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Genericity of chaos for colored graphs

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Abstract: To each colored graph one can associate its closure in the universal space of isomorphism classes of pointed colored graphs, and this subspace can be regarded as a generalized subshift. Based on this correspondence, we introduce two definitions for chaotic (colored) graphs, one of them analogous to Devaney's. We show the equivalence of our two novel definitions of chaos, proving their topological genericity in various subsets of the universal space.

Keywords: graph coloring, symbolic dynamics, Cantor set, subshift, chaos theory

MSC2010: 37B10, 37D45, and 05C15

1 Introduction

Chaotic dynamical systems are one of the central objects in modern mathematics. The aim of this paper is to study chaotic dynamical systems in a setting that generalizes both symbolic dynamics and one-dimensional foliated spaces: the universal space of pointed, colored graphs. In what follows, we offer a very brief account on both these mathematical fields.

Given a countable group G and a finite set F of colors, G acts naturally on the left on the Cantor space F^G of F -valued colorings of G by the formula $(g \cdot \phi)(g') = \phi(g^{-1}g')$. The subsets of F^G that are both closed and saturated are called *subshifts*, and they constitute the main subject of study in symbolic dynamics [27]; in particular, for every coloring $\phi: G \rightarrow F$, the closure of its orbit under the G -action is a subshift. Properties similar to those that characterize chaos in a continuous setting have been studied for many years in the context of symbolic dynamics (see e.g. [16]); for example, it was shown in [21] that, for F finite, F^G has density of periodic orbits if and only if G is residually finite.

An n -dimensional *foliated space* or *lamination* is a topological space endowed with a partition into connected topological manifolds (the *leaves*) of dimension n ; furthermore, we require that this partition is locally modeled on the partition of $\mathbb{R}^n \times Z$ into the equivalence classes $\mathbb{R}^n \times \{z\}$, where Z is some locally compact separable metric space (See [15, Ch. 11] for a detailed definition). Foliated spaces can be regarded as dynamical objects where the leaves play the role that in a classical dynamical system would belong to the orbits; they are usually modeled by their holonomy pseudogroups (see [15, Sec. 2.2]). In a similar vein, we may also consider *laminations by graphs*, where, instead of topological manifolds, the leaves are discrete sets endowed with a graph structure. Laminations by graphs may be used as discrete and simpler models of foliated spaces; e.g., they were implicitly used by Ghys [22] to obtain examples of laminations by Riemann surfaces. Laminations by graphs have also been studied from a dynamical point of view (see [2, 9, 28]); in particular, laminations with chaotic properties were constructed in [29].

We can generalize symbolic dynamical systems by using graphs in a way similar to [2, 22, 28, 29]: Consider the *Gromov space of pointed colored graphs*, denoted by $\mathfrak{G}_*$ (we use this notation because of its similarity

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with the Gromov space of pointed metric spaces, see [10]). As a set, $\widehat{\mathcal{G}}_*$ consists of the pointed isomorphism classes $[X, x, \phi]$ of triples (X, x, ϕ) , where X is a connected graph with finite vertex degrees, $x \in V(X)$ is a distinguished vertex ($V(X)$ denotes the vertex set of X), and $\phi: V(X) \rightarrow \mathcal{E}$ is a (vertex-)coloring. By an isomorphism $h: (X, x, \phi) \rightarrow (Y, y, \psi)$ we mean a graph isomorphism $h: X \rightarrow Y$ such that $h(x) = y$ and $\phi = \psi \circ h$. We assume for the purposes of this paper that $\mathcal{E}$ is a Cantor space. In [3] the reader can enjoy a nice survey on both properties of $\widehat{\mathcal{G}}_*$ and open problems on the subject.

Associated to every colored graph (X, ϕ) , there is a canonical map $\ell_{X, \phi}: V(X) \rightarrow \widehat{\mathcal{G}}_*$ defined by the formula $x \mapsto [X, x, \phi]$. Let $[X, \phi]$ denote the isomorphism class of the colored graph (X, ϕ) (see Section 3.2), so that we have $\text{im}(\ell_{X, \phi}) = \text{im}(\ell_{Y, \psi})$ if $[X, \phi] = [Y, \psi]$, and $\text{im}(\ell_{X, \phi}) \cap \text{im}(\ell_{Y, \psi}) = \emptyset$ otherwise. By abuse of notation, we may identify the subset $\text{im}(\ell_{X, \phi})$ and the equivalence class $[X, \phi]$, and we will consider both to be endowed with the canonical graph structure of the quotient $X/\text{Aut}(X, \phi)$ (Sec. 3.2). Therefore $\widehat{\mathcal{G}}_*$ is a space partitioned into graphs, where each “leaf” corresponds to an equivalence class $[X, \phi]$. This is analogous to the smooth case, where one considers a Gromov space of Riemannian manifolds (see [1, 5, 6]). It is easy to see that the partition of $\widehat{\mathcal{G}}_*$ is not locally a product, and thus, properly speaking, $\widehat{\mathcal{G}}_*$ is not a lamination by graphs.

Let us explain how we can formalize the notion of chaotic colored graphs in this context. Based on the preceding discussion, one can associate to each colored graph (X, ϕ) the closed, saturated subset $\overline{[X, \phi]} \subset \widehat{\mathcal{G}}_*$, which can be regarded as a generalized dynamical system. The topology on $\widehat{\mathcal{G}}_*$ is given by the following notion of convergence: Let (X, x, ϕ) be a pointed, colored graph, and let $D_X(x, r)$ denote the *disk* or *closed ball* of center x and radius r . Then $[X, x, \phi]$ is the limit of the sequence $[X_n, x_n, \phi_n]$ if, for every $r \in \mathbb{N}$ and $\epsilon > 0$, there are graph isomorphisms

$$h_n: D_{X_n}(x_n, r) \rightarrow D_X(x, r),$$

defined for n large enough, so that $h_n(x_n) = x$ and $d(\phi_n(y), \phi(h_n(y))) \leq \epsilon$ for all $y \in D_{X_n}(x_n, r)$.

Our setting is a generalization of symbolic dynamics in the following sense: If we fix the underlying graph to be the Cayley graph of a countable group G with a system of generators, then we can canonically identify each class $[G, g, \phi]$ with $[G, e_G, g^{-1} \cdot \phi]$. Note that we are considering Cayley graphs with unlabeled edges. If we restrict our attention to graph isomorphisms given by translations (instead of the full isomorphism group of the Cayley graph), then we recover the notion of subshift.

One can define analogously a space $\mathcal{G}_*$ of isomorphism classes of pointed graphs $[X, x]$ with canonical injections $\ell_X: V(X) \rightarrow \mathcal{G}_*$. All the preceding considerations apply in a similar manner.

Before introducing our definitions of chaos for graphs, let us recall what is perhaps the most commonly used definition of chaos:

Definition 1.1 (Devaney chaos [20]). Let (X, d) be a metric space. A map $f: X \rightarrow X$ is *chaotic* if it satisfies the following three conditions:

1. for any non-empty open $U, V \subset X$, there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ (f is *topologically transitive*);
2. the periodic orbits are dense in X (f has *density of periodic orbits*); and
3. there is $c > 0$ such that, for every $x \in X$ and $r > 0$, there are $y \in X$ and $n \in \mathbb{N}$ satisfying

$$d(x, y) < r \quad \text{and} \quad d(f^n(x), f^n(y)) \geq c$$

(f is *sensitive to initial conditions*).

This definition of chaos can be generalized to group actions $G \curvearrowright (X, d)$ by replacing f^n ($n \in \mathbb{N}$) with $g \in G$. Chaotic group actions have been studied from various points of view: see [13, 16, 30, 32–34] for the study on general theory and [12, 14, 31] for the construction of chaotic group actions on various spaces.

It was proved shortly after this definition was introduced that the third condition actually follows from the first two [7]; that is, sensitivity to initial conditions is redundant in the definition of Devaney chaos. This result was later generalized to groups and semigroups acting on metric spaces [26, 33]; in particular, this shows that the definition of chaos in [11] is equivalent to Devaney chaos. For this reason, some authors have used definitions of chaos that only require (1) and (2), particularly in settings where sensitivity to initial conditions might not be easy to formulate; as an example, Churchill [18] and Bazaikin, Galaev, and Zhukova [8] have

defined chaotic foliations to be those with a dense leaf and a dense subset of compact (respectively, closed) leaves. This discussion leads us to introduce two different definitions of chaos for (colored) graphs, one of them including sensitivity to initial conditions and the other one ignoring it.

We begin with the definition that mirrors that of [8, 18]. Following the heuristic rule that orbits of dynamical systems should correspond to leaves, the following notion corresponds to periodic orbits in our context: a colored graph (X, ϕ) is *quasi-transitive* if $[X, \phi]$ is a finite set. By abuse of terminology, we will also refer to the orbit $[X, \phi]$ as quasi-transitive.

