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Positive answers to Koch's problem in special cases

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Abstract: A topological semigroup is monothetic provided it contains a dense cyclic subsemigroup. The Koch problem asks whether every locally compact monothetic monoid is compact. This problem was opened for more than sixty years, till in 2018 Zelenyuk obtained a negative answer. In this paper we obtain a positive answer for Koch's problem for some special classes of topological monoids. Namely, we show that a locally compact monothetic topological monoid S is a compact topological group if and only if S is a submonoid of a quasitopological group if and only if S has open shifts if and only if S is non-viscous in the sense of Averbukh. The last condition means that any neighborhood U of the identity 1 of S and for any element $a \in S$ there exists a neighborhood V of a such that any element $x \in S$ with $(xV \cup Vx) \cap V \neq \emptyset$ belongs to the neighborhood U of 1 .

Keywords: Koch's problem, monothetic semigroup, non-viscous monoid, topological semigroup, semitopological semigroup, cancellative semigroup, locally compact semigroup, countably compact semigroup, feebly compact semigroup, Tkachenko-Tomita group

MSC: 22A15, 54D30

1 Introduction

In this paper natural numbers are positive integers and a "space" means a "topological space". Any space in the paper is assumed to be Hausdorff, unless it satisfies others explicitly stated separation axioms. For the semigroup operation we use the multiplicative notation or the additive notation when the operation is necessarily commutative. By a *topologized semigroup* we understand a semigroup endowed with a topology.

1.1 Koch's problem.

A topologized semigroup S is called *monothetic* (cf., [1, p.10]) provided it contains a dense cyclic subsemigroup, that is $S = \overline{\{a^n : n \geq 1\}}$ for some element $a \in S$, which is called a *topological generator* of the semigroup S . It is easy to check that each monothetic semitopological semigroup is commutative. Compact monothetic (semi)topological groups are compact Abelian groups whose character groups are subgroups of the circle

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(endowed with the discrete topology). Pontrjagin's theorem [2] states that each locally compact monothetic topological group is compact. Compact monothetic topological semigroups were described by Hewitt [3], see also [1, p. 123–130]. In particular, a locally compact monothetic topological semigroup with a minimal ideal is compact, see [1, Theorem 3.13]. Whether a counterpart of Pontrjagin's theorem holds for locally compact monothetic topological monoids was a well-known problem, posed by Koch [4]. Remark that if such a monoid is compact, then it is a (topological) group [4], [5]. In [6] Zelenyuk constructed a countable locally compact cancellative monothetic topological semigroup which is neither compact, nor discrete. It is easy to see that a countable locally compact monothetic topological monoid is discrete, so we cannot attach the identity to Zelenyuk's semigroup. It took him thirty years to reach a final (negative) answer to Koch's problem in [7].

On the other hand, Koch's problem has an affirmative answer in some special cases. For instance, for a totally disconnected monoid S , which was known already to Koch [4]. If S is a group, then S is a topological group by Ellis' Theorem [8]. Recently Guran and Kisil [9] improved this result showing that a locally compact cancellative monothetic topological monoid with open shifts is a compact topological group.

In the present paper we obtain positive answers to Koch's Problem for more general classes of topological monoids. In particular, we show that a monothetic topological monoid is cancellative provided it has open shifts or is non-viscous.

1.2 Embeddings of semigroups into groups.

Each submonoid of a group is a cancellative semigroup. On the other hand, it is well-known (see, for instance, [10]) that a commutative cancellative semigroup can be embedded into the group of its quotients.

Unfortunately, this algebraic construction can have no obvious topological counterparts.

The problem of recognizing topological semigroups that embed into topological groups is a well-known and old research problem. A list of papers published before 1990 is contained in [11]. Positive results are known for some special cases, see [12], [13], [14].

A recent advance [15, 16] by Averbukh (also, see [17, 18]) allowed to find a surprisingly simple characterization of monothetic topological monoids which can be embedded into a topological group. Namely, such monoids are cancellative, T_3 , and non-viscous. The last property is a simple condition expressed in terms of elements and neighborhoods. Moreover, this notion turned out to be useful in obtaining a positive answer to the Koch problem for a partial case. Namely, a monothetic non-viscous topological monoid S is cancellative (see Proposition 3.5) and if S is locally compact, then it is a compact topological group, see Theorem 3.9.

2 Definitions

A semigroup S is *cancellative* provided for any elements $a, x, y \in S$, the equality $ax = ay$ or $xa = ya$ implies $x = y$.

A topologized semigroup S is called

- a *left-topological semigroup* provided for every $a \in S$ the left shift $S \rightarrow S, x \mapsto ax$, is continuous;
- a *semitopological semigroup* provided the multiplication $S \times S \rightarrow S, (x, y) \mapsto xy$, is separately continuous;
- a *topological semigroup* if the multiplication $S \times S \rightarrow S$ is continuous.

A topology on a semigroup S is called a *semigroup topology* on S if it turns S into a topological semigroup.

