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A fixed point theorem involving rational expressions without using Picard iteration

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Abstract: In this paper, we consider a certain fixed point theorem that contains some rational expressions. The main aim of this paper is to prove a fixed point theorem without using the Picard iteration.

Keywords: Picard Iteration, fixed point

MSC: 47H10, 54H25, 46J10, 46J15

1 Introduction and Preliminaries

Let T be a self-mapping on a Banach space $(X, \|\cdot\|)$ and $x_0 \in X$. The sequence $\{x_p\}$ is called Picard sequence if $x_p = Tx_{p-1}$ for $n = 1, 2, \dots$. In this case, the mapping T is said to be Picard operator.

In [1], the authors set out a different general principle for obtaining fixed points. We recall the main results of their paper.

Theorem 1. [1] Let T be self-mapping on a nonempty closed convex subset C of a Banach space X . If for some $x_0 \in C$ there exists a constant c , $0 \leq c < 1$ such that

$$\|x_{p+2} - x_{p+1}\| \leq c\|x_{p+1} - x_p\| \text{ for all } p = 0, 1, 2, \dots, \quad (1.1)$$

where

$$x_{p+1} := \frac{1}{2}(x_p + Tx_p). \quad (1.2)$$

Then, $\{x_p\}$ converges to a point u in C . If, in addition, there exist the nonnegative constants $\alpha, \beta, \gamma, \delta$, $0 \leq \gamma < 1$ such that

$$\|Tx_p - Tu\| \leq \alpha\|x_p - u\| + \beta\|x_p - Tx_p\| + \gamma \max\{\|u - Tu\|, \|x_p - Tu\|, \delta\|u - Tx_p\|\} \quad (1.3)$$

for all p sufficiently large, then u is a fixed point of T .

A mapping $\varphi : [0, \infty) \rightarrow [0, \infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \rightarrow 0$, $n \rightarrow \infty$, for any $t \in [0, \infty)$. We denote by Φ , the class of the comparison function $\varphi : [0, \infty) \rightarrow [0, \infty)$. For more details and examples, see e.g. [2]–[6]. Among them, we recall the following essential result.

Lemma 2. (Berinde [2], Rus [3]) If $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a comparison function, then:

- (1) each iterate φ^k of φ , $k \geq 1$, is also a comparison function;
- (2) φ is continuous at 0;
- (3) $\varphi(t) < t$, for any $t > 0$.

Later, Berinde [2] introduced the concept of *(c)-comparison function* in the following way.

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Definition 3. (Berinde [2]) A function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is said to be a (c)-comparison function if
 (c₁) φ is monotone increasing,
 (c₂) the series $\sum_{j=1}^{\infty} \varphi^j(t)$ is convergent for any $t \in [0, \infty)$ (here φ^j denotes the j -th iteration of φ).

By Ψ we will denote the set of all (c)-comparison functions.

Very recently, Karapınar ([7]) announced the following interesting result.

Theorem 4. [7] On a nonempty closed convex subset C of a Banach space X , let T be a self-mapping. If for some $x_0 \in C$, there exists a $\varphi \in \Psi$ such that

$$\|x_{p+2} - x_{p+1}\| \leq \varphi(\|x_{p+1} - x_p\|) \text{ for all } p = 0, 1, 2, \dots, \quad (1.4)$$

where

$$x_{p+1} := \frac{1}{2}(x_p + Tx_p). \quad (1.5)$$

Then, $\{x_p\}$ converges to a point u in C . If, in addition, there exist the nonnegative constants $\alpha, \beta, \gamma, \delta$, $0 \leq \gamma < 1$, such that

$$\|Tx_p - Tu\| \leq \alpha\|x_p - u\| + \beta\|x_p - Tx_p\| + \gamma \max\{\|u - Tu\|, \|x - Tu\|, \delta\|u - Tx_p\|\}, \quad (1.6)$$

for all p sufficiently large, then u is a fixed point of T .