Definition 1.2. A colored graph (X, ϕ) is *chaotic* if the union of quasi-transitive classes is dense in $[X, \phi]$ and (X, ϕ) itself is not quasi-transitive.

Our second notion of chaotic graphs is more involved because, given a colored graph (X, ϕ) , the closure $\overline{[X, \phi]}$ may not be a genuine lamination by graphs in general, so the holonomy pseudogroup cannot be defined in the usual way; this means that we need to obtain another dynamical model for $\overline{[X, \phi]}$. In Section 4 we show how to associate to $\overline{[X, \phi]}$ a semigroup $\mathcal{P}_{X, \phi}$ in a way similar to the holonomy pseudogroup of a foliated space. Then we may think of dynamical properties of $\mathcal{P}_{X, \phi}$ as attributes of $[X, \phi]$, since of course the latter determines the former. It is easy to see that $\mathcal{P}_{X, \phi}$ has no periodic orbits, so it is reasonable to consider the density of quasi-transitive orbits instead of the density of periodic orbits. All these considerations guide us to the following principle: we will say that $[X, \phi]$ is a chaotic colored graph if $\mathcal{P}_{X, \phi}$ satisfies topological transitivity, density of quasi-transitive orbits and sensitivity on initial conditions. We call a colored graph (X, ϕ) *Devaney chaotic* if the semigroup $\mathcal{P}_{X, \phi}$ associated to (X, ϕ) satisfies these three conditions of Devaney (Definition 4.7). Our first main contribution is showing that these two notions of chaos are, in fact, equivalent:

Theorem 1.3. A colored graph (X, ϕ) is chaotic if and only if it is Devaney chaotic.

The second main contribution of the paper is studying the topological genericity of the aforementioned notion of chaos for several subsets of $\widehat{\mathcal{G}}_*$. We will use the following notation: Let $\Delta \in \mathbb{N}$, let $\widehat{\mathcal{G}}_*(\Delta)$ denote the subset of classes whose underlying graph X satisfies $\deg X \leq \Delta$, and similarly for $\mathcal{G}_*(\Delta)$. We also add ∞ as a superscript to denote the subset of classes whose underlying graph is infinite. Finally, we say that a colored graph (X, ϕ) is *aperiodic* if the map $\ell_{X, \phi}: V(X) \rightarrow \widehat{\mathcal{G}}_*$ is injective, or, equivalently, $\text{Aut}(X, \phi) = \{\text{id}_X\}$; we can define aperiodicity for (uncolored) graphs analogously. Again, we refer to the class $[X, \phi]$ as aperiodic if (X, ϕ) is. It follows trivially from the definitions that, for X infinite, aperiodicity implies that X is not quasi-transitive.

Recall that a property is (topologically) generic if it holds on a *residual subset*, where a subset is residual if it contains a countable intersection of open dense sets. This notion is well-behaved for *Baire spaces*. *Polish spaces*, i.e. separable and completely metrizable spaces, are all Baire spaces by the Baire Category Theorem. The following theorem is the main result of the paper.

Theorem 1.4. The following subsets are generic in the respective ambient spaces:

- (a) the chaotic classes in $\mathcal{G}_*^\infty$, $\mathcal{G}_*^\infty(\Delta)$, $\widehat{\mathcal{G}}_*^\infty$ and $\widehat{\mathcal{G}}_*^\infty(\Delta)$, for $\Delta \geq 3$;
- (b) the aperiodic classes in $\widehat{\mathcal{G}}_*^\infty$, $\widehat{\mathcal{G}}_*^\infty(\Delta)$; and
- (c) the aperiodic chaotic classes in $\widehat{\mathcal{G}}_*^\infty$ and $\widehat{\mathcal{G}}_*^\infty(\Delta)$ for $\Delta \geq 3$.

All the subspaces appearing in Theorem 1.4 are Polish by Lemma 3.4, and in fact $\mathcal{G}_*^\infty(\Delta)$ and $\widehat{\mathcal{G}}_*^\infty(\Delta)$ are Cantor spaces. The reason for imposing $\Delta \geq 3$ is the following: For $\Delta = 1$ we have $\widehat{\mathcal{G}}_*^\infty(1) = \emptyset$, so that the corresponding result is vacuously true. For $\Delta = 2$, all connected infinite graphs are isomorphic to either $\mathbb{N}$ or $\mathbb{Z}$ with the obvious graph structures. Since the graph $\mathbb{N}$ is obviously aperiodic, the quasi-transitive classes cannot be dense in either $\widehat{\mathcal{G}}_*(2)$ nor $\mathcal{G}_*(2)$. Therefore, Theorem 1.4 (a) is false for $\Delta = 2$.

Remark 1. It is common to see in the literature a variation of $\widehat{\mathcal{G}}_*$ consisting of classes of pointed graphs endowed with both a vertex coloring and an edge coloring, see e.g. [3]. The definitions of quasi-transitive, aperi-

riodic, chaotic and chaotic classes can be obviously translated to this setting, and the same applies to the statement of Theorem 1.4. The proof presented hereafter can be adapted without any difficulty.

2 Examples

The concepts that we have introduced thus far can be illustrated by the following simple examples, the first of which is inspired by Champernowne's number [17].

Example 2.1. Consider the lexicographical order on the set of finite sequences of 0s and 1s:

$$0 < 1 < 00 < 01 < 11 < 000 < 001 < \dots$$

We concatenate these sequences in order to construct the infinite sequence

$$a_0 a_1 a_2 a_3 a_4 a_5 a_6 \dots = 01000111000001 \dots$$

Let ϕ be the coloring of $\mathbb{Z}$ with two colors, 0 and 1, defined by

$$\phi(n) = \begin{cases} 0 & \text{for } n < 0, \\ a_n & \text{for } n \geq 0. \end{cases}$$

Then ϕ is a chaotic coloring of the integers. Indeed, since the non-positive integers determine the only infinite ray colored by 0 and adjacent to a 1, it is clear that $(\mathbb{Z}, \phi)$ is aperiodic. To see that $(\mathbb{Z}, \phi)$ is chaotic it is enough to note that, for every finite word w with values in $\{0, 1\}$, every concatenation w^n will eventually appear on a_n .

Figure 1: An illustration of Example 2.1

From the last example $(\mathbb{Z}, \phi)$, we can construct a chaotic (uncolored) graph X . Such graph X is obtained by adding a free edge to each vertex of $\mathbb{Z}$ colored with 1 and by forgetting the coloring ϕ .

The following is a straightforward generalization of Example 2.1.

Example 2.2. Let $C \subset \mathcal{E}$ be a countable subset with at least two elements. Let

$$\mathcal{W} = \{c_{i_1} c_{i_2} \dots c_{i_k} \mid c_{i_j} \in C\}$$

be the set of finite words over the alphabet C . Fix a well-order¹ $\mathcal{W} = \{w_n\}_{n \in \mathbb{N}}$ and take $c_0 \in C$. Now we concatenate the elements of $\mathcal{W}$ using the well-order to define a coloring ϕ on $\mathbb{Z}$. Namely,

$$\phi(n) = \begin{cases} c_0 & \text{for } n < 0, \\ w_m(i) & \text{for } n = i + \sum_{l < m} |w_l|, 1 \leq i \leq |w_m|; \end{cases}$$

where $|w|$ denotes the length of the corresponding finite word. The proof that ϕ is chaotic proceeds as in Example 2.1.

¹ Recall that a well-order is a total order such that every non-empty subset has a least element.

Finally, we present an example of a colored graph whose isomorphism class is dense in $\widehat{\mathcal{G}}_\star^\infty$.

Example 2.3. Let $C \subset \mathcal{E}$ be a countable, dense subset. Consider the set of isomorphism classes of aperiodic pointed colored graphs with finite vertex sets and whose colorings take values in C , and let P be a set of pointed colored graphs containing exactly one representative of each such class. Since every vertex set is finite, the number of graph structures on a finite vertex set is also finite, and the set of colors is countable, P is a countable set. Hence, we can choose an enumeration of the form $P = \{(Y_i, y_i, \phi_i)\}_{i \in \mathbb{Z}}$. Let $\psi: \mathbb{Z} \rightarrow C$ be a bijective coloring, which is *a fortiori* aperiodic, and let X be the colored graph constructed as follows: take the disjoint union of $(\mathbb{Z}, \psi)$ and the family $\{(Y_i, y_i, \phi_i)\}_{i \in \mathbb{Z}}$, and add an edge joining y_i to $i \in \mathbb{Z}$ for every $i \in \mathbb{Z}$.