A topologized group G is called

- a *semitopological group* provided the multiplication $S \times S \rightarrow S$ is separately continuous;
- a *quasitopological group* if G is a semitopological group with continuous inversion $G \rightarrow G, x \mapsto x^{-1}$;
- a *paratopological group* if the multiplication $S \times S \rightarrow S$ is continuous;
- a *topological group* if G is a paratopological group and a quasitopological group.

A standard example of a paratopological group failing to be a topological group is the Sorgenfrey line, that is the real line endowed with the Sorgenfrey topology (generated by the base consisting of half-intervals $[a, b)$, $a < b$).

Since compact spaces have very good properties, they play an important role in general topology. These properties are so good that even spaces satisfying slightly weaker conditions are still useful. Therefore, topologists often investigate various classes of compact-like spaces and relations between them, see, for instance, basic [19, Chap. 3] and general works [20], [21], [22]. In the present paper we will consider some of compact-like properties.

We recall that a space X is called

- *sequentially compact* if each sequence of X contains a convergent subsequence;
- *countably compact at a subset A of X* if each infinite subset B of A has an accumulation point x in the space X (the latter means that each neighborhood of x contains infinitely many points of the set B);
- *countably compact* if X is countably compact at itself;
- *countably precompact* if X is countably compact at a dense subset of X ;
- *feebly compact* if each locally finite family of open subsets of the space X is finite.

Relations between various classes of compact-like spaces are well-studied. Some of them are presented in Diagram 3 of [24, p. 17], in Diagram 1 of [23, p. 58] (for Tychonoff spaces), in Diagram 3.6 of [22, p. 611], and in Diagram of [25].

In particular, the following inclusions hold.

- Each compact space is countably compact.
- Each sequentially compact space is countably compact.
- Each countably compact space is countably precompact.
- Each countably precompact space is feebly compact.

In these terms, a space X is compact if and only if X is countably compact and Lindelöf. A Tychonoff space X is feebly compact iff it is pseudocompact, that is iff each continuous real-valued function on X is bounded.

3 The non-viscosity

Following Averbukh [15] we define a topologized monoid S to be *non-viscous* if for any neighborhood U of the identity 1 of S and for any element $a \in S$ there exists a neighborhood V of a such that any element $x \in S$ with $(xV \cup Vx) \cap V \neq \emptyset$ belongs to U .

Remark 3.1. It is easy to see that any non-viscous T_1 topologized monoid S satisfies the following Condition: for any $a, x \in S$ if $ax = a$ or $xa = a$ then $x = 1$. In particular, the only idempotent of S is the identity. On the other hand, a commutative monoid generated by two elements p and q with the only condition $p + q = p$ has only an identity idempotent but does not satisfy Condition. By [18, Proposition 1.1], each compact topological monoid satisfying Condition is non-viscous. Condition also assures that S is non-viscous provided S is discrete. Condition holds for every cancellative monoid, so the additive monoid of non-negative real numbers endowed with the usual topology, but with isolated zero is a viscous locally compact cancellative topological monoid. On the other hand, the direct product of additive semigroup $\mathbb{N}$ by the semilattice $(\{0, 1\}, \min)$ with adjoint identity satisfies Condition but is not cancellative. Nevertheless, it is easy to check that in a finite monoid whose only idempotent is identity, each element is invertible, so this monoid is a group, see, for instance, [26, § 3].

Lemma 3.2. *Let S be a monothetic semitopological monoid. Let $S = \overline{P}$, where P is the semigroup generated by an element $p \in S$. If $\overline{x} + \overline{P} \cap P \neq \emptyset$ for each $x \in S$, then S is cancellative.*

Proof. Suppose to the contrary that there exist elements a, b , and x of the semigroup S such that $a \neq b$, but $a + x = b + x$. Since the semigroup S is Hausdorff, the set $Y = \{y \in S : a + y = b + y\}$ is closed. Since $Y \supset x + P$

and $\overline{x + P} \cap P \neq \emptyset$, there exists a natural number n such that $np \in Y$. Since Y is an ideal, $mp \in Y$ for each $m \geq n$. Since $a \neq b$ and S is a Hausdorff semitopological monoid, there exists a neighborhood U of the identity such that sets $a + U$ and $b + U$ are disjoint. Since the semigroup S is monothetic, there exists a natural number $m \geq n$ such that $mp \in U$. Since $mp \in Y$, $a + mp = b + mp \in (a + U) \cap (b + U)$, a contradiction. $\square$

Corollary 3.3. *A monothetic semitopological monoid S is cancellative, if for each $x \in S$ the set $\overline{x + S}$ has non-empty interior in S .*

Proof. Let P be a dense subsemigroup of S as in Lemma 3.2. It suffices to remark that since P is dense in S , $x + P$ is dense in $x + S$ for each $x \in S$, so $\overline{x + S} = \overline{x + P}$. $\square$

Corollary 3.4. *A monothetic semitopological monoid with open shifts is cancellative.*

Proposition 3.5. *A monothetic non-viscous topological monoid S is cancellative.*

