR}(f) = \sup\{h_{AKM}(f|_K) : K \in \mathcal{K}(X, f)\}.$$

If $\mathcal{K}(X, f)$ is empty, then $h_{CR}(f) = 0$. This definition was introduced by Cánovas and Rodríguez in [4], and obviously $h_{CR}(f) \leq h_M(f)$. In addition, since the compact subsets of X are compact subsets of any compactification X^* , we immediately have that $h_{CR}(f) \leq h^*(f)$, and then for the map $f(n) = n + 1$ defined in Example 2, we get that $h_{CR}(f) = h^*(f) = 0$. In addition, it is clear from the definition of $h_{CR}(f)$ that if a map f holds that $K_f = \emptyset$, then $h_{CR}(f) = 0$, that is, this definition is blind for the dynamics which is not essentially compact. One of the most interesting problems in this setting is when the inequality $h_{CR}(f) \leq h^*(f)$ can be replaced by an equality. Some examples will be introduced along this paper.

It fulfills all the properties (TE1)–(TE4) except for the surjectivity case of (TE1) which is satisfied when φ is a proper map, that is, $\varphi^{-1}(K)$ is compact for all $K \in \mathcal{K}(X)$. If φ is not proper, the following example can be found in [4].

Example 6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x$ for all $x \in \mathbb{R}$ and let $g : S^1 \rightarrow S^1$ defined also by $g(x) = 2x$ for all $x \in S^1$. Then g is a factor of f and $h_{CR}(f) = 0$ while $h_{CR}(g) = h_{AKM}(g) = \log 2$ (see [1, Chapter 4]).

It is also remarkable that the formula $h_{CR}(f) = \max\{h_{CR}(f|_{X_i}) : i = 1, 2, \dots, k\}$, where X_i are invariant subsets of X whose union is X , is not true in general. Again the following example can be seen in [4].

Example 7. Let $f : X \rightarrow X$ be a minimal homeomorphism defined on a compact set with positive topological entropy (see [14]). Let $x \in X$ and we define $X_1 := \text{FullOrb}_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$ and $X_2 := X \setminus X_1$. It is clear that both sets are invariant by f and that they have no compact invariant subsets (in case of that they have compact invariant subsets, the map f is not minimal). Therefore $h_{CR}(f|_{X_i}) = 0$, $i = 1, 2$, while $h_{CR}(f) = h_{AKM}(f) > 0$.

It is clear that following the steps of the proof of Proposition 5 we can prove the equality $h_{CR}(f) = \max\{h_{CR}(f|_{X_i}) : i = 1, 2, \dots, k\}$ when X_i are closed subsets of X .

6 Proper maps

Now, we introduce the so called co-compact topological entropy introduced first by Patrao in [13] (see also [15]) based on open covers with special properties. Namely, an open cover $\mathcal{A} = \{A_i : i \in \mathcal{J}\}$ of a normal topological space X is co-compact if for any $A_i \in \mathcal{A}$ the subset $X \setminus A_i$ is compact. Co-compact covers have finite subcovers. We assume that $f : X \rightarrow X$ is proper. Then $f^{-1}(\mathcal{A})$ is a co-compact cover of X and then Adler, Konheim and McAndrew's definitions makes sense in this setting, that is, the co-compact topological entropy of f relative to the co-compact cover $\mathcal{A}$ is

$$h(f, \mathcal{A}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{N} \left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A}) \right),$$

and the co-compact topological entropy of f is then

$$h_P(f) = \sup\{h(f, \mathcal{A}) : \mathcal{A} \text{ is a co-compact open cover of } X\}.$$

It is easy to see that $h_P(f) \leq h_H(f)$. The following result establishes the relationship between $h_P(f)$ and $h_{CR}(f)$.

Theorem 8. If $f : X \rightarrow X$ is a proper map, then $h_{CR}(f) \leq h_P(f)$.