It is easy to see that (X, ϕ) is aperiodic: Note that $\mathbb{Z}$ is the unique bi-infinite simple path in X and therefore it is preserved by any automorphism f of X . Since $\phi|_{\mathbb{Z}} = \psi$ is aperiodic, f fixes each vertex of $\mathbb{Z}$ and preserves (Y_i, y_i) for each i . Each (Y_i, y_i, ϕ_i) is aperiodic, and therefore so is (X, ϕ) . By construction, (X, ϕ) has an isomorphic copy of every aperiodic colored finite graph as a subgraph, from which we conclude that $\overline{[X]} = \widehat{\mathcal{G}}_\star^\infty$. So (X, ϕ) is chaotic if and only if the quasi-transitive classes are dense in $\widehat{\mathcal{G}}_\star^\infty$. This can be proved directly by using a construction similar to that illustrated in Figure 2. Since the result also follows trivially from Theorem 1.4 (a), we leave the details of the proof to the interested reader.

3 Preliminaries on graphs and colorings

3.1 Graphs

An undirected graph $X \equiv (V(X), E(X))$ consists of a set $V(X)$ and a family $E(X)$ of subsets $e \subset X$ with² $|e| = 2$. The elements of $V(X)$ and $E(X)$ are called *vertices* and *edges*, respectively. If an edge e contains a vertex x , it is said that e and x are *incident*. The *degree* $\deg x$ of a vertex x is the number of edges incident to x . The *degree* of X is $\deg X = \sup_{x \in V(X)} \deg x$. Two different vertices are *adjacent* if they define an edge, and two different edges are *consecutive* if they have a common vertex. For $n \in \mathbb{N}$, a *path of length n* from x to y in X is a sequence of n consecutive edges joining x to y ; in terms of their vertices, it can be considered as a sequence $(z_0, \dots, z_n)$, where $z_0 = x$, $z_n = y$, and z_{i-1} and z_i are adjacent vertices for all $i = 1, \dots, n$. If any two vertices of X can be joined by a path, then X is called *connected*. A *subgraph* $Y \equiv (V(Y), E|_Y)$ of X , denoted by $Y \subset X$, is a graph such that $V(Y) \subset V(X)$ and endowed with the induced graph structure $E|_Y = \{\{x, y\} \in E \mid x, y \in V(Y)\}$.

Let $X' \equiv (X', E')$ be another graph. A bijection $f: V(X) \rightarrow V(X')$ is an *isomorphism (of graphs)* if it induces a bijection $E \rightarrow E'$; we will use the notation $f: X \rightarrow Y$ for an isomorphism. Given distinguished points, $x_0 \in V(X)$ and $x'_0 \in V(X')$, a (*pointed*) *isomorphism* $f: (X, x_0) \rightarrow (X', x'_0)$ is an isomorphism $f: X \rightarrow X'$ satisfying $f(x_0) = x'_0$. If there is an isomorphism $X \rightarrow X'$ (respectively, $(X, x_0) \rightarrow (X', x'_0)$), then these structures are called *isomorphic*, and the notation $X \cong X'$ (respectively, $(X, x_0) \cong (X', x'_0)$) may be used. The composition of isomorphisms is another isomorphism. An isomorphism $X \rightarrow X$ (respectively, $(X, x_0) \rightarrow (X, x_0)$) is called an *automorphism* of X (respectively, (X, x_0)). The group of automorphisms of X (respectively, (X, x_0)) is denoted by $\text{Aut}(X)$ (respectively, $\text{Aut}(X, x_0)$).

For the purposes of this paper, we will only consider connected graphs with finite vertex degrees. In this case there is a canonical metric structure d , where $d(x, y)$ is the minimum length of paths in X from x to y . For convenience, we will use *disks* or *closed balls* defined with non-strict inequalities, $D(x, r) = \{y \in X \mid d(x, y) \leq r\}$. Also, let $S(x, r) = \{y \in X \mid d(x, y) = r\}$ denote the *sphere* of center x and radius r . We may add the ambient metric space or graph as a subscript " $D_X(x, r)$ " to avoid ambiguity. The following basic result will be used implicitly throughout the paper.

² The cardinality of a set X is denoted by $|X|$.

Lemma 3.1. *If X is connected and has finite vertex degrees, then its disks are finite³ and X is countable.*

3.2 Colorings

Let $\mathcal{E}$ be a Cantor space; namely, a compact, metrizable, perfect, and totally disconnected topological space. Fix an ultrametric $d_{\mathcal{E}}$ inducing the topology of $\mathcal{E}$, we will denote $d_{\mathcal{E}}$ simply by d when no confusion may arise. Let X be a graph. A map $\phi : V(X) \rightarrow \mathcal{E}$ is called a (vertex) coloring of X , and (X, ϕ) is called a colored graph. If $Y \subset X$, then the simplified notation $(Y, \phi) = (Y, \phi|_{V(Y)})$ will be used. The following concepts for colored graphs are the obvious extensions of their graph versions: (pointed) isomorphisms, denoted by $f : (X, \phi) \rightarrow (X', \phi')$ and $f : (X, x_0, \phi) \rightarrow (X', x'_0, \phi')$, isomorphic (pointed) colored graphs, denoted by $(X, \phi) \cong (X', \phi')$ and $(X, x_0, \phi) \cong (X', x'_0, \phi')$, and automorphism groups of (pointed) colored graphs, denoted by $\text{Aut}(X, \phi)$ and $\text{Aut}(X, x_0, \phi)$. Let G be either $\text{Aut}(X, \phi)$ and $\text{Aut}(X, x_0, \phi)$, then $V(X)/G$ has a canonical graph structure; namely, two different classes $[x]$ and $[y]$ are adjacent if there are $u \in [x]$ and $v \in [y]$ such that u and v are adjacent in X .

For $n \in \mathbb{N}$ and $\epsilon > 0$, an (n, ϵ) -equivalence $h : (X, x, \phi) \rightarrow (Y, y, \psi)$ is a pointed graph isomorphism $h : (D_X(x, n), x) \rightarrow (D_Y(y, n), y)$ such that $d(\phi(u), \psi(h(u))) < \epsilon$ for every $u \in D_X(x, n)$. We use the modified arrow $\rightarrow$ to emphasize that h is actually a partial map. It will be said that $(X, x, \phi), (Y, y, \psi) \in \hat{\mathcal{G}}_*$ are (n, ϵ) -equivalent if there is an (n, ϵ) -equivalence $h : (X, x, \phi) \rightarrow (Y, y, \psi)$. The following lemma follows immediately from the ultrametric triangle inequality.

Lemma 3.2. *If $f : (X_1, x_1, \phi_1) \rightarrow (X_2, x_2, \phi_2)$ and $g : (X_2, x_2, \phi_2) \rightarrow (X_3, x_3, \phi_3)$ are (n, ϵ) and (m, δ) -equivalences, respectively, then $g \circ f : (X_1, x_1, \phi_1) \rightarrow (X_3, x_3, \phi_3)$ is a $(\min\{n, m\}, \max\{\epsilon, \delta\})$ -equivalence, where $\text{dom}(g \circ f) = D_{X_1}(x_1, \min\{n, m\})$.*

Let $\hat{\mathcal{G}}_*$ be the set⁴ of isomorphism classes, $[X, x, \phi]$, of pointed connected colored graphs, (X, x, ϕ) , whose vertices have finite degree. For $n \in \mathbb{N}$ and $\epsilon > 0$, let

$$\hat{\mathcal{N}}_{n, \epsilon} = \{ ([X, x, \phi], [Y, y, \psi]) \in \hat{\mathcal{G}}_*^2 \mid [X, x, \phi] \text{ and } [Y, y, \psi] \text{ are } (n, \epsilon)\text{-equivalent} \}.$$

Remark 2. Since the property of being (n, ϵ) -equivalent is clearly invariant by colored graph isomorphisms, it makes sense to say whether two classes are (n, ϵ) -equivalent or not. Thus the subsets $\hat{\mathcal{N}}_{n, \epsilon}$ are well-defined. In the same way, we will speak indistinctly of properties of (pointed, colored) graphs or of isomorphism classes when these properties are invariant by isomorphisms.

The sets $\hat{\mathcal{N}}_{n, 1/n}$, $n \in \mathbb{Z}^+$, form a base of entourages of a uniformity on $\hat{\mathcal{G}}_*$ that is easily seen to be complete. Since this base is countable, the induced uniformity is metrizable. A metric inducing this uniformity is given by

$$d([X, x, \phi], [Y, y, \psi]) = 2^{-n},$$

where n is the largest integer such that $([X, x, \phi], [Y, y, \psi]) \in \hat{\mathcal{N}}_{n, 1/n}$. Lemma 3.2 shows that d is actually an ultrametric.

Lemma 3.3. *$\hat{\mathcal{G}}_*$ and $\mathcal{G}_*$ are totally disconnected, separable, and completely metrizable spaces, hence Polish. Moreover, $\hat{\mathcal{G}}_*^\infty$ and $\mathcal{G}_*^\infty$ are also perfect.*

Proof. By the preceding discussion, $\hat{\mathcal{G}}_*$ and $\mathcal{G}_*$ are totally disconnected and completely metrizable spaces. To prove separability, note that elements $[X, x, \phi]$ with both $V(X)$ and $\text{im } \phi$ finite form a countable dense subset.