Proof. Suppose to the contrary that there exist elements a, b , and x of the semigroup S such that $a \neq b$, but $a + x = b + x = y$. Since S is a Hausdorff topological semigroup, there exist neighborhoods V_a of a , V_b of b , and U of 1 such that the sets $V_a + U$ and $V_b + U$ are disjoint. Since S is non-viscous, there exists a neighborhood V of the point y such that for any $y_1, y_2 \in V$ and $z \in S$ such that $y_1 + z = y_2$ we have $z \in U$. Since S is a topological semigroup, there exist neighborhoods W_a of a , W_b of b , and V_x of x such that $W_a + V_x \subset V$ and $W_b + V_x \subset V$. Let $S = \overline{P}$, where P is the semigroup generated by an element $p \in S$. There exist natural numbers n_a, n_b and n_x such that $n_ap \in V_a \cap W_a$, $n_bp \in V_b \cap W_b$, and $n_xp \in V_x$. Then $n_ap + n_xp \in V$ and $n_bp + n_xp \in V$. Without loss of generality we may assume that $n_b > n_a$, so $n = n_b - n_a > 0$. Since $(n_ap + n_xp) + np = n_bp + n_xp$, $np \in U$. So $V_b + U \supset V_b \ni n_bp = n_ap + np \in V_a + U$, a contradiction. $\square$

Example 3.6. Averbukh proved [18, Theorem 2.2] that a monothetic topological monoid S can be embedded into a topological group if and only if S is T_3 , cancellative, and non-viscous. By Proposition 3.5 the cancellativity holds automatically, but neither non-viscosity nor monotheticity can be dropped.

Indeed, Sorgenfrey circle S (see Example 3.17) is a monothetic T_3 paratopological group. Since a subgroup of a topological group is a topological group too, S cannot be embedded in a topological group.

Now let $S = [0, \infty)$ be the (non-viscous) submonoid of the Sorgenfrey arrow, see Example 3.17. To derive a contradiction, assume that S is a submonoid of a topological group G . Then there exists a neighborhood U of the identity 0 of G such that $(1 \cdot U) \cap S \subset [1, 2)$. Since G is a topological group, there exists a number $0 < \varepsilon < 1$ such that $U \supset [0, \varepsilon)^{-1}$. Then $(1 \cdot U) \cap S \supset (1 - \varepsilon, 1)$, a contradiction.

Lemma 3.7. *A T_1 monothetic left topological monoid S containing an isolated point p_0 is a finite group.*

Proof. Let $S = \overline{P}$, where P is the semigroup generated by an element $p \in S$. Then $p_0 = p^n$ for some natural n and $p^n U = \{p^n\}$ for some neighborhood U of the identity. Pick any $p^m \in U$ with $m > 0$. Then $p^{m+n} = p^n$, which implies that the semigroup P is finite. Since S is a T_1 space with a finite dense set P , we see that $S = P$ is a finite monoid. Since S is monothetic, S is a group (see, for instance, [26, § 3]). $\square$

Lemma 3.8. *A locally compact non-viscous monothetic topological monoid S has open shifts.*

Proof. By Lemma 3.7 it suffices to consider the case when S has no isolated points.

Let $S = \overline{P}$, where P is the semigroup generated by an element $p \in S$. Let U be any compact neighborhood of the zero and $a \in S$ be any element. Since S is non-viscous, there exists a neighborhood V of the point a such that for any $a_1, a_2 \in V$ and $x \in S$ with $a_1 + x = a_2$ we have $x \in U$. Since the semigroup S is monothetic, the set $V \cap P$ is dense in V . Let $np \in V \cap P$ be any element. Since S has no isolated points, the set $V' = V \cap (P \setminus \{mp : 0 \leq m \leq N\})$ is dense in V , too. Let $v \in V'$ be any element. There exists an element $mp \in P$ such that $mp + np = v$. Therefore $mp \in U$, so $v \in U + np$. Thus $V' \subset U + np$ and $V \subset \overline{V'} \subset \overline{U + np} = U + np$.

We claim that $V \subset a + U$. To derive a contradiction, assume that there exists an element $v \in V \setminus (a + U)$. By the compactness of U , there exists an open neighborhood W of a such that $v \notin W + U$. Pick an arbitrary point $np \in W \cap V \cap P$. By the previous paragraph, $v \in V \subset np + U \subset W + U$, a contradiction. Thus $a \in V \subset a + U$ and $a + U$ is a neighborhood of the point a .

Now let $y \in S$ be an any element and V be any non-empty open subset of S . Let $z \in y + V$ be an arbitrary point. Then $z = y + v$ for some point $v \in V$. Pick a compact neighborhood U of the zero such that $v + U \subset V$. Then $z = y + v \in y + v + U = z + U \subset y + V$, and by the previous paragraph the set $z + U$ is a neighborhood of the point z . $\square$

Now we can give a partial answer to the Koch problem.

Theorem 3.9. *For a locally compact monothetic topological monoid S , the following conditions are equivalent:*

1. S is a compact topological group;
2. S has open shifts;
3. S is non-viscous.

Proof. The implication (1) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (2) follows from Lemma 3.8. To prove that (2) $\Rightarrow$ (1), assume that S has open shifts. By Corollary 3.4, the monothetic monoid S is cancellative. By Theorem 8 of [9], S is either discrete or a compact topological group. If S is discrete, then S is a finite group by Lemma 3.7. $\square$

We shall call an element g of a semitopological monoid S *topologically periodic*, if for each neighborhood U of the identity of S there exists a natural number n such that $g^n \in U$. A semitopological monoid S is called *topologically periodic*, provided each its element is topologically periodic. It is easy to see that each (pre)compact topological group G is topologically periodic.