2 Main results

First, we demonstrate the following useful Lemma:

Lemma 5. Let C be a nonempty closed convex subset of a Banach space X and the sequence $\{x_p\}$ in C . If for some $x_0 \in C$ there exists a function $\varphi \in \Psi$ such that

$$\|x_{p+2} - x_{p+1}\| \leq \varphi(\|x_{p+1} - x_p\|) \text{ for all } p = 0, 1, 2, \dots, \quad (2.1)$$

then the sequence $\{x_p\}$ converges to a point $u \in C$.

Proof. Let $x_0 \in C$ be such that the inequality (2.1) holds. Thus, recursively, from the inequality (2.1) and taking into account the monotony of the function φ , it follows that

$$\|x_{p+1} - x_p\| \leq \varphi^p(\|x_1 - x_0\|) \text{ for all } p = 0, 1, 2, \dots \quad (2.2)$$

and by (c₂), we have that $\lim_{p \rightarrow \infty} \|x_{p+1} - x_p\| = 0$, so the sequence $\{x_p\}$ is asymptotically regular. Moreover, by the triangle inequality and taking (c₂) into account, we get

$$\begin{aligned} \|x_{p+n} - x_p\| &\leq \sum_{j=p}^{p+n-1} \|x_{j+1} - x_j\| \leq \sum_{j=p}^{p+n-1} \varphi^j(\|x_1 - x_0\|) \\ &= U_{n+p-1} - U_{p-1} \rightarrow 0 \text{ as } p \rightarrow \infty, \end{aligned} \quad (2.3)$$

where $U_p = \sum_{j=n}^p \varphi^j(\|x_1 - x_0\|)$. Thereupon, $\{x_p\}$ is a Cauchy sequence and since, the subset C is closed, we deduce that $\{x_p\}$ converges to a point in $u \in C$. $\square$

Theorem 6. Let C be a nonempty closed convex subset of a Banach space X , T be a self-mapping on C , x_0 be arbitrary in C and the sequence $\{x_n\}$ in C defined as

$$x_{p+1} := \frac{1}{2}(x_p + Tx_p). \quad (2.4)$$

Then the sequence $\{x_p\}$ is convergent provided that there exists $\varphi \in \Phi$ such that

$$\|x_{p+2} - x_{p+1}\| \leq \varphi(\|x_{p+1} - x_p\|) \text{ for all } p = 0, 1, 2, \dots. \quad (2.5)$$

Moreover if $u = \lim_{p \rightarrow \infty} x_p$ and there exist the constants $c_1, c_2 \geq 0$, $0 \leq \lambda < 1$, such that

$$\begin{aligned} \|Tx_p - Tu\| &\leq c_1 \|x_p - u\| + c_2 \max \left\{ \frac{\|x_p - Tx_p\|(1 + \|u - Tu\|)}{1 + \|x_p - u\|}, \|u - Tx_p\| \right\} \\ &\quad + \lambda \max \left\{ \frac{\|u - Tu\|(1 + \|x_p - Tx_p\|)}{1 + \|x_p - u\|}, \|x_p - Tu\| \right\} \end{aligned} \quad (2.6)$$

for p sufficiently large, then u is a fixed point of T .

Proof. Let $x_0 \in C$ and the sequence $\{x_p\}$ be defined by (2.4). Thus, keeping in mind (2.10), by Lemma 5 it follows that $\{x_p\}$ is a convergent sequence. Let $u \in C$ be the limit of x_p . We claim that u is a fixed point of the mapping T . Indeed, we can easily see that (2.4) can be rewritten as $Tx_p - x_p = 2(x_{p+1} - x_p)$. Therefore,

$$\|x_p - Tx_p\| = 2\|x_{p+1} - x_p\| \leq 2\varphi^p(\|x_1 - x_0\|) \rightarrow 0 \text{ as } p \rightarrow \infty$$