³ This means that X is a proper metric space, in the sense that its closed balls are compact.

⁴ These graphs are countable, and therefore we can assume that their underlying sets are contained in $\mathbb{N}$. In this way, $\hat{\mathcal{G}}_*$ becomes a well defined set.

The subspaces $\widehat{\mathcal{G}}_\star^\infty$ and $\mathcal{G}_\star^\infty$ are totally disconnected because this is a hereditary property, and separable because separability is hereditary for metric spaces. In order to prove that $\widehat{\mathcal{G}}_\star^\infty$ is perfect we can modify $[X, x, \phi]$ in a region arbitrarily far from x , thus obtaining a non-constant sequence converging to $[X, x, \phi]$. The proof for $\mathcal{G}_\star$ proceeds similarly. $\square$

Removing the colorings from the notation, we get the Polish space $\mathcal{G}_\star$ of isomorphism classes of pointed connected graphs. In this way we also obtain canonical maps $\ell_X : V(X) \rightarrow \mathcal{G}_\star$ for connected graphs X , defining a canonical partition of $\mathcal{G}_\star$ by the sets $[X] := \text{im}(\ell_X)$. Note that the forgetful map $\hat{p} : \widehat{\mathcal{G}}_\star \rightarrow \mathcal{G}_\star$ is continuous. The space $\mathcal{G}_\star$ is a subspace of the Gromov space $\mathcal{M}_\star$ of isomorphism classes of pointed proper metric spaces [23], [24, Chapter 3].

Lemma 3.4. *The topological spaces $\mathcal{G}_\star^\infty(\Delta)$ and $\widehat{\mathcal{G}}_\star^\infty(\Delta)$ are totally disconnected, separable, perfect, compact, and metrizable, and therefore Cantor spaces.*

Proof. The spaces $\mathcal{G}_\star^\infty(\Delta)$ and $\widehat{\mathcal{G}}_\star^\infty(\Delta)$ are totally disconnected because $\widehat{\mathcal{G}}_\star$ and $\mathcal{G}_\star$ are by Lemma 3.3 and this is a hereditary property. In fact, $\mathcal{G}_\star^\infty(\Delta)$ and $\widehat{\mathcal{G}}_\star^\infty(\Delta)$ are closed subsets of $\mathcal{G}_\star$ and $\widehat{\mathcal{G}}_\star$, respectively, hence Polish by Lemma 3.3 and [19, Prop. 8.1.2]. To prove compactness note that, for every $n \in \mathbb{N}$, the disks $(D_X(x, n), x)$ with $[X, x] \in \mathcal{G}_\star(\Delta)$ represent only finitely many isomorphism classes. Therefore, for every sequence $[X_n, x_n]$ in $\mathcal{G}_\star^\infty(\Delta)$, we can find a convergent subsequence by a diagonal argument. Finally, the compactness of $\mathcal{E}$ ensures that, given colorings ϕ_n , we can find a subsequence $n(k)$ such that $[X_{n(k)}, x_{n(k)}, \phi_{n(k)}]$ converges in $\widehat{\mathcal{G}}_\star^\infty(\Delta)$. $\square$

Unraveling the definition of the topology of $\widehat{\mathcal{G}}_\star$, we obtain the following characterization for chaotic pointed colored graphs.

Lemma 3.5. *Let (X, x, ϕ) be a pointed colored graph which is not quasi-transitive. Then (X, ϕ) is chaotic if and only if there is a sequence of quasi-transitive pointed colored graphs (Y_n, y_n, ψ_n) satisfying:*

- (a) (X, x, ϕ) is $(n, 1/n)$ -equivalent to (Y_n, y_n, ψ_n) , and
- (b) for all $n, m \in \mathbb{Z}_+$, there is some $x_{n,m} \in V(X)$ such that $(X, x_{n,m}, \phi)$ is $(m, 1/m)$ -equivalent to (Y_n, y_n, ψ_n) .

4 Chaotic graphs in the sense of Devaney

In this section we introduce a notion analogous to Devaney chaos for colored graphs. The main difficulty in this task is obtaining a suitable dynamical model for the closure $\overline{[X, \phi]}$. We begin by showing why the definition of a pseudogroup used in [2, 9, 22, 28, 29] to study the dynamics on the space of pointed graphs with bounded vertex degrees and coloring of edges by a finite number of colors is not suitable for our setting, and then continue by introducing a semigroup action associated to the colored graph. Finally, we spell out our definition of Devaney chaos for colored graphs, showing in Theorem 1.3 that this is actually equivalent to Definition 1.2.

Let us provide a brief sketch of the pseudogroup model for the graph matchbox manifold used in [29]: Let G be the Cayley graph of a finitely generated group with respect to a finite set of generators, and let $\mathcal{T}$ be the set of subtrees of G that contain the identity. By a subtree we mean a subgraph that is itself a tree; i.e., any two vertices are connected by a unique path. This space $\mathcal{T}$ has a natural topology induced by a metric [29, Def. 2.1]; intuitively, two trees are close if they agree on very large balls in G centered at the identity element. We consider every tree in $\mathcal{T}$ as a pointed tree with the identity element in G as the distinguished point, denoting thus the elements of $\mathcal{T}$ as pairs (T, e) . Let $g \in G$, then g induces a homeomorphism between open subsets of $\mathcal{T}$ as follows: $g(T, e)$ is defined if and only if T contains the vertex of G that corresponds to g , and $g(T, e) = (T', e)$ if and only if the left translation by g , considered as a graph automorphism of G , induces a pointed graph isomorphism $(T, e) \rightarrow (T', g)$. The homeomorphisms between open subsets of $\mathcal{T}$ induced by the elements of G generate a pseudogroup in $\mathcal{T}$.

We may think of the maps defined in the previous paragraph as being the composition of two transformations: first, we move the distinguished point e across an edge $\{e, g\}$, $(T, e) \mapsto (T, g)$; and then we send the distinguished point back to e by left multiplication, $(T, g) \mapsto (g^{-1}(T), e)$. The second step is irrelevant for our purposes, since we are considering graphs with arbitrary distinguished points. The first transformation, however, needs a way of identifying edges for graphs that are sufficiently close.

We show now why this approach does not work in our setting: Consider a pointed colored graph (X, x, ϕ) , and take a vertex $x' \in V(X)$ adjacent to x . We want to define a local map $f_{x,x'}$ from an open neighborhood of $[X, x, \phi]$ in $\widehat{\mathcal{G}}_*$ to an open neighborhood of $[X, x', \phi]$ in $\widehat{\mathcal{G}}_*$ such that $f_{x,x'}([X, x, \phi]) = [X, x', \phi]$. This is the same as saying that we want to be able to identify the edge $\{x, x'\}$ with edges in neighboring graphs, perhaps only on a small open neighborhood of $[X, x, \phi]$. Recall that a neighborhood system for $[X, x, \phi]$ of $\widehat{\mathcal{G}}_*$ is given by

$$\widehat{\mathcal{N}}_{n,\epsilon}[X, x, \phi] = \{[Y, y, \psi] \in \widehat{\mathcal{G}}_* \mid [Y, y, \psi] \text{ is } (n, \epsilon)\text{-equivalent to } [X, x, \phi]\},$$

with $n \in \mathbb{N}^5$ and $\epsilon > 0$. Given $n \geq 2$, $\epsilon > 0$, and $[Y, y, \psi] \in \widehat{\mathcal{N}}_{n,\epsilon}[X, x, \phi]$, we may tentatively set $f_{x,x'}([Y, y, \psi]) = [Y, h(x'), \psi]$, where h is an (n, ϵ) -equivalence $[X, x, \phi] \rightarrow [Y, y, \psi]$. However, we can see that this $f_{x,x'}$ is not well-defined if there is an automorphism σ of (X, x, ϕ) such that $\sigma(x') \neq x'$ and $\text{Aut}(Y, y, \psi) = \{e\}$. In this case, there are two equivalences $h, h \circ \sigma: [X, x, \phi] \rightarrow [Y, y, \psi]$ such that $h(x') \neq h \circ \sigma(x')$ and $[Y, h(x'), \psi] \neq [Y, h \circ \sigma(x'), \psi]$.