Proposition 3.10. *Each non-viscous countably compact topologized monoid S is topologically periodic.*

Proof. Let g be any element of S and U be any neighborhood of the identity 1 of S . Let a be a cluster point of the sequence $\{g^n : n \in \mathbb{N}\}$. Let V be a neighborhood of a such that for any elements $a_1, a_2 \in V$, if $a_1 x = a_2$ then $x \in U$. There exist natural numbers $m < n$ such that $g^m, g^n \in V$. Then $g^{n-m} \in U$. $\square$

Proposition 3.11. *Let S be a topological monoid such that $S \times S$ is countably compact. Then each left invertible element g of S is topologically periodic.*

Proof. Pick an element $h \in S$ such that $hg = 1$. Let (a, b) be a cluster point of the sequence (h^n, g^n) . Assume that $ab \neq 1$. Then $S \setminus \{1\}$ is a neighborhood of ab . There exist neighborhoods U and V of a and b , respectively such that $UV \subset S \setminus \{1\}$. Since $U \times V$ is a neighborhood of (a, b) in $S \times S$, there exists natural n such that $(h^n, g^n) \in U \times V$. Then $1 = h^n g^n \in S \setminus \{1\}$, a contradiction. Thus $ab = 1$. Let W be any neighborhood of 1. There exist neighborhoods U and V of a and b , respectively such that $UV \subset W$. Since $U \times V$ is a neighborhood of (a, b) in $S \times S$, there exist natural numbers $m < n$ such that $(h^m, g^n) \in U \times V$. Then $g^{n-m} = h^m g^n \in W$. $\square$

Proposition 3.12. *Each topologically periodic semitopological monoid S is cancellative.*

Proof. Let a and b be distinct elements of the semigroup S and $x \in S$. Since $a \neq b$, there exists a neighborhood U of the identity of S such that $aU \cap bU = \emptyset = Ua \cap Ub$. Since the element x is topologically periodic, there exists a positive integer n such that $x^n \in U$. Then $ax^n \in aU$, and $bx^n \in bU$, so $ax^n \neq bx^n$. Thus $ax \neq bx$. Similarly we can show that $xa \neq xb$. $\square$

The following example shows that local compactness is essential in Theorem 3.9, It also concerns [18, Theorem 2.2] and shows that a monoid with open shifts, contained in a topological group, may fail to be a group.

Example 3.13. Let g be any non-periodic topologically periodic element of a topological group G . For instance, g can be any nonperiodic element of any (pre)compact topological group G . In particular, we can

take as G the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and as g any element of infinite order in $\mathbb{T}$. Put $S = \{g^n : n \in \omega\}$. Since S is a subsemigroup of a topological group and $1 \in S$, S is a cancellative topological monoid. Moreover, it is easy to check that each topological group is non-viscous and a submonoid of a non-viscous monoid is non-viscous. Since g is topologically periodic, S is monothetic. Since g is non-periodic, S is not a group. For any element $h \in S$ the set hS has finite complement in S and hence is open in S . Let U be any open subset of S . Since a shift in a topological group is a homeomorphism onto its image, the set hU is open in hS , so it is open in S too.

However, a non-viscous monothetic submonoid of a topological group needs not have open shifts.

Example 3.14. Let $s : \mathbb{T} \rightarrow \mathbb{Q}$ be a surjective homomorphism, $S' = \{z \in \mathbb{T} : s(z) > 0\}$ and $S = S' \cup \{0\}$. Then $\mathbb{T} = S' - S'$, both sets S' and $-S'$ are dense in $\mathbb{T}$, but $S' \cap (-S') = \emptyset$. Thus S is a non-viscous monoid which is not open in $\mathbb{T}$. Since the set S' is uncountable, it contains a non-periodic element, which generates a cyclic semigroup which is dense in $\mathbb{T}$, so in S too. Hence the monoid S is monothetic. Pick any element $g \in S'$. Then the set $S' \setminus (g + S') = s^{-1}((0, s(g)))$ is dense in $\mathbb{T}$ and in S . Thus the set $g + S$ is not open in S .

We can generalize Proposition 1.2 from [18], which states that a non-viscous paratopological group is a topological group as follows.

