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Common fixed point results via simulation type functions in non-Archimedean modular metric spaces and applications

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Abstract: In this study, we demonstrate the existence and uniqueness of common fixed points of a generalized (α, β) -simulation contraction on a non-Archimedean modular metric space. We achieve some consequences in non-Archimedean modular metric spaces as an application, using the structure of a directed graph. Eventually, we contemplate the existence of solutions to a class of functional equations standing up dynamic programming with the help of our outcomes.

Keywords: Non-Archimedean modular metric space, Simulation function, Cyclic (α, β) -admissible pair

MSC: 47H10, 54H25, 37C25

1 Introduction

Subsequently, the letters $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ will specify the set of all real numbers, the set of all nonnegative real numbers, and the set of all positive integer numbers, respectively.

In 2010, Chistyakov [1, 2] set up a new structure named modular metric, which is an extension of metric and a linear modular.

Definition 1.1 [1, 3] Let M be non-void set and $\delta : (0, \infty) \times M \times M \rightarrow [0, \infty]$ be a function fulfilling the following statements: for all $\lambda, \mu > 0$ and $p, q, k \in M$,

δ_1 . $p = q$ if and only if $\delta_\lambda(p, q) = 0$;

δ_2 . $\delta_\lambda(p, q) = \delta_\lambda(p, q)$;

δ_3 . $\delta_{\lambda+\mu}(p, q) \leq \delta_\lambda(p, k) + \delta_\mu(k, q)$.

So, δ is entitled modular metric in M , and in the present case, M_δ is a modular metric space. If we substitute (δ_1) by

δ_4 . $\delta_\lambda(p, p) = 0$ for all $\lambda > 0$ and $p \in M$,

then δ is said to be a pseudomodular metric on M . A modular metric δ on M is named regular if the following weaker version of (δ_1) is provided for some $\lambda > 0$;

δ_5 . $p = q$ if and only if $\delta_\lambda(p, q) = 0$.

As well, δ is a convex function if, for $\lambda, \mu > 0$ and $p, q, k \in M$, the inequality holds:

δ_6 . $\delta_{\lambda+\mu}(p, q) \leq \frac{\lambda}{\lambda+\mu} \delta_\lambda(p, k) + \frac{\mu}{\lambda+\mu} \delta_\mu(k, q)$.

For all $\lambda, \mu > 0$ and $p, q, k \in M_\delta$, if the following property

δ_7 . $\delta_{\max\{\lambda, \mu\}}(p, q) \leq \delta_\lambda(p, k) + \delta_\mu(k, q)$

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is put in a place of (δ_3) , then, we attain a new space with some new features. This new structure M_δ is named a non-Archimedean modular metric space. For simplicity, throughout the study, we will write:

$$\delta_\lambda(p, q) = \delta(\lambda, p, q)$$

for all $\lambda > 0$ and $p, q \in M$.

Definition 1.2 [1, 2] Presume that the function δ be a pseudomodular on M and $\delta_0 \in M$ be a fixed element. The hereinbelow two sets M_δ and M_δ^* are called modular spaces (around δ_0).

$$M_\delta = M_\delta(p_0) = \{p \in M : \delta_\lambda(p, p_0) \text{ as } \lambda \rightarrow \infty\}$$

and

$$M_\delta^* = M_\delta^*(p_0) = \{p \in M : \exists \lambda = \lambda(p) > 0 \text{ such that } \delta_\lambda(p, p_0) < \infty\}.$$

It is noticeable that $M_\delta \subset M_\delta^*$, but this inclusion may be proper in general. Presume that from [1, 2], δ is modular on M ; we derive that the modular space M_δ can be equipped with a (nontrivial) metric, induced by δ and presented by:

$$d_\delta(p, q) = \inf \{\lambda > 0 : \delta_\lambda(p, q) < \lambda\}$$

for all $p, q \in M_\delta$.

Note that if δ is a convex modular on M , then according to [1, 2], the two modular spaces coincide, i.e., $M_\delta = M_\delta^*$, and this common set can be endowed with the metric d_δ^* given by:

$$d_\delta^*(p, q) = \inf \{\lambda > 0 : \delta_\lambda(p, q) < 1, \forall p, q \in M_\delta\},$$

which is called Luxemburg distance.

Definition 1.3 [4] Let M_δ be a modular metric space, S be a subset of M_δ and $(p_n)_{n \in \mathbb{N}}$ be a sequence in M_δ . Then,

- A sequence $(p_n)_{n \in \mathbb{N}}$ is called δ -convergent to $p \in M_\delta$ if and only if $\delta_\lambda(p_n, p) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda > 0$, p is said to be the δ -limit of (p_n) .
- A sequence $(p_n)_{n \in \mathbb{N}}$ is called δ -Cauchy if $\delta_\lambda(p_n, p_m) \rightarrow 0$, as $m, n \rightarrow \infty$ for all $\lambda > 0$.
- A subset S is called δ -complete if any δ -Cauchy sequence in S is δ -convergent to the point of S .