Proof. If $\mathcal{K}(X, f) = \emptyset$, then $h_{CR}(f) = 0$ and there is nothing to prove. So, assume $K \in \mathcal{K}(X, f)$ and let $\mathcal{A} = \{A_1, \dots, A_k\}$ be a finite open cover of K . Then, there exist $A'_1, \dots, A'_k$ such that $A_i = A'_i \cap K$ for all $i = 1, \dots, k$. In addition, we have that $A_i = (A'_i \cup (X \setminus K)) \cap K$ for $i = 1, \dots, k$. Then, the open cover $\mathcal{B} = \{(A'_i \cup (X \setminus K)), i = 1, \dots, k\}$ is cocompact since the closed sets $X \setminus (A'_i \cup (X \setminus K))$, $i = 1, \dots, k$, are contained in the compact set K . Now, let $\mathcal{J}$ be the family of indexes $(i_0, \dots, i_{n-1}) \in \{1, \dots, k\}^n$ such that $\{\bigvee_{i=0}^{n-1} f^{-i}(A'_{i_j} \cup (X \setminus K)) : (i_0, \dots, i_{n-1}) \in \mathcal{J}\}$ is a finite subcover of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{B})$ with minimal cardinality. As K is invariant by f , we have that $\{\bigvee_{i=0}^{n-1} f^{-i}(A'_{i_j}) :$

$(i_0, \dots, i_{n-1}) \in \mathcal{I}\}$ is a finite subcover of $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A})$. Then $\mathcal{N}\left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A})\right) \leq \mathcal{N}\left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{B})\right)$ and hence

$$h(f|_K, \mathcal{A}) \leq h_P(f, \mathcal{B}) \leq h_P(f),$$

which implies that $h(f|_K) \leq h_P(f)$, and then $h_{CR}(f) \leq h_P(f)$, which finishes the proof. $\square$

It is unclear whether the converse inequality holds. It depends on the dynamics of f outside the compact invariant subsets of f and $h_{CR}(f)$ cannot measure anything outside $\mathcal{K}(X, f)$.

Concerning properties (TE1), the surjective and proper case was proved in [13] and [15]. From this we get the invariance by conjugacy. It is unclear whether the injective property of (TE1) holds even when $\varphi(X)$ is closed in Y . The property (TE2) works as for the Misiurewicz case as stated in [15]. It is proved in [15] that if $Y \subset X$ is closed and invariant, then $h_P(f|_Y) \leq h_P(f)$. The equality $h_P(f) = \max\{h_P(f|_{Y_i}) : i = 1, \dots, k\}$ for a collection of finite closed subsets Y_i invariant by f follows as in the proof of Proposition 5. Finally, for (TE4), we easily get the inequality $h_P(f \times g) \leq h_P(f) + h_P(g)$.

7 Final remarks and conclusion

The product map formula (TE4) deserves a comment. It was proved in [8] using inverse limits, close to Friedland's approach included in this paper. It is unclear whether it works for most of the definitions. Even for Bowen's like definitions of entropy, in the case of metric spaces, a counterexample disproving it was recently found (see [11]).

Although some preliminary results have been stated, it is unclear in general what is the influence of sets K_f , C_f and E_f in the entropy notions that we have considered. We know that $h_H(f) = +\infty$ when $E_f \neq \emptyset$ and that $h_{CR}(f) = 0$ when $K_f = \emptyset$, but we do not know how the emptiness or not of such subsets influence the remaining entropy notions.

This author knows that this paper is in some sense minimalist, in the sense that if we add metric or uniform structures we can obtain richer results. However, the list of possible definitions increases a lot. In addition, we fix here the basic ideas for possible definitions of topological entropy which are extending the space to a compact space, taking supremum over compact subsets and fixing open covers with special properties. We have studied the basic properties with the only use of open covers following the seminal ideas from [1]. The reader should have noticed that there are many holes or open questions in this review, the most important one is to establish a connection between all the introduced notions or disprove that connections.

Acknowledgement: Authors have been partially supported by the Grants MTM2014-52920-P and MTM2017-84079-P from Agencia Estatal de Investigación (AEI) y Fondo Europeo de Desarrollo Regional (FEDER).

I wish to thank to professor Dikranjan for pointed me out the existence of reference [5].

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