Proposition 3.15. *A non-viscous left-topological group G is a topological group.*

Proof. It suffices to show that the map $\delta : G \times G \rightarrow G$, $\delta : (x, y) \mapsto xy^{-1}$, is continuous. Fix any points $x, y \in G$ and an open neighborhood $W \subset G$ of xy^{-1} . Since G is left-topological, $yx^{-1}W$ is an open neighborhood of $yx^{-1}xy^{-1} = 1 = yy^{-1}$. By the non-viscousity of G , the point y has an open neighborhood V_y such that $V_y V_y^{-1} \subset yx^{-1}W$ and hence $(xy^{-1}V_y)V_y^{-1} \subset W$. Since G is left-topological, the set $V_x := xy^{-1}V_y$ is an open neighborhood of x . Since $V_x V_y^{-1} \subset W$, the map δ is continuous at (x, y) . $\square$

Proposition 3.16. *Let S be a non-viscous topological monoid S such that S is sequentially compact or S^{2^c} is countably compact. Then S is a topological group.*

Proof. By Proposition 3.10, S is topologically periodic. By Proposition 3.12, S is cancellative. If S is sequentially compact, then S is a topological group by [27, Theorem 6]. If S^{2^c} is countably compact, then by Theorem 3.1 and the remark at the beginning of Section 3.2 of [28], S is a group. By Proposition 3.15, S is a topological group. $\square$

Example 3.17. Let $\mathbb{S}$ be the Sorgenfrey line, that is the real line endowed with the Sorgenfrey topology (generated by the base consisting of half-intervals $[a, b)$, $a < b$). It is easy to check that the submonoid $S = [0, \infty)$ of $\mathbb{S}$ is non-viscous.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle whose group operation is the multiplication of complex numbers. Define a map $f : \mathbb{S} \rightarrow \mathbb{T}$, $f : x \mapsto e^{2\pi xi}$. It is well-known that f is a homomorphism with the kernel $\mathbb{Z}$. Therefore the group $G = \mathbb{S}/\mathbb{Z}$ endowed with the quotient topology is a paratopological group (see, for instance, [29, Proposition 1.12]). We call it *the Sorgenfrey circle*. The group G is monothetic. Since G is not a topological group, it is viscous by Proposition 3.15.

Now define an action $\mathbb{Z}_2 \rightarrow \text{Aut}(\mathbb{T})$ of the two-element group $\mathbb{Z}_2$ on $\mathbb{T}$ by putting $\sigma(1)(z) = z^{-1}$ for each $z \in \mathbb{T}$. The first author observed in [30] that the semidirect product $H = \mathbb{T} \rtimes \mathbb{Z}_2$ naturally endowed with the topology of the Aleksandrov "two-arrows" space (see, for instance, [19, 3.10.C]) is a left-topological group. The dense subgroup $\mathbb{T} \times \{0\}$ of H can be identified with the Sorgenfrey circle. Any non-periodic element of H generates a dense subsemigroup in H , which implies that H is monothetic, but not Abelian. So, H is a monothetic first-countable compact left-topological group, which is not semitopological. By Proposition 3.15, the left-topological group H is viscous.

4 Embedded variations

To obtain another partial affirmative answer to the Koch problem, we need the following

Lemma 4.1. *For any topologically periodic element g of a quasitopological group G and $n \in \mathbb{Z}$, the set $H = \overline{\{g^m : m \geq n\}}$ is a group.*

Proof. If g is periodic then H is a finite group, so we can assume that g is not periodic. Let l be any integer and U be any neighborhood of an element g^l . Since $1 \cdot g^l = g^l$, there exists a neighborhood V of 1 such that $Vg^l \subset U$. Since g is topologically periodic, there exists a number $k \geq n - l$ such that $g^k \in V$. Then $g^k g^l \in \{g^m : m \geq n\} \cap U$. Thus $g^l \in H$. It follows that H is the closure of the subgroup $\{g^l : l \in \mathbb{Z}\}$ in the quasitopological group G , so by [32, Prop. 1.4.13] H is a group. $\square$

Theorem 4.2. *Assume that a monothetic topological monoid S is locally compact at the unit 1. If S is a submonoid of a quasitopological group G , then S is a compact topological group.*

Proof. Let g be a topological generator of the monoid S and $H = \overline{\{g^m : m \geq 1\}}^G$. Clearly the element g is topologically periodic. By Lemma 4.1, H is a (quasitopological) group. It is easy to see that $\bar{S}^G = H$. Let U be an open (in S) neighborhood of the identity such that $\bar{U}^S$ is compact. Then $S \supset \bar{U}^S = \bar{U}^H$ is a neighborhood of the identity in $H = \bar{S}^H$. Let $h \in H$ be any element. Since H is a quasitopological group, the set hS^{-1} is a neighborhood in H of the element h , so $hS^{-1} \cap S \neq \emptyset$, thus $h \in SS = S$. Therefore $H = S$ is both a quasitopological group and a topological semigroup, that is, a topological group. Since H is locally compact, it is compact by the Pontrjagin dichotomy. $\square$

Remark that Theorem 4.2 generalizes [15, Cor. 1.15] to quasitopological groups and that by [31] every locally compact topological group G is closed in any quasitopological group which contains G as a subgroup.

Corollary 4.3. *Assume that a topologically periodic topological monoid S is locally compact at the unit 1. If S is a submonoid of a quasitopological group G , then S is a topological group.*

Proof. Let a be an arbitrary element of S . Then S_a , the smallest closed subsemigroup of S containing a is a monothetic topological monoid, whose identity has a neighborhood with compact closure. Since S_a is contained in a quasitopological group, by Theorem 4.2, it is a group. In particular, S_a (and so, S) contains the inverse of a , so S is a group, see [26, § 3]. Thus S is a paratopological group with continuous inversion, because S is a subgroup of a quasitopological group. $\square$

A counterpart of Theorem 4.2 fails for T_1 case, as the following example shows.