But, since $\lim_{p \rightarrow \infty} x_p = u$, taking into account the uniqueness of the limit we get that $\lim_{p \rightarrow \infty} Tx_p = u$. Now, letting the limit of (2.11) as $p \rightarrow \infty$, we have

$$\begin{aligned} \|u - Tu\| &= \lim_{p \rightarrow \infty} \|Tx_p - Tu\| \\ &\leq \lim_{p \rightarrow \infty} \left[c_1 \|x_p - u\| + c_2 \max \left\{ \|u - Tx_p\|, \frac{\|x_p - Tx_p\|(1 + \|u - Tu\|)}{1 + \|x_p - u\|} \right\} + \right. \\ &\quad \left. + \lambda \max \left\{ \|x_p - Tu\|, \frac{\|u - Tu\|(1 + \|x_p - Tx_p\|)}{1 + \|x_p - u\|} \right\} \right] \\ &= \lambda \|u - Tu\|, \end{aligned} \quad (2.7)$$

which is a contradiction. Thereupon, $u = Tu$. □

Adding a supplementary condition in Theorem 6, we can assure the uniqueness of the fixed point.

Theorem 7. *If in Theorem 6 the constants $c_1, c_2, \lambda \geq 0$ are such that $c_1 + c_2 + \lambda < 1$ then the fixed point of the mapping T is unique.*

Proof. Let $\{x_p\}$ be the sequence defined by (2.4). We know by the previous proof, that $\{x_p\}$ is convergent to a point $u \in C$. More than that,

$$\lim_{p \rightarrow \infty} x_p = \lim_{p \rightarrow \infty} Tx_p = u \text{ and } Tu = u.$$

Supposing than there exists a point $v \in C$ such that $Tv = v \neq u$, by (2.11) we have

$$\begin{aligned} \|Tx_p - v\| &= \|Tx_p - Tv\| \leq c_1 \|x_p - v\| + c_2 \max \left\{ \frac{\|x_p - Tx_p\|(1 + \|v - Tv\|)}{1 + \|x_p - v\|}, \|v - Tx_p\| \right\} \\ &\quad + \lambda \max \left\{ \frac{\|v - Tv\|(1 + \|x_p - Tx_p\|)}{1 + \|x_p - v\|}, \|x_p - Tv\| \right\} \\ &= c_1 \|x_p - v\| + c_2 \max \left\{ \frac{\|x_p - Tx_p\|(1 + \|v - v\|)}{1 + \|x_p - v\|}, \|v - Tx_p\| \right\} \\ &\quad + \lambda \max \left\{ \frac{\|v - v\|(1 + \|x_p - Tx_p\|)}{1 + \|x_p - v\|}, \|x_p - v\| \right\} \\ &= c_1 \|x_p - v\| + c_2 \max \left\{ \frac{\|x_p - Tx_p\|}{1 + \|x_p - v\|}, \|v - Tx_p\| \right\} \\ &\quad + \lambda \max \{0, \|x_p - v\|\}. \end{aligned} \quad (2.8)$$

Taking the limit as $p \rightarrow \infty$ in (2.8) we get

$$\|u - v\| \leq c_1\|u - v\| + c_2\|u - v\| + \lambda\|u - v\| = (c_1 + c_2 + \lambda)\|u - v\| < \|u - v\|.$$

This is a contradiction, so that $u = v$. $\square$

Corollary 8. Let C be a nonempty closed convex subset of a Banach space X , T be a self-mapping on C , x_0 be arbitrary in C and the sequence $\{x_n\}$ in C defined as

$$x_{p+1} := \frac{1}{2}(x_p + Tx_p). \quad (2.9)$$

Then the sequence $\{x_p\}$ is convergent provided that there exists $0 \leq \kappa < 1$ such that

$$\|x_{p+2} - x_{p+1}\| \leq \kappa \cdot \|x_{p+1} - x_p\| \text{ for all } p = 0, 1, 2, \dots. \quad (2.10)$$

Moreover if $u = \lim_{p \rightarrow \infty} x_p$ and there exist the constants $c_1, c_2 \geq 0$, $0 \leq \lambda < 1$, such that

$$\begin{aligned} \|Tx_p - Tu\| &\leq c_1\|x_p - u\| + c_2 \max \left\{ \frac{\|x_p - Tx_p\|(1 + \|u - Tu\|)}{1 + \|x_p - u\|}, \|u - Tx_n\| \right\} \\ &\quad + \lambda \max \left\{ \frac{\|u - Tu\|(1 + \|x_p - Tx_p\|)}{1 + \|x_p - u\|}, \|x_n - Tu\| \right\} \end{aligned} \quad (2.11)$$

for all p sufficiently large, then u is a fixed point of T .