This discussion shows that, for an approach similar to [2, 9, 22, 28, 29] to work in our setting, we need some sort of ambient rigid structure. For example, in [29], this came from considering subgraphs of a fixed graph. We introduce now a rigid version of $\widehat{\mathcal{G}}_*$ consisting of graphs with a fixed vertex set. Let $\mathcal{S}_*^\infty$ be the set of all pointed colored connected infinite graphs with vertex set $\mathbb{N}$; thus, elements of $\mathcal{S}_*^\infty$ are of the form (E, x, ϕ) , where E is the edge set of a graph structure on $\mathbb{N}$; thus, elements of $\mathcal{S}_*^\infty$ are of the form (E, x, ϕ) , where E is the edge set of a graph structure on $\mathbb{N}$, $\phi: \mathbb{N} \rightarrow \mathcal{E}$ is a coloring, and $x \in \mathbb{N}$ is a distinguished point. Let $\mathfrak{S}_\mathbb{N}$ be the group of bijections from $\mathbb{N}$ to itself. We have an obvious action of $\mathfrak{S}_\mathbb{N}$ on $\mathcal{S}_*^\infty$ given by $g(E, x, \phi) = (gE, g(x), \phi \circ g^{-1})$ for $g \in \mathfrak{S}_\mathbb{N}$ and $(E, x, \phi) \in \mathcal{S}_*^\infty$, where

$$gE = \{\{g(y), g(z)\} \mid \{y, z\} \in E\}.$$

Note that, for every $[X, x, \phi] \in \widehat{\mathcal{G}}_*^\infty$, there is a bijection between $V(X)$ and $\mathbb{N}$, so there is some element $(E, y, \psi) \in \mathcal{S}_*^\infty$ with $[(\mathbb{N}, E), x, \psi] = [X, x, \phi]$. Moreover, two elements $(E, x, \phi), (F, y, \psi) \in \mathcal{S}_*^\infty$ satisfy $[(\mathbb{N}, E), x, \phi] = [(\mathbb{N}, F), y, \psi]$ if and only if they lie on the same $\mathfrak{S}_\mathbb{N}$ -orbit. Thus, we may identify $\widehat{\mathcal{G}}_*^\infty$ with the quotient $\mathcal{S}_*^\infty / \mathfrak{S}_\mathbb{N}$, obtaining a canonical map

$$\pi: \mathcal{S}_*^\infty \rightarrow \mathcal{S}_*^\infty / \mathfrak{S}_\mathbb{N} = \widehat{\mathcal{G}}_*^\infty.$$

Given $(E, x, \phi) \in \mathcal{S}_*^\infty$ and $n \in \mathbb{N}$, let $D_E(x, n)$ denote the closed ball of center x and radius n with respect to the metric in $\mathbb{N}$ induced by the graph structure E (note that, in general, this metric does not coincide with the usual metric on $\mathbb{N}$). For $n \in \mathbb{N}$ and $\epsilon > 0$, let $\mathcal{V}_{n,\epsilon}$ be the set of all pairs $((E_0, x_0, \phi_0), (E_1, x_1, \phi_1)) \in \mathcal{S}_*^\infty \times \mathcal{S}_*^\infty$ such that

- $x_0 = x_1$,
- $D_{E_0}(x_0, n) = D_{E_1}(x_1, n)$,
- the restrictions of E_0 and E_1 to $D_{E_0}(x_0, n)$ coincide, and
- $d(\phi_0(y), \phi_1(y)) < \epsilon$ for any y in $D_{E_0}(x_0, n)$.

Remark 3. Note that we have defined $\mathcal{S}_*^\infty$ so that we are only considering pointed colored graphs with fixed vertex set $\mathbb{N}$, so the first and second conditions require equality as elements and subsets of $\mathbb{N}$, respectively. Similarly, since E_0 and E_1 are edge sets on $\mathbb{N}$, their restrictions to $D_{E_0}(x_0, n)$ are subsets of the powerset of $D_{E_0}(x_0, n)$, so the third condition requires that these subsets are equal.

We consider the uniformity on $\mathcal{S}_*^\infty$ whose base is $\{\mathcal{V}_{n,\epsilon}\}_{n \in \mathbb{N}, \epsilon > 0}$. It is easy to see that the $\mathfrak{S}_\mathbb{N}$ -action is uniformly equicontinuous.

For $n \geq 1$, $\epsilon > 0$, $(E, x, \phi) \in \mathcal{S}_*^\infty$ and a vertex x' E -adjacent to x , we can define $f_{x,x',n,\epsilon,E} : \mathcal{S}_*^\infty \rightarrow \mathcal{S}_*^\infty$ as follows:

$$f_{x,x',n,\epsilon,E}(F, y, \psi) = \begin{cases} (F, x', \psi) & \text{if } (F, y, \psi) \in \mathcal{V}_{n,\epsilon}(E, x, \phi), \\ (F, y, \psi) & \text{else.} \end{cases}$$

This map is continuous because $\mathcal{V}_{n,\epsilon}(E, x, \phi)$ is clopen. Let $\mathcal{P}$ be the semigroup acting on $\mathcal{S}_*^\infty$ generated by all $f_{x,x',n,\epsilon,E}$ and the $\mathfrak{S}_\mathbb{N}$ -action. The next lemma follows immediately from the definition:

Lemma 4.1. *The $\mathcal{P}$ -orbit of (E, x, ϕ) is $\bigcup_{y \in \mathbb{N}} \mathfrak{S}_\mathbb{N}(E, y, \phi)$. In particular, $\pi(\mathcal{P}(E, x, \phi)) = [(\mathbb{N}, E), \phi]$.*

Lemma 4.2. *If $(E, x, \phi) \in \mathcal{S}_*^\infty$ is such that $[(\mathbb{N}, E), \phi]$ is quasi-transitive, then its $\mathcal{P}$ -orbit is closed.*

Proof. Consider a sequence of elements in the $\mathcal{P}$ -orbit of (E, x, ϕ) , which by Lemma 4.1 may be written as $g_n(E, x_n, \phi)$ for some sequences $g_n \in \mathfrak{S}_\mathbb{N}$, $x_n \in \mathbb{N}$, and assume that it converges to some $(F, y, \phi) \in \mathcal{S}_*^\infty$. Since $[(\mathbb{N}, E), \phi]$ is quasi-transitive, we may assume that the sequence x_n attains only finitely many different values; hence, after passing to a subsequence, we may furthermore assume that x_n is the constant sequence z for some $z \in \mathbb{N}$.

By the preceding paragraph, there is a sequence $g_n(E, z, \phi)$ converging to (F, y, ψ) . By the definition of $\mathcal{V}_{n,\epsilon}$, for every $m > 0$ and n large enough, we have that

- $y = g_n(z)$,
- $D_F(y, m) = g_n(D_E(z, m))$,
- the restrictions of F and $g_n E$ to $D_F(y, m)$ coincide, and
- $d(\phi(u), \phi(g_n(u))) < 1/m$ for every $u \in D_E(z, m)$.

A simple diagonal argument now yields the existence of a bijection $g \in \mathfrak{S}_\mathbb{N}$ such that $g(E, z, \phi) = (F, y, \psi)$. Hence (F, y, ψ) is in the $\mathcal{P}$ -orbit of (E, x, ϕ) . Since (F, y, ψ) was an arbitrary point in the closure of $\mathcal{P}(E, x, \phi)$, the orbit is closed. $\square$

Let (X, ϕ) be an connected colored infinite graph. Fix a bijection from $V(X)$ to $\mathbb{N}$. Choose a distinguished point x , and let (E, x, ϕ) denote the corresponding element of $\mathcal{S}_*^\infty$. Let $\mathcal{P}_{X,\phi}$ be the restriction of $\mathcal{P}$ to the closure $C_{X,\phi}$ of the $\mathcal{P}$ -orbit of (E, x, ϕ) . Obviously $C_{X,\phi}$ and $\mathcal{P}_{X,\phi}$ are independent of the choice of x and the bijection from $V(X)$ to $\mathbb{N}$. We will use the semigroup action $\mathcal{P}_{X,\phi} \curvearrowright C_{X,\phi}$ as a dynamical model for $[X, \phi]$; the reader familiar with the subject can recognize that we are using ideas similar to the definition of the holonomy pseudogroup of a foliation.

In the introduction, we defined chaotic colored graphs (X, ϕ) as those exhibiting density of quasi-transitive classes in $[X, \phi]$ while not being quasi-transitive. Before giving our definition of Devaney chaotic colored graphs, let us adapt Definition 1.1 to the setting of semigroup actions [33]:

Definition 4.3. Let $\mathcal{Q}$ be a topological semigroup continuously acting on a uniform space T .

1. $\mathcal{Q}$ is said to be *topologically transitive* if, for any nonempty open sets U, V in T , there exists $\gamma \in \mathcal{Q}$ such that $\gamma(U) \cap V \neq \emptyset$.
2. A $\mathcal{Q}$ -orbit O is called *periodic* if the stabilizer S of O in $\mathcal{Q}$ is right syndetic; namely, there exists a compact set K in $\mathcal{Q}$ such that $KS = K^{-1}S = \mathcal{Q}$. Here $K^{-1}S = \{\gamma \in \mathcal{Q} \mid \exists k \in K, k\gamma \in S\}$. We say that $\mathcal{Q}$ has *density of periodic orbits* if the union of all periodic $\mathcal{Q}$ -orbits is dense in T .
3. $\mathcal{Q}$ has *sensitive dependence on initial conditions* if there exists an entourage U on T such that $\forall x \in T$ and every neighborhood N of x in T , there exist $y \in N$ and $\gamma \in \mathcal{Q}$ such that $(\gamma x, \gamma y) \notin U$.