Example 4.4. An additive group $\mathbb{Z}$ endowed with the cofinite topology is a T_1 compact quasitopological group containing a monothetic compact topological semigroup S of non-negative integers. The monoid S cannot be embedded in a T_1 topological group, because S is not Hausdorff.

Remark 4.5. Theorem 4.2 suggests the question whether we can weaken the local compactness at 1 to the (local) countable or feeble compactness. Our answers to this question coincide with the respective answers to a known question whether a compact-like cancellative topological semigroup S is a group.

Bokalo and Guran [27] proved that each sequentially compact cancellative topological semigroup is a topological group.

For pseudocompact case there is a relatively simple counterexample. Let H be the Tychonoff product of an uncountable family of nontrivial compact topological groups, G be the Σ -product contained in H . Assume that H is monothetic and pick an element $g \in H$ such that the subgroup of H generated by g is dense in H . Clearly, $g \in G$. Since G is a dense countably compact subset of H , the subsemigroup S of H generated by G and g is countably precompact, so it is pseudocompact. Since $S \ni g$, it is easy to see that the subsemigroup

S is monothetic. By Exercise 9.6.f from [32], a compact Abelian group K is monothetic iff the Pontrjagin dual group K^* is isomorphic to a subgroup of the group $\mathbb{T}$ endowed with the discrete topology. We can also directly show that a group $\mathbb{T}^\kappa$ endowed with the Tychonoff product topology is monothetic for each $\kappa \leq \mathfrak{c}$. It is well-known that the real line considered as the linear space over the field $\mathbb{Q}$ has a Hamel basis of cardinality $\mathfrak{c}$ and hence contains a linearly independent subset $T = \{t_\alpha : \alpha < \kappa\}$ consisting of pairwise distinct real numbers t_α , $\alpha < \kappa$. Put $g = (g_\alpha) = (e^{2\pi i t_\alpha}) \in \mathbb{T}^\kappa$. By [33, Example 65], for each sequence $t_{\alpha_1}, \dots, t_{\alpha_r}$ of mutually different elements of T , each sequence $d_1, \dots, d_r$ of real numbers, and each $\varepsilon > 0$ there exist a sequence $n_1, \dots, n_r$ of integer numbers and an integer number m such that $|mt_{\alpha_i} - d_i - n_i| < \varepsilon$ for each index $i = 1, \dots, r$. This implies that the set of powers of g is dense in the group $\mathbb{T}^\kappa$.

But the countably compact case is much harder, because it is closely related with the following question by van Douwen [34] from 1980, which is, according to references from [35, p. 2], considered central in the theory of topological groups.

Question 4.6. *Is there an infinite countably compact group without non-trivial convergent sequences?*

For a history of related results see introductions of [36] and [35]. In particular, in 1976 Hajnal and Juhász in [37] constructed the first example of such a group under the Continuum Hypothesis. In 2004 García-Ferreira, Tomita, and Watson in [38] obtained such a group from a selective ultrafilter. In 2009 Szeptycki and Tomita in [39] showed that in the Random model there exists such a group, giving the first example that does not depend on selective ultrafilters. Finally, Hrušák, van Mill, Ramos-García, and Shelah obtained a positive answer to van Douwen's question in ZFC, see [35] for "a presentable draft of the proof". The built group is Boolean and at p.17 the authors say that they do not know how to modify (in ZFC) their construction to get a non-torsion example and formulate the question about an existence of such a group in ZFC.

We denote by TT a bit stronger axiomatic assumption: there is an infinite torsion-free Abelian countably compact topological group without non-trivial convergent sequences. An example of such a group was constructed by Tkachenko [40] under the Continuum Hypothesis. Later, the Continuum Hypothesis was weakened to the Martin's Axiom for σ -centered posets by Tomita in [28], for countable posets in [41], and finally to the existence of continuum many incomparable selective ultrafilters in [42].

The proof of [43, Lemma 6.4] implies that under TT there exists a group topology on a free Abelian group F generated by the cardinal $\mathfrak{c}$ such that for each countable infinite subset M of the group F there exists an element $\alpha \in \overline{M} \cap \mathfrak{c}$ such that $M \subset \langle \alpha \rangle$. Let $S_0 \subset F$ be the free Abelian semigroup generated by the set $\mathfrak{c}$. The mentioned property of the group F implies that S_0 is countably compact. Let $g \in S_0$ be an arbitrary non-zero element and S be the closure in S_0 of the semigroup generated by g . Since the group F is countably compact, it is precompact and so topologically periodic. Then the element g is topologically periodic. Thus S is a countably compact monothetic topological monoid, which is contained in a topological group, but S is not a group, because $-g \in F \setminus S_0 \subset F \setminus S$.

But Tomita in [28] proved that if the power $S^{2^{\mathfrak{c}}}$ of a cancellative topological semigroup S is countably compact then S is a group. As far as we know, a question whether there exists a cancellative topological semigroup S such that S (or $S \times S$) is countably compact, but S is not a group, is still open in ZFC.