Proof. Put $\varphi(t) = \kappa t$, with $0 \leq \kappa < 1$, in Theorem 6. $\square$

Corollary 9. Let C be a nonempty closed convex subset of a Banach space X , and $T, G : C \rightarrow C$ be two mappings such that $Gx = \frac{x+Tx}{2}$, for any $x \in C$. Supposing that there exists $0 < \gamma < 2$ such that

$$\|Tx - Ty\| \leq \gamma \max \left\{ \|x - y\|, \frac{1}{2} \frac{\|x - Tx\|(1 + \|y - Ty\|)}{1 + \|x - y\| + \|x - Ty\|}, \|y - Tx\|, \frac{1}{4} \frac{\|y - Ty\|(1 + \|x - Tx\|)}{1 + \|x - y\|}, \frac{1}{2} \|x - Ty\| \right\}, \quad (2.12)$$

for any $x, y \in C$. If there exist two positive real numbers a, b , with $a < \frac{2}{\gamma}$ and $b < 1$ such that

$$\begin{aligned} \|x - TGx\| &\leq a\|T^2x - TGx\|, \text{ for any } x \in C; \\ \|T^2x - \omega\| &\leq b\|x - \omega\|, \text{ for } \omega \in \{Tx, Gx\}, \end{aligned} \quad (2.13)$$

then there exists at least one point $u \in C$ such that $Tu = u$.

Proof. First of all, by considering the definition of the mapping G , we can easily obtain the following relations:

$$\begin{aligned} \|x - Tx\| &= \|x - 2Gx + x\| = 2\|Gx - x\|, \\ \|Gx - Tx\| &= \|Gx - 2Gx + x\| = \|Gx - x\|, \\ \|Gx - TGx\| &= \|Gx - 2G^2x + Gx\| = 2\|G^2x - Gx\|. \end{aligned} \quad (2.14)$$

Let x_0 be an arbitrary but fixed point in C and the sequence $\{x_p\}$ defined as follows:

$$x_1 = Gx_0, \dots, x_p = Gx_{p-1} = G^p x_0. \quad (2.15)$$

Using this notation, the relations (2.14) become

$$\begin{aligned} \|x_p - Tx_p\| &= 2\|x_p - x_{p+1}\|, \\ \|x_{p+1} - Tx_p\| &= \|x_{p+1} - x_p\|, \\ \|x_{p+1} - Tx_{p+1}\| &= 2\|x_{p+2} - x_{p+1}\|. \end{aligned} \quad (2.16)$$

Thus,

$$\begin{aligned}
 \|x_{p+2} - x_{p+1}\| &= \frac{1}{2} \|x_{p+1} - Tx_{p+1}\| = \frac{1}{4} \|2x_{p+1} - 2Tx_{p+1}\| = \frac{1}{4} \|(2x_{p+1} - Tx_{p+1}) - Tx_{p+1}\| \\
 &\leq \frac{1}{4} [\|2x_{p+1} - Tx_{p+1} - Tx_p\| + \|Tx_p - Tx_{p+1}\|] \\
 &= \frac{1}{4} [\|x_p + Tx_p - Tx_{p+1} - Tx_p\| + \|Tx_p - Tx_{p+1}\|] \\
 &= \frac{1}{4} [\|x_p - Tx_{p+1}\| + \|Tx_p - Tx_{p+1}\|] \\
 &\leq \frac{1}{2} \max \{ \|x_p - Tx_{p+1}\|, \|Tx_p - Tx_{p+1}\| \}.
 \end{aligned}$$