It is easy to check that none of the $\mathcal{P}$ -orbit of $\mathcal{S}_*^\infty$ is periodic, and therefore Definition 4.3(2) is never satisfied. Hence, in order to define chaotic graphs, we need to modify the condition of density of periodic orbits. We can see that this phenomenon comes from the fact that the map $\mathcal{S}_*^\infty \rightarrow \widehat{\mathcal{G}}_*^\infty$ is not proper. Instead of the

compactness of the $\mathcal{P}$ -orbits in $\mathcal{S}_*^\infty$, we consider the compactness in $\widehat{\mathcal{G}}_*^\infty$. Since $[Y, \psi]$ is compact if and only if it is finite, we can regard the orbits corresponding to quasi-transitive graphs as periodic orbits. So, instead of density of periodic orbits, we will consider the following condition.

Definition 4.4. We say that $(E, x, \phi) \in \mathcal{S}_*^\infty$ is quasi-transitive if the underlying colored graph $((\mathbb{N}, E), \phi)$ is quasi-transitive. Then $\mathcal{P}_{X, \phi}$ has *density of quasi-transitive graphs* if the union of quasi-transitive graphs is dense in $C_{X, \phi}$.

We observe the following.

Lemma 4.5. An element $(E, x, \phi) \in \mathcal{S}_*^\infty$ is quasi-transitive if and only if there exists a finite subset K in $\mathcal{P}$ such that $(K\mathfrak{S}_{\mathbb{N}})^{-1}S = \mathcal{P}$, where S is the stabilizer of (E, x, ϕ) in $\mathcal{P}$.

Proof. Suppose first that (E, x, ϕ) is quasi-transitive, then there is a finite set $\{x_i\}_{i=1}^N$ with $x_1 = x$ and such that $[(\mathbb{N}, E), \phi] = \{[(\mathbb{N}, E), x_n, \phi]\}_{n=1}^N$. For every $n \in \{1, \dots, N\}$, choose a path

$$\tau_n := (y_0 = x_n, y_1, \dots, y_{L-1}, y_L = x)$$

in $(\mathbb{N}, E)$. Then let k_n be a composition of the form $h_L \circ \dots \circ h_1$, where $h_i = f_{y_{i-1}, y_i, m, \epsilon, E}$ for some $m \geq 1, \epsilon > 0$. Let p be an arbitrary map in $\mathcal{P}$. By Lemma 4.1, there is some $s \in \mathfrak{S}_{\mathbb{N}}$ such that $sp(E, x, \phi) = (E, x_n, \phi)$ for some $n \in \{1, \dots, N\}$. Since x_n and x are the initial and ending points of τ_n , we have $k_n(E, x_n, \phi) = (E, x, \phi)$, hence $k_nsp(E, x, \phi) = (E, x, \phi)$. Since p was arbitrary, we get $(K\mathfrak{S}_{\mathbb{N}})^{-1}S = \mathcal{P}$ by letting $K = \{k_n\}_{n=1}^N$.

Suppose now that there is a finite set K such that $(K\mathfrak{S}_{\mathbb{N}})^{-1}S = \mathfrak{S}_{\mathbb{N}}^{-1}(K^{-1}S) = \mathcal{P}$. Since elements in $\mathfrak{S}_{\mathbb{N}}$ have inverses, $\mathfrak{S}_{\mathbb{N}}^{-1}(K^{-1}S) = \mathfrak{S}_{\mathbb{N}}(K^{-1}S)$. Hence $\mathcal{P}(E, x, \phi) \subset \mathfrak{S}_{\mathbb{N}}K^{-1}(E, x, \phi)$, so $\pi(\mathcal{P}(E, x, \phi)) \subset \pi(K^{-1}(E, x, \phi))$. But $K^{-1}(E, x, \phi)$ is finite, and now the result follows from Lemma 4.1. $\square$

Corollary 4.6. For $(E, x, \phi) \in \mathcal{S}_*^\infty$, if $((\mathbb{N}, E), \phi)$ is quasi-transitive, then there is an equicontinuous subset $H \subset \mathcal{P}$ such that $H^{-1}S = \mathcal{P}$.

Proof. Since $\mathfrak{S}_{\mathbb{N}}$ acts equicontinuously on $\mathcal{S}_*^\infty$ and K is finite, $K\mathfrak{S}_{\mathbb{N}}$ is equicontinuous. $\square$

Now we introduce Devaney chaotic colored graphs.

Definition 4.7. We call a connected colored infinite graph (X, ϕ) *Devaney chaotic* if the associated semigroup $\mathcal{P}_{X, \phi}$ satisfies the topological transitivity, density of quasi-transitive graphs, and sensitivity on initial conditions.

Note that, by construction, $\mathcal{P}_{X, \phi}$ always has a dense orbit and is topologically transitive. We define Devaney chaos for uncolored graphs similarly. Note that a colored graph is chaotic (Definition 1.2) if and only if the associated semigroup satisfies density of quasi-transitive graphs. Since Devaney chaotic colored graphs are not quasi-transitive, they are chaotic.

Now let us prove Theorem 1.3, which says that two notions of chaotic colored graphs are equivalent. Let (X, ϕ) be a connected colored infinite graph, and consider the semigroup $\mathcal{P}_{X, \phi}$; as mentioned above, $\mathcal{P}_{X, \phi}$ is always topologically transitive. On the other hand, Banks-Brooks-Cairns-Davis-Stacey [7] proved that, for continuous group actions on metric spaces, topological transitivity and density of periodic orbits imply sensitive dependence on initial conditions. Schneider-Kerckhoff-Behrisch-Siegmund [33] generalized this result to continuous actions of semigroups on uniform Hausdorff spaces. Combining the argument of [33] with basic properties of quasi-transitive graphs (Lemma 4.2, Corollary 4.6), we obtain the following.

Proposition 4.8. If (X, ϕ) is not quasi-transitive and $\mathcal{P}_{X, \phi}$ has density of quasi-transitive graphs, then $\mathcal{P}_{X, \phi}$ has sensitive dependence on initial conditions.

Proof. The proof is similar to [33, Theorem 4.7]. In this proof we denote the elements of $\mathcal{S}_*^\infty$ by Greek letters. Let d be a $\mathfrak{S}_{\mathbb{N}}$ -invariant distance on $\mathcal{S}_*^\infty$ inducing the uniformity, and take any $\alpha \in \mathcal{S}_*^\infty$. Let us show first that there

exist $\epsilon > 0$ and a quasi-transitive graph $\beta_0 \in C_{X,\phi}$ such that $d(\alpha, \mathcal{P}_{X,\phi}(\beta_0)) \geq 4\epsilon$. Since (X, ϕ) is not quasi-transitive and $\mathcal{P}_{X,\phi}$ has density of quasi-transitive graphs, there exists a quasi-transitive graph $\beta_0 \in C_{X,\phi}$ whose orbit is not equal to $C_{X,\phi}$. By Lemma 4.2, the orbit of β_0 is closed, so, by density of quasi-transitive graphs, we have a quasi-transitive graph $\beta'_0 \in C_{X,\phi}$ whose orbit is disjoint from the orbit of β_0 . Therefore, we have either $d(\alpha, \mathcal{P}_{X,\phi}(\beta_0)) \geq 4\epsilon$ or $d(\alpha, \mathcal{P}_{X,\phi}(\beta'_0)) \geq 4\epsilon$ for ϵ small enough, which implies the claim.

Take any open neighborhood U of α in $\mathcal{S}_*^\infty$ and let $V = U \cap D_{\mathcal{S}_*^\infty}(\alpha, \epsilon)$. By the density of quasi-transitive graphs, there exists a quasi-transitive $\gamma \in V$. We have by Corollary 4.6 an equicontinuous subset $H \subset \mathcal{P}_{X,\phi}$ such that $H^{-1}S = \mathcal{P}_{X,\phi}$, where S is the stabilizer of γ . Also, by the last paragraph, there exists $\epsilon > 0$ and a quasi-transitive graph $\beta \in C_{X,\phi}$ such that $d(\alpha, \mathcal{P}_{X,\phi}(\beta)) \geq 4\epsilon$. The equicontinuity of H now yields the existence of some open neighborhood W of β such that $tW \subset D_{\mathcal{S}_*^\infty}(t\beta, \epsilon)$ for every $t \in H$. Since $\mathcal{P}_{X,\phi}$ is topologically transitive, there are $\delta \in V$ and $g \in \mathcal{P}_{X,\phi}$ with $g\delta \in W$. But $H^{-1}S = \mathcal{P}_{X,\phi}$, so we have $tg \in S$ for some $t \in H$, and then we have $d(tg\delta, t\beta) < \epsilon$ because $g\delta \in W$. Assume that we have $d(tg\alpha, tg\delta) < \epsilon$ and $d(tg\alpha, tg\gamma) < \epsilon$. Since $tg\gamma = \gamma$, it follows that

$$d(\alpha, t\beta) \leq d(\alpha, \gamma) + d(tg\gamma, tg\alpha) + d(tg\alpha, tg\delta) + d(tg\delta, t\beta) < 4\epsilon,$$

which contradicts the fact that $d(\alpha, \mathcal{P}_{X,\phi}(\beta)) \geq 4\epsilon$. Therefore we have either $d(tg\alpha, tg\delta) \geq \epsilon$ or $d(tg\alpha, tg\gamma) \geq \epsilon$, which shows that $\mathcal{P}_{X,\phi}$ has sensitive dependence on initial conditions. $\square$

For a colored graph (X, ϕ) that is not quasi-transitive, it is clear that $\mathcal{P}_{X,\phi}$ has density of quasi-transitive graphs if and only if the union of quasi-transitive graphs in $[X, \phi]$ is dense in $[X, \phi]$. Thus, the last proposition implies that chaotic graphs are Devaney chaotic. Therefore the proof of Theorem 1.3 is concluded.