Definition 1.4 [4] Let M_δ be a modular metric space. The mapping $K : M_\delta \rightarrow M_\delta$ is δ -continuous if $\delta_\lambda(p_n, p) \rightarrow 0$, provided to $\delta_\lambda(Kp_n, Kp) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 1.5 [5] A simulation function is a mapping $\zeta : [0, \infty)^2 \rightarrow \mathbb{R}$ admitting the following features:

- ζ_1 . $\zeta(0, 0) = 0$,
- ζ_2 . $\zeta(p, q) < p - q$, for all $p, q > 0$,
- ζ_3 . if $\{q_n\}$ and $\{p_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} q_n = \lim_{n \rightarrow \infty} p_n = l$ and $l \in (0, \infty)$, then

$$\limsup_{n \rightarrow \infty} \zeta(q_n, p_n) < 0.$$

Argoubi et al.[6] modified the definition of simulation function and so introduced it as:

Definition 1.6 A simulation function is a mapping $\zeta : [0, \infty)^2 \rightarrow \mathbb{R}$ satisfying the followings:

- $\zeta(p, q) < p - q$, for all $p, q > 0$,
- if $\{q_n\}$ and $\{p_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} q_n = \lim_{n \rightarrow \infty} p_n > 0$, and $q_n < p_n$, then $\limsup_{n \rightarrow \infty} \zeta(q_n, p_n) < 0$.

Example 1.7 [6] The function $\zeta_c : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is a simulation function specified by

$$\zeta_c(q, p) = \begin{cases} 1 & \text{if } (q, p) = (0, 0), \\ cp - q & \text{otherwise,} \end{cases}$$

where $c \in (0, 1)$.

One may point to [5]-[12] for some detailed information about simulation functions.

Definition 1.8 [13] Let $G : [0, \infty)^2 \rightarrow \mathbb{R}$ is called C -class function when it has continuity property and admits the following features:

- i. $G(p, q) \leq p$,
- ii. $G(p, q) = p$ implies that either $p = 0$ or $q = 0$, for all $p, q \in [0, \infty)$.

Definition 1.9 [14] Let $\zeta : [0, \infty)^2 \rightarrow \mathbb{R}$ be a function admitting the following features:

- a. $\zeta(q, p) < G(p, q)$ for all $t, s > 0$, where $G : [0, \infty)^2 \rightarrow \mathbb{R}$ is a C -class function,
- b. if $\{q_n\}$ and $\{p_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} q_n = \lim_{n \rightarrow \infty} p_n > 0$, and $q_n < p_n$, then $\limsup_{n \rightarrow \infty} \zeta(q_n, p_n) < C_G$.

Then ζ is named a C_G -simulation function.

Definition 1.10 [14] A mapping $G : [0, \infty)^2 \rightarrow \mathbb{R}$ has the property C_G , if there exists a $C_G \geq 0$ such that

- i. $G(p, q) > C_G$ implies $p > q$,
- ii. $G(q, q) \leq C_G$ for all $q \in [0, \infty)$.

Let T_G denotes the family of all C_G -simulation functions $\zeta : [0, \infty)^2 \rightarrow \mathbb{R}$.

Recently, Alizadeh et al. [15] defined the concept of cyclic (α, β) -admissible mapping as indicated below:

Definition 1.11 Let M be a non-void set, K be a self-mapping on M and $\alpha, \beta : M \rightarrow [0, \infty)$ be two mappings. K is named a cyclic (α, β) -admissible mapping if

- (i) for some $p \in M$, $\alpha(p) \geq 1$ implies $\beta(Kp) \geq 1$;
- (ii) for some $p \in M$, $\beta(p) \geq 1$ implies $\alpha(Kp) \geq 1$.

Besides, they demonstrated fixed point theorems by using the structure of cyclic (α, β) -admissible mapping.

Example 1.12 [15] Let $K : \mathbb{R} \rightarrow \mathbb{R}$ be a function described as $Kp = -(p + p^3)$. Presume that $\alpha, \beta : \mathbb{R} \rightarrow \mathbb{R}_+$ are furnishing mappings for all $p, q \in \mathbb{R}_+$ such that

$$\alpha(p) = e^p \text{ and } \beta(q) = e^{-q}.$$

Then K is a cyclic (α, β) -admissible mapping.

The generalized cyclic (α, β) -admissible mappings have been generalized by Latif et al. [16], as follows.

Definition 1.13 Let M be a non-void set, K, L be two self-mappings on M and $\alpha, \beta : M \rightarrow [0, \infty)$ be two mappings. (K, L) is called a cyclic (α, β) -admissible pair if

- (i) for some $p \in M$ $\alpha(p) \geq 1$ implies $\beta(Kp) \geq 1$;
- (ii) for some $p \in M$ $\beta(p) \geq 1$ implies $\alpha(Lp) \geq 1$.

Remark 1.16 In the above definition, if we acquire $K = L$, then K is a cyclic (α, β) -admissible mapping.

Throughout this study, δ will be used as a convex and regular function.

2 Main results

Initially, this section aims to innovate the generalized (α, β) -simulation contraction. Besides, common fixed point results are procured satisfying generalized (α, β) -simulation contraction in non-Archimedean modular metric spaces.

Definition 2.1 Let M_δ be a non-Archimedean modular metric space, $K, L : M_\delta \rightarrow M_\delta$ be two self-mappings and $\alpha, \beta : M_\delta \rightarrow [0, \infty)$ be two functions. The pair (K, L) is named a generalized (α, β) -simulation contraction if there exists $\zeta \in