In case that $\max \{ \|x_p - Tx_{p+1}\|, \|Tx_p - Tx_{p+1}\| \} = \|Tx_p - Tx_{p+1}\|$, by (2.12) and keeping in mind (2.16), we obtain

$$\begin{aligned}
 \|x_{p+2} - x_{p+1}\| &\leq \frac{1}{2} \|Tx_p - Tx_{p+1}\| \\
 &\leq \frac{\gamma}{2} \max \left\{ \|x_p - x_{p+1}\|, \frac{1}{2} \frac{\|x_p - Tx_p\| (1 + \|x_{p+1} - Tx_{p+1}\|)}{1 + \|x_p - x_{p+1}\| + \|x_p - Tx_{p+1}\|}, \|x_{p+1} - Tx_p\|, \right. \\
 &\quad \left. \frac{1}{4} \frac{\|x_{p+1} - Tx_{p+1}\| (1 + \|x_p - Tx_p\|)}{1 + \|x_p - x_{p+1}\|}, \frac{1}{2} \|x_p - Tx_{p+1}\| \right\} \\
 &\leq \frac{1}{2} \max \left\{ \|x_p - x_{p+1}\|, \frac{1}{2} \frac{2 \|x_p - x_{p+1}\| (1 + \|x_{p+1} - x_p\| + \|x_p - Tx_{p+1}\|)}{1 + \|x_p - x_{p+1}\| + \|x_p - Tx_{p+1}\|}, \|x_{p+1} - x_p\|, \right. \\
 &\quad \left. \frac{1}{4} \frac{2 \|x_{p+1} - x_{p+2}\| (1 + \|x_p - x_{p+1}\|)}{1 + \|x_p - x_{p+1}\|}, \frac{1}{2} [\|x_p - x_{p+1}\| + \|x_{p+1} - Tx_{p+1}\|] \right\} \quad (2.17) \\
 &\leq \frac{\gamma}{2} \max \left\{ \|x_p - x_{p+1}\|, \|x_p - x_{p+1}\|, \|x_{p+1} - x_p\|, \right. \\
 &\quad \left. \frac{1}{2} \|x_{p+1} - x_{p+2}\|, \frac{1}{4} [\|x_p - x_{p+1}\| + 2 \|x_{p+1} - x_{p+2}\|] \right\} \\
 &\leq \frac{\gamma}{2} \max \left\{ \|x_p - x_{p+1}\|, \|x_p - x_{p+1}\|, \|x_{p+1} - x_p\|, \right. \\
 &\quad \left. \frac{1}{2} \|x_{p+1} - x_{p+2}\|, \frac{1}{2} \max \{ \|x_p - x_{p+1}\|, 2 \|x_{p+1} - x_{p+2}\| \} \right\} \\
 &\leq \frac{\gamma}{2} \max \{ \|x_p - x_{p+1}\|, \|x_{p+1} - x_{p+2}\| \}.
 \end{aligned}$$

Of course, since in case that $\max \{ \|x_p - x_{p+1}\|, \|x_{p+1} - x_{p+2}\| \} = \|x_{p+1} - x_{p+2}\|$, we get a contradiction, it follows that

$$\|x_{p+2} - x_{p+1}\| \leq \frac{\gamma}{2} \|x_{p+1} - x_p\|, \quad (2.18)$$

for $p \in \mathbb{N}$.

In the second case, when $\max \{ \|x_p - Tx_{p+1}\|, \|Tx_p - Tx_{p+1}\| \} = \|x_p - Tx_{p+1}\|$, we have

$$\|x_{p+2} - x_{p+1}\| \leq \frac{1}{2} \|x_p - Tx_{p+1}\|.$$

But, from (2.13), we get

$$\begin{aligned}
 \|x_p - Tx_{p+1}\| &\leq a \|T^2 x_p - Tx_{p+1}\|, \\
 \|T^2 x_p - Tx_p\| &\leq b \|x_p - Tx_p\| = 2b \|x_p - x_{p+1}\|, \\
 \|T^2 x_p - x_{p+1}\| &\leq b \|x_p - x_{p+1}\|,
 \end{aligned} \quad (2.19)$$