5 Proof of Theorem 1.4

Let us start by proving Theorem 1.4 (a). Let $\Delta \geq 3$ and fix an arbitrary $\xi \in \Xi$. Given a pointed colored graph (X, x, ϕ) with X infinite, and $n \in \mathbb{N}$, define the pointed colored graph

$$(K, k, \psi)_{X,x,\phi,n} := (K_{X,x,\phi,n}, k_{X,x,\phi,n}, \psi_{X,x,\phi,n})$$

as follows: Since X is infinite, we can choose $y \in S(x, n)$ such that $\deg_{D(x,n)} y < \deg X$. Let $(\mathbb{Z}, \chi)$ be the Cayley graph of the integers with χ the constant coloring with value ξ . Consider a $\mathbb{Z}$ -indexed family $(Y_z, x_z, \phi_z)_{z \in \mathbb{Z}}$ of copies of the pointed colored disk $(D_X(x, n), x, \phi)$, and let y_z denote the corresponding copies of y . Let (K, ψ) be the disjoint union of $(\mathbb{Z}, \chi)$ and the family $(Y_z, x_z, \phi_z)_{z \in \mathbb{Z}}$ with additional edges joining y_z to z for each $z \in \mathbb{Z}$ (see Figure 2). Let $k = x_0 \in Y_0$.

The previous construction depends on the choice of $y \in S(x, n)$. Let us assume that we have a fixed choice of such y for every class $[X, x, \phi] \in \widehat{\mathcal{G}}_*^\infty$ and $n \in \mathbb{N}$, so that we have a well-defined map $([X, x, \phi], n) \mapsto [K, k, \psi]_{X,x,\phi,n}$. The definition makes the following lemma obvious.

Lemma 5.1. *If $[X, x, \phi]$ is $(m, 1/m)$ -equivalent to $[Y, y, \psi]$, then $[K, k, \chi]_{X,x,\phi,m}$ is $(m, 1/m)$ -equivalent to $[K, k, \chi]_{Y,y,\psi,m}$.*

Figure 2: Construction of K from X and $\mathbb{Z}$

Lemma 5.2. *For every pointed colored graph (X, x, ϕ) with $V(X)$ infinite and $n \in \mathbb{N}$, the colored graph $(K, \psi)_{X, x, \phi, n}$ is quasi-transitive.*

Proof. For each $m \in \mathbb{N}$, there is an obvious colored graph isomorphism sending each $v \in Y_z$, $z \in \mathbb{Z}$ to the corresponding point in Y_{z+m} , and sending $z \in \mathbb{Z}$ to $z + m \in \mathbb{Z}$. $\square$

Let $\widehat{V}(n, r, m)$ be the subset consisting of classes $[X, x, \phi] \in \widehat{\mathcal{G}}_\star^\infty$ satisfying that there is some $y \in V(X)$ such that $d(x, y) \geq r$ and $[X, y, \phi]$ is $(m, 1/m)$ -equivalent to $[K, k, \psi]_{X, x, \phi, n}$.

Proposition 5.3. *For $n, r, m \in \mathbb{Z}^+$, the sets $\widehat{V}(n, r, m)$ and $V(n, r, m)$ are open, dense subsets of $\widehat{\mathcal{G}}_\star^\infty$ and $\mathcal{G}_\star^\infty$, respectively. For $\Delta \geq 3$, $\widehat{V}(n, r, m) \cap \widehat{\mathcal{G}}_\star^\infty(\Delta)$ and $V(n, r, m) \cap \mathcal{G}_\star^\infty(\Delta)$ are open, dense subsets of $\widehat{\mathcal{G}}_\star^\infty(\Delta)$ and $\mathcal{G}_\star^\infty(\Delta)$, respectively.*

Proof. We prove first that $\widehat{V}(n, r, m)$ is open. Let $[X, x, \phi] \in \widehat{V}(n, r, m)$, so that there is some $y \in V(X)$ such that $d(x, y) \geq r$ and there is an $(m, 1/m)$ -equivalence

$$f: (X, y, \phi) \rightarrow (K, k, \psi)_{X, x, \phi, n}.$$

Then

$$\mathcal{N}_{n+m+d(x, y), 1/\max\{m, n\}}[X, x, \phi] \subset \widehat{V}(n, r, m).$$

Indeed, let

$$(Z, z, \chi) \in \mathcal{N}_{n+m+d(x, y), 1/\max\{m, n\}}[X, x, \phi],$$

and let

$$h: (X, x, \phi) \rightarrow (Z, z, \chi)$$

be an $(n + m + d(x, y), 1/\max\{m, n\})$ -equivalence. By the triangle inequality we have $D_X(y, n + m) \subset \text{dom } h$, so the restriction $h|_{D_X(y, n + m)}: (X, y, \phi) \rightarrow (Z, h(y), \chi)$ is an $(n + m, 1/\max\{m, n\})$ -equivalence. Lemma 3.2 yields that

$$f \circ h^{-1}: (Z, h(y), \chi) \rightarrow (K, k, \psi)_{X, x, \phi, n}$$

is an $(m, 1/m)$ -equivalence. Finally, we get $[Z, z, \chi] \in \widehat{V}(n, r, m)$ by Lemma 5.1.

Let us prove that $\widehat{V}(n, r, m)$ is dense in $\widehat{\mathcal{G}}_\star^\infty$. Let $[X, x, \phi] \in \widehat{\mathcal{G}}_\star$, let $l \in \mathbb{Z}^+$, and let (K, k, ψ) denote $(K, k, \psi)_{X, x, \phi, n}$. We will now define a graph $(l, 1/l)$ -equivalent to $[X, x, \phi]$ and contained in $\widehat{V}(n, r, m)$. Take the disjoint union of $(D_X(x, \max\{n, l\}), \phi)$, $(D_K(k, m), \psi)$ and an arbitrarily colored infinite semi-ray $\mathbb{N}$. Then take points $y \in S_X(x, n)$ and $z \in S_K(k, m)$ and add edges connecting y and z to $0 \in \mathbb{N}$. It is trivial to check that such a graph satisfies the required conditions. To prove that $\widehat{V}(n, r, m) \cap \widehat{\mathcal{G}}_\star^\infty(\Delta)$ is dense in $\widehat{\mathcal{G}}_\star^\infty(\Delta)$, note that $D_X(x, n)$ and $D_K(k, m)$ are disks in infinite graphs of degree $\leq \Delta$. Therefore we can perform the same construction but choosing $y \in S_X(x, n)$ and $z \in S_K(k, m)$ of degree $< \Delta$, and the resulting graph in $\widehat{V}(n, r, m)$ will have degree $\leq \Delta$.

The same proof applies to $V(n, r, m)$ by ignoring all references to colorings. $\square$

The following result is an immediate consequence of Lemmas 3.5 and 5.2, and Proposition 5.3. It concludes the proof of Theorem 1.4 (a).