By $\mathfrak{u}$ we denote the smallest character on a free ultrafilter on ω and by $\mathfrak{p}$ the smallest character of a free filter $\mathcal{F}$ on ω that has no infinite pseudointersection (which is an infinite subset $I \subset \omega$ such that $I \setminus F$ is finite for every $F \in \mathcal{F}$). It is known [44, 6.15] that $\mathfrak{p} \leq \text{cf}(\mathfrak{c})$ and $\omega_1 \leq \mathfrak{p} \leq \mathfrak{u} \leq \mathfrak{c}$. Martin's Axiom implies $\mathfrak{p} = \mathfrak{u} = \mathfrak{c}$ (more precisely, $\mathfrak{p} = \mathfrak{c}$ is equivalent to $\text{MA}_{\sigma\text{-centered}}$, the Martin Axiom for σ -centered posets, see [44, 7.12]).

A topological space is called κ -compact for a cardinal κ if every its open cover of size $\leq \kappa$ has a finite subcover. In [28] Tomita proved that under $\mathfrak{u} = \omega_1$, each Tychonoff ω_1 -compact cancellative topological semigroup is a group. On the other hand, he showed that under $\mathfrak{c} = \mathfrak{p} > \omega_1$ (which is equivalent to $\text{MA}_{\sigma\text{-centered}} + \neg\text{CH}$), the topological group $\mathbb{T}^{\mathfrak{c}}$ contains an ω_1 -compact subsemigroup S_0 , which is not a group. Thus S_0 contains a non-periodic element g . Let S be the closure in S_0 of the semigroup generated by g . Since the group $\mathbb{T}^{\mathfrak{c}}$ is compact, it is topologically periodic. Then the element g is topologically periodic. Thus S is a ω_1 -compact monothetic topological monoid, which is contained in a topological group, but which is not a group, be-

cause $-g \in \mathbb{T}^c \setminus S_0 \subset \mathbb{T}^c \setminus S$. Therefore the answer to our question for Tychonoff ω_1 -compact semigroups is independent of ZFC.

Theorem 4.6(i) in [45] implies the following

Lemma 4.7. *A subgroup G of a compact semitopological semigroup S is a topological group.*

Remark that by Theorem 0.5 from [46], a cancellative compact semitopological semigroup is a topological group.

Pfister [47] showed that each T_3 locally countably compact paratopological group is a topological group. The following proposition generalizes this result a bit.

Let $\mathcal{C}$ be a class of spaces. A space is *locally $\mathcal{C}$* if each its point has a closed neighborhood in the class $\mathcal{C}$.

Proposition 4.8. *A subgroup G of a locally countably compact T_3 topological semigroup S with open left shifts is a topological group.*

Proof. In order to prove that G is a topological group it suffices to show that the inversion map on G is continuous at its identity e . Suppose to the contrary that there exists a neighborhood U of e such that for each neighborhood V of e there exists an element $g \in V \cap G$ with $g^{-1} \notin U$. Using that S is a T_3 topological semigroup, we can inductively construct a sequence $\{U_i\}$ of neighborhoods of e such that $\overline{U_0}$ is a countably compact subset of U and $\overline{U_{i+1}^2} \subset U_i$ for each i . By the assumption, for each i there exists a point $x_i \in U_i \cap G$ such that $x_i^{-1} \notin U$. For each natural k put $y_k = x_1 \cdots x_k \in U_1 \cdots U_k \subset U_1^2 \subset \overline{U_0}$. Let $y \in \overline{U_0}$ be a cluster point of the sequence $\{y_k\}$. There exists $k > 1$ such that $y_{k-1} \in yU_1$. Then $x_k^{-1} = y_k^{-1}y_{k-1} \in y_k^{-1}yU_1$. Clearly $y_k^{-1}y$ is a cluster point of the sequence $\{y_k^{-1}y_j : j \in \mathbb{N}\}$. If $j > k$ then $y_k^{-1}y_j = x_{k+1} \cdots x_j \in U_{k+1} \cdots U_j \subset U_k$, so we see that $y_k^{-1}y \in \overline{U_k} \subset U_{k-1}$. So $x_k^{-1} \in y_k^{-1}yU_1 \subset U_{k-1}U_1 \subset U_1U_1 \subset U_0 \subset U$, a contradiction with $x_k^{-1} \notin U$. $\square$

We recall that a group G endowed with a topology is *left (resp., right) precompact*, if for each neighborhood U of the identity of G there exists a finite subset F of G such that $FU = G$ (resp., $UF = G$). It is easy to check (see, for instance, [48, Pr.3.1] or [49, Pr.2.1]) that a paratopological group G is left precompact if and only if G is right precompact, so we shall often call left precompact paratopological groups *precompact*. Moreover, it is well-known [32] that a topological group G is precompact if and only if G is a subgroup of a compact topological group. Remark that a topological group G is precompact if and only if for any neighborhood U of the identity of the group G there exists a finite set $F \subset G$ such that $G = FUF$. This equivalence answers a question of Roelke and was rediscovered by many mathematicians: Uspenskij [50, 15.81], Solecki, Bouziad and Troallic [51, 4.3].