and then

$$\begin{aligned}
 \|x_{p+2} - x_{p+1}\| &= \frac{1}{2} \|x_p - Tx_{p+1}\| \leq \frac{a}{2} \|T^2 x_p - Tx_{p+1}\| \\
 &\leq \frac{a\gamma}{2} \max \left\{ \|Tx_p - x_{p+1}\|, \frac{1}{2} \frac{\|Tx_p - T^2 x_p\| (1 + \|x_{p+1} - Tx_{p+1}\|)}{1 + \|Tx_p - x_{p+1}\| + \|Tx_p - Tx_{p+1}\|}, \|x_{p+1} - T^2 x_p\|, \right. \\
 &\quad \left. \frac{1}{4} \frac{\|x_{p+1} - Tx_{p+1}\| (1 + \|Tx_p - T^2 x_p\|)}{1 + \|x_p - x_{p+1}\|}, \frac{1}{2} \|Tx_p - Tx_{p+1}\| \right\} \\
 &= \frac{a\gamma}{2} \max \left\{ \|x_p - x_{p+1}\|, \frac{1}{2} \frac{2b\|x_p - x_{p+1}\| (1 + \|x_{p+1} - Tx_p\| + \|Tx_p - Tx_{p+1}\|)}{1 + \|Tx_p - x_{p+1}\| + \|Tx_p - Tx_{p+1}\|}, \right. \\
 &\quad \left. b\|x_{p+1} - x_p\|, \frac{1}{4} \frac{2\|x_{p+1} - x_{p+2}\| (1 + 2b\|x_p - x_{p+1}\|)}{1 + \|x_p - x_{p+1}\|}, \right. \\
 &\quad \left. \frac{\gamma}{2} \max \{ \|x_p - x_{p+1}\|, \|x_{p+1} - x_{p+2}\| \} \right\} \\
 &\leq \frac{a\gamma}{2} \max \{ \|x_{p+1} - x_p\|, b\|x_{p+1} - x_p\|, \|x_{p+1} - x_{p+2}\| \} \\
 &= \frac{a\gamma}{2} \max \{ \|x_{p+1} - x_p\|, \|x_{p+1} - x_{p+2}\| \}.
 \end{aligned}$$

Consequently, since $\frac{a\gamma}{2} < 1$, we get

$$\|x_{p+1} - x_{p+2}\| \leq \frac{a\gamma}{2} \|x_{p+1} - x_p\|. \quad (2.20)$$

Thereupon, if we denote $\kappa = \max \{ \frac{\gamma}{2}, \frac{a\gamma}{2} \}$, from (2.18) and (2.18) we have

$$\|x_{p+1} - x_{p+2}\| \leq \kappa \|x_{p+1} - x_p\|, \quad (2.21)$$

for any $p \in \mathbb{N}$.

On the other hand, from the inequality (2.12), for $x = x_p$ and $y = u$, we get

$$\begin{aligned}
 \|Tx_p - Tu\| &\leq \gamma \max \left\{ \|x_p - u\|, \frac{1}{2} \frac{\|x_p - Tx_p\| (1 + \|u - Tu\|)}{1 + \|x_p - u\| + \|x_p - Tu\|}, \|u - Tx_p\|, \right. \\
 &\quad \left. \frac{1}{4} \frac{\|u - Tu\| (1 + \|x_p - Tx_p\|)}{1 + \|x_p - u\|}, \frac{1}{2} \|x_p - Tu\| \right\} \\
 &\leq \gamma \|x_p - u\| + \frac{\gamma}{2} \max \left\{ \frac{\|x_p - Tx_p\| (1 + \|u - Tu\|)}{1 + \|x_p - u\| + \|x_p - Tu\|}, \|u - Tx_p\| \right\} + \\
 &\quad + \frac{\gamma}{4} \max \left\{ \frac{\|u - Tu\| (1 + \|x_p - Tx_p\|)}{1 + \|x_p - u\|}, \|x_p - Tu\| \right\} \\
 &\leq \gamma \|x_p - u\| + \frac{\gamma}{2} \max \left\{ \frac{\|x_p - Tx_p\| (1 + \|u - Tu\|)}{1 + \|x_p - u\|}, \|u - Tx_p\| \right\} + \\
 &\quad + \frac{\gamma}{4} \max \left\{ \frac{\|u - Tu\| (1 + \|x_p - Tx_p\|)}{1 + \|x_p - u\|}, \|x_p - Tu\| \right\}.
 \end{aligned}$$