Proposition 5.4. *The sets $\bigcap_{n,r,m \in \mathbb{N}} \widehat{V}(n, r, m)$ and $\bigcap_{n,r,m \in \mathbb{N}} V(n, r, m)$ are generic in $\widehat{\mathcal{G}}_\star^\infty$ and $\mathcal{G}_\star^\infty$, respectively. Their intersections with $\widehat{\mathcal{G}}_\star^\infty(\Delta)$ and $\mathcal{G}_\star^\infty(\Delta)$ are also generic. All these sets consist of aperiodic chaotic classes.*

Let us now proceed with the proof of Theorem 1.4 (b). For $n \in \mathbb{Z}_+$, let

$$W(n) = \{ [X, x, \phi] \in \widehat{\mathcal{G}}_\star^\infty \mid \forall x, y \in D(x, n), x \neq y \Rightarrow \phi(x) \neq \phi(y) \}.$$

Proposition 5.5. *$W(n)$ and $W(n) \cap \widehat{\mathcal{G}}_\star^\infty(\Delta)$ are open, dense subsets of $\widehat{\mathcal{G}}_\star^\infty$ and $\widehat{\mathcal{G}}_\star^\infty(\Delta)$, respectively.*

Proof. Let us prove that $W(n)$ is open. Let $[X, x, \phi] \in W(n)$ and

$$\epsilon = \inf \{ d(\phi(y), \phi(z)) \mid y, z \in D(x, n), y \neq z \} > 0.$$

Then

$$\widehat{\mathcal{N}}_{n, \epsilon/3}[X, x, \phi] \cap \widehat{\mathcal{G}}_\star^\infty \subset W(n).$$

Indeed, let $h: (Y, y, \psi) \rightarrow (X, x, \phi)$ be an $(n, \epsilon/3)$ -equivalence, and suppose for the sake of contradiction that there are $u, v \in D(y, n)$, $u \neq v$ with $\psi(u) = \psi(v)$. Then we get

$$d(\phi(h(u)), \phi(h(v))) \leq d(\phi(h(u)), \psi(u)) + d(\phi(h(v)), \psi(v)) + d(\psi(u), \psi(v)) \leq \frac{2\epsilon}{3} < \epsilon,$$

by the definition of $(n, \epsilon/3)$ -equivalence and the triangle inequality. This clearly contradicts the definition of ϵ .

To show that $W(n)$ is dense, let $[X, x, \phi] \in W(n)$ and modify the coloring ϕ in the following way: for every $y \in D(x, n)$, choose $\hat{\phi}(y) \in \mathcal{E}$ so that

$$0 < d_\mathcal{E}(\hat{\phi}(y), \phi(y)) < \epsilon$$

and the restriction of $\hat{\phi}$ to $D(x, n)$ is injective. Clearly this implies $[X, x, \hat{\phi}] \in W(n)$, and $[X, x, \hat{\phi}]$ is (m, ϵ) -equivalent to $[X, x, \phi]$ for every $m \in \mathbb{N}$. Since $[X, x, \hat{\phi}] \in \widehat{\mathcal{G}}_\star^\infty(\Delta)$ if $[X, x, \phi] \in \widehat{\mathcal{G}}_\star^\infty(\Delta)$, it follows that $W(n) \cap \widehat{\mathcal{G}}_\star^\infty(\Delta)$ is dense in $\widehat{\mathcal{G}}_\star^\infty(\Delta)$. $\square$

Corollary 5.6. *The set $\bigcap_{n \in \mathbb{Z}_+} W(n)$ consists of aperiodic classes.*

Proof. Let $[X, x, \phi] \in \bigcap_{n \in \mathbb{N}} W(n)$, and suppose for the sake of contradiction that there is some non-trivial $f \in \text{Aut}(X, \phi)$. Then there are $y, z \in X$, $y \neq z$, such that $f(y) = z$. But this yields $\phi(y) = \phi(z)$, which in turn implies

$$[X, x, \phi] \notin W(\max\{d(x, y), d(x, z)\}),$$

a contradiction. $\square$

Figure 3: In any graph that contains such a pattern, there is a non-trivial isomorphism given by the interchange of a and b .

This concludes the proof of Theorem 1.4 (b). Note that for this result we do not need the assumption $\Delta \geq 3$. The corresponding result is not true if we restrict our attention to non-colored graphs.

Proposition 5.7. *The aperiodic classes are not dense in $\mathcal{G}_*^\infty$.*

Proof. It is clear that $\mathcal{G}_*^\infty$ contains a graph X exhibiting the local pattern shown in Figure 3. Let $x \in V(X)$, and let n be large enough so that $B_X(x, n)$ also contains the pattern in Figure 3. Let (Y, y) be any pointed graph n -equivalent to (X, x) , with n -equivalence h . Then there is an automorphism of Y that interchanges $h(a)$ and $h(b)$ and leaves all other vertices fixed. This shows that every pointed graph n -equivalent to (X, x) has a non-trivial automorphism, and so the aperiodic classes cannot be dense. $\square$

The same counterexample applies for colored graphs if the space of possible values has an isolated point ξ by using that color to decorate both a and b .

To finish the proof of Theorem 1.4 (c), note that for infinite graphs aperiodic implies non-quasi-transitive. Therefore both the set

$$\bigcap_{n,r,m \in \mathbb{N}} \widehat{V}(n, r, m) \cap \bigcap_{n \in \mathbb{N}} W(n)$$

and its intersection with $\widehat{\mathcal{G}}_*^\infty(\Delta)$, $\Delta \geq 3$, consist of chaotic classes and are generic in $\widehat{\mathcal{G}}_*^\infty$ and $\widehat{\mathcal{G}}_*^\infty(\Delta)$, respectively.

6 An example of a chaotic colored graph

The colored graph $[X, \phi]$ defined in Example 2.3 satisfies $\overline{[X, \phi]} = \widehat{\mathcal{G}}_*^\infty$, so that it being chaotic can be seen as a weakening of Theorem 1.4 (a). To prove density of quasi-transitive classes we take arbitrary finite patterns and embed them in a periodic configuration, as illustrated in Figure 2. The same ideas can be used to obtain less trivial examples. In this section we detail the construction of a chaotic graph whose closure is the family of classes $[Y, \psi]$ with every vertex $y \in Y$ satisfying $\deg(y) = 1$ or 3 .

Let $C \subset \mathcal{E}$ be a countable, dense subset, and let $\mathcal{F} \subset \widehat{\mathcal{G}}_*$ consist of the classes $[Y, y, \psi]$ such that

- (i) $[Y, y, \psi]$ is aperiodic,
- (ii) Y is a finite graph,
- (iii) $\text{im}(\psi) \subset C$,
- (iv) $\deg(w) = 1$ or 3 for every $w \in Y$, and
- (v) $\deg(y) = 1$.

Let F be a set of colored graphs that contains exactly one representative for each class in $\mathcal{F}$. Since F is countable, we can choose an enumeration of the form $F = \{(Y_z, y_z, \psi_z)\}_{z \in \mathbb{Z}}$. Let $\chi: \mathbb{Z} \rightarrow F$ be a bijective coloring, which is also aperiodic, and let X be the colored graph constructed as follows: take the disjoint union of $(\mathbb{Z}, \chi)$ and the family $\{(Y_z, y_z, \psi_z)\}_{z \in \mathbb{Z}}$, and add an edge joining y_z to $z \in \mathbb{Z}$ for every $z \in \mathbb{Z}$. Since the colored graphs (Y_z, y_z, ψ_z) are aperiodic, the proof that (X, ϕ) is aperiodic follows exactly as in Example 2.3.

Let us prove that (X, ϕ) is aperiodic and chaotic. Let $x_0 := 0 \in \mathbb{Z} \subset V(X)$. For $n \in \mathbb{N}$, let (H_n, ϕ_n) be the colored subgraph of (X, ϕ) with vertex set

$$\{-n, -n+1, \dots, n\} \cup \bigcup_{-n < z < n} Y_z,$$

where by $\{-n, -n+1, \dots, n\}$ we mean the obvious subset of the copy of $\mathbb{Z}$ contained inside $V(X)$ (see Figure 4). Note that $D(x_0, n) \subset V(H_n)$. Let $x_n = n \in V(H_n)$. Then (H_n, x_n, ϕ) satisfies conditions (i)–(v). Let us embed (H_n, ϕ) into a quasi-transitive graph (Z_n, ξ_n) as follows: the disjoint union of a $\mathbb{Z}$ -indexed family of copies of (H_n, ϕ) , denoted by $H_{n,z}$, and a single copy of $\mathbb{Z}$, and identify the point corresponding to x_n in $V(H_{n,z})$ with $z \in \mathbb{Z}$ (see Figure 5). The proof that (Z_n, ξ_n) is quasi-transitive proceeds as in Lemma 5.2. Also (Z_n, ξ_n) contains an isomorphic copy of $(D(x_0, n), \phi)$ by construction. For each $m \in \mathbb{N}$, the colored subgraph of (Z_n, ξ_n) with

Figure 4: An example of the graph H_2 Figure 5: An example of the graph Z_2

vertex set

$$\{-m, -m+1, \dots, m\} \cup \bigcup_{-m < z < m} H_{n,z}$$

satisfies again conditions (i)–(v). This means that we can find an isomorphic copy inside (X, ϕ) , and thus $[Z_n, \chi_n] \subset \overline{[X, \phi]}$. To recap, we have proved that for each disk $D(x, n)$ we can find a quasi-transitive graph (Z_n, χ_n) containing a copy of $(D(x, n), \phi)$ and $[Z_n, \chi_n] \subset \overline{[X, \phi]}$. Therefore (X, ϕ) is chaotic.

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