Proposition 4.9. *A precompact subgroup G of a locally compact semitopological semigroup S is a topological group.*

Proof. We have that $\overline{G}$ is a locally compact semitopological semigroup. Let U be a compact neighborhood (in $\overline{G}$) of the identity of the group G . Since the group G is precompact, there exists a finite subset F of the group G such that $G = F(U \cap G)$. Thus $FU \supset G$ and FU is a closed subset of $\overline{G}$, so $FU = \overline{G}$ and the set $\overline{G}$ is compact. Then by Lemma 4.7, G is a topological group. $\square$

Example 4.10. In [52, Ex. 3] it is constructed a monothetic second countable paratopological group G such that each power of G is countably precompact but G is not a topological group.

Proposition 4.9 does not extend to first countable countably precompact paratopological groups. Our example will be based on the group from [52, Example 3] and the embedding described at the end of the proof of [53, Theorem 3]. But for brevity and clarity we describe the full construction here. Let $S = \mathbb{T} \times \mathbb{T}$, τ be the usual topology on S , and $s : \mathbb{T} \rightarrow \mathbb{Q}$ be a (discontinuous) homomorphism onto the additive group of

rational numbers. For each natural n put

$$V_n = \{(g, h) \in \mathbb{T} \times \mathbb{T} : \max\{|\arg g|, |\arg h|\} < 1/n\} \in \tau,$$

$$U_n = \{(g, 1) \in \mathbb{T} \times \mathbb{T} : 0 \leq \arg g < 1/n\} \cup \{(g, h) \in V_n : s(h) > 0\}.$$

Observing that $\{1\} \cup \{h \in \mathbb{T} : s(h) > 0\}$ is a semigroup, it is easy to check that the family $\{U_n\}$ satisfies Pontrjagin conditions (see, [29, Proposition 1.1]) and it is a base at the identity of a first countable semigroup topology σ on the group S .

The subgroup $G = \mathbb{T} \times \{1\}$ of S is naturally topologically isomorphic to Sorgenfrey circle (see Example 3.17), so it is precompact.

We claim that the space (S, σ) is countably pracomact. To show this choose an element $b = e^{\psi i} \in \mathbb{T}$ such that $s(b) > 0$. Also pick $\varphi \in \mathbb{R} \setminus \psi\mathbb{Q}$ and put $a = e^{\varphi i}$. Put $A = \{n(a, b) : n \in \mathbb{N}\} \subset S$. Example 65 from [33] implies that the set A is dense in (S, τ) . Let $U \in \sigma$ be any non-empty set. The definition of the set U_n implies that there exists a non-empty set $V \in \tau$ and a number $k \in \mathbb{N}$ such that $\{(g, h) \in V : s(h) > k\} \subset U$. The density of A in (D, τ) implies that there exists $m \in \mathbb{N}$ such that $ms(b) > k$ and $m(a, b) \in V$. Then $m(a, b) \in U$. Thus the set A is dense in (S, σ) .

Now let B be any infinite subset of A . Let (g, h) be an accumulation point of B in (S, τ) . Let $(g, h) + U_n \in \sigma$ be a basic neighborhood of (g, h) . A set $B_n = ((g, h) + V_n) \cap B$ is infinite. Since $s(b) > 0$, the set $C_n = \{(c, d) \in B_n : s(d) \leq s(h)\}$ is finite. Since $B_n \setminus C_n \subset (g, h) + U_n$, we conclude that (g, h) is an accumulation point of B in (S, σ) .

There exists a pseudocompact quasitopological group G of period 2, which is not a paratopological group, see ([54], [55]). But Reznichenko in [46, Theorem 2.5] showed that each semitopological group $G \in \mathcal{N}$ is a topological group, where $\mathcal{N}$ is a family of all pseudocompact spaces X such that (X, X) is a *Grothendieck pair*, that is if each continuous image of X in $C_p(Y)$ has the compact closure in $C_p(Y)$. In particular, a pseudocompact space X belongs to $\mathcal{N}$ provided X has one of the following properties: countable compactness, countable tightness, separability, X is a k -space, see [46].

We can prove a straightforward generalization of Pontrjagin's theorem for compact-like groups.

Proposition 4.11. *Let $\mathcal{C}$ be a class of feebly compact spaces closed with respect to homeomorphisms. Also, assume that each space X belongs to $\mathcal{C}$ provided there exists finitely many its subspaces $X_1, \dots, X_n \in \mathcal{C}$ such that for any point $x \in X$ there exists X_i which is a neighborhood of x . Then each locally $\mathcal{C}$ monothetic topological G group is in $\mathcal{C}$.*

Proof. Theorem 6.9.27 from [32] implies that the Raïkov completion $\hat{G}$ of G is a locally compact topological group. Since $\hat{G}$ is monothetic, it is compact by Pontrjagin's theorem. The group G is precompact as a subgroup of a precompact topological group by Proposition 3.7.4 from [32]. Since the group G is locally $\mathcal{C}$, there exists an open neighborhood U of the identity e of G such that $\bar{U} \in \mathcal{C}$. Since the group G is precompact, there exists a finite subset F of G such that $G = FU$. Now the property of the class $\mathcal{C}$ implies $G \in \mathcal{C}$. $\square$

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