Moreover, choosing $c_1 = \gamma$, $c_2 = \frac{\gamma}{2}$ and $\lambda = \frac{\gamma}{4}$, for p sufficiently large, we get (2.11) and then, by Theorem 6 it follows that u is a fixed point of T . □

Corollary 10. Let C be a nonempty closed convex subset of a Banach space X , and $T : C \rightarrow C$ be a mapping such that $T^2 = I$. If there exists $0 \leq \gamma < 2$ such that

$$\|Tx - Ty\| \leq \gamma \max \left\{ \|x - y\|, \frac{1}{2} \frac{\|x - Tx\| (1 + \|y - Ty\|)}{1 + \|x - y\|}, \|y - Tx\|, \right. \\
 \left. \frac{1}{4} \frac{\|y - Ty\| (1 + \|x - Tx\|)}{1 + \|x - y\|}, \frac{1}{2} \|x - Ty\| \right\}, \quad (2.22)$$

for any $x, y \in C$ then we can find at least one point $u \in C$ such that $Tu = u$.

Proof. Let x_0 be an arbitrary but fixed point in C and the sequence $\{x_p\}$ defined by (2.15). On the other hand, by (2.16) we have:

$$\begin{aligned}\|x_p - Tx_p\| &= 2\|x_p - x_{p+1}\|, \\ \|x_{p+1} - Tx_p\| &= \|x_{p+1} - x_p\|, \\ \|x_{p+1} - Tx_{p+1}\| &= 2\|x_{p+2} - x_{p+1}\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|x_{p+2} - x_{p+1}\| &= \frac{1}{2}\|x_{p+1} - Tx_{p+1}\| = \frac{1}{4}\|2x_{p+1} - 2Tx_{p+1}\| = \frac{1}{4}\|(2x_{p+1} - Tx_{p+1}) - Tx_{p+1}\| \\ &\leq \frac{1}{4}[\|2x_{p+1} - Tx_{p+1} - Tx_p\| + \|Tx_p - Tx_{p+1}\|] \\ &= \frac{1}{4}[\|x_p + Tx_p - Tx_{p+1} - Tx_p\| + \|Tx_p - Tx_{p+1}\|] \\ &= \frac{1}{4}[\|x_p - Tx_{p+1}\| + \|Tx_p - Tx_{p+1}\|] \\ &\leq \frac{1}{2} \max \{ \|x_p - Tx_{p+1}\|, \|Tx_p - Tx_{p+1}\| \},\end{aligned}$$

and we distinguish two cases:

Case 1. $\max \{ \|x_p - Tx_{p+1}\|, \|Tx_p - Tx_{p+1}\| \} = \|Tx_p - Tx_{p+1}\|$.

Then, by (2.22), keeping in mind (2.16) and following the same arguments as in Corollary 9 we get

$$\begin{aligned}\|x_{p+2} - x_{p+1}\| &\leq \frac{1}{2}\|Tx_p - Tx_{p+1}\| \\ &\leq \frac{\gamma}{2} \max \{ \|x_p - x_{p+1}\|, \|x_{p+1} - x_{p+2}\| \} \\ &\leq \frac{\gamma}{2} \|x_{p+1} - x_p\|,\end{aligned}\tag{2.23}$$

for $p \in \mathbb{N}$ (since $\frac{\gamma}{2} < 1$).

Case 2. $\max \{ \|x_p - Tx_{p+1}\|, \|Tx_p - Tx_{p+1}\| \} = \|x_p - Tx_{p+1}\|$. We have

$$\begin{aligned}\|x_{p+2} - x_{p+1}\| &\leq \frac{1}{2}\|x_p - Tx_{p+1}\| = \frac{1}{2}\|T^2x_p - Tx_{p+1}\| \\ &\leq \frac{\gamma}{2} \max \left\{ \|Tx_p - x_{p+1}\|, \frac{1}{2} \frac{\|Tx_p - T^2x_p\|(1 + \|x_{p+1} - Tx_{p+1}\|)}{1 + \|Tx_p - x_{p+1}\| + \|Tx_p - Tx_{p+1}\|}, \|x_{p+1} - T^2x_p\|, \right. \\ &\quad \left. \frac{1}{4} \frac{\|x_{p+1} - Tx_{p+1}\|(1 + \|Tx_p - T^2x_p\|)}{1 + \|x_p - x_{p+1}\|}, \frac{1}{2}\|Tx_p - Tx_{p+1}\| \right\} \\ &= \frac{\gamma}{2} \max \left\{ \|x_p - x_{p+1}\|, \frac{1}{2} \frac{\|Tx_p - x_p\|(1 + \|x_{p+1} - Tx_p\| + \|Tx_p - Tx_{p+1}\|)}{1 + \|Tx_p - x_{p+1}\| + \|Tx_p - Tx_{p+1}\|}, \right. \\ &\quad \left. \|x_{p+1} - x_p\|, \frac{1}{4} \frac{2\|x_{p+1} - x_{p+2}\|(1 + \|Tx_p - x_p\|)}{1 + \|x_p - x_{p+1}\|}, \right. \\ &\quad \left. \frac{\gamma}{2} \max \{ \|x_p - x_{p+1}\|, \|x_{p+1} - x_{p+2}\| \} \right\} \\ &\leq \frac{\gamma}{2} \max \{ \|x_{p+1} - x_p\|, \|x_{p+1} - x_p\|, \|x_{p+1} - x_{p+2}\| \} \\ &= \frac{\gamma}{2} \max \{ \|x_{p+1} - x_p\|, \|x_{p+1} - x_{p+2}\| \}.\end{aligned}$$

Consequently, since $\frac{\gamma}{2} < 1$, we get

$$\|x_{p+1} - x_{p+2}\| \leq \frac{\gamma}{2} \|x_{p+1} - x_p\|.\tag{2.24}$$

and letting $\kappa = \max \{ \frac{\gamma}{2}, \frac{\alpha\gamma}{2} \}$, from (2.23) and (2.24) we have

$$\|x_{p+1} - x_{p+2}\| \leq \kappa \|x_{p+1} - x_p\|,\tag{2.25}$$

for any $p \in \mathbb{N}$.

Again, letting $x = x_p$ and $y = u$ in (2.22), we get

$$\begin{aligned} \|Tx_p - Tu\| &\leq \mathcal{T} \max \left\{ \|x_p - u\|, \frac{1}{2} \frac{\|x_p - Tx_p\|(1 + \|u - Tu\|)}{1 + \|x_p - u\| + \|x_p - Tu\|}, \|u - Tx_p\|, \right. \\ &\quad \left. \frac{1}{4} \frac{\|u - Tu\|(1 + \|x_p - Tx_p\|)}{1 + \|x_p - u\|}, \frac{1}{2} \|x_p - Tu\| \right\} \\ &\leq \mathcal{T} \|x_p - u\| + \frac{\mathcal{T}}{2} \max \left\{ \frac{\|x_p - Tx_p\|(1 + \|u - Tu\|)}{1 + \|x_p - u\| + \|x_p - Tu\|}, \|u - Tx_p\| \right\} + \\ &\quad + \frac{\mathcal{T}}{4} \max \left\{ \frac{\|u - Tu\|(1 + \|x_p - Tx_p\|)}{1 + \|x_p - u\|}, \|x_p - Tu\| \right\} \\ &\leq \mathcal{T} \|x_p - u\| + \frac{\mathcal{T}}{2} \max \left\{ \frac{\|x_p - Tx_p\|(1 + \|u - Tu\|)}{1 + \|x_p - u\|}, \|u - Tx_p\| \right\} + \\ &\quad + \frac{\mathcal{T}}{4} \max \left\{ \frac{\|u - Tu\|(1 + \|x_p - Tx_p\|)}{1 + \|x_p - u\|}, \|x_p - Tu\| \right\}. \end{aligned}$$

Moreover, choosing $c_1 = \mathcal{T}$, $c_2 = \frac{\mathcal{T}}{2}$ and $\lambda = \frac{\mathcal{T}}{4}$, for p sufficiently large, we get (2.11) and then, by Theorem 6 it follows that u is a fixed point of T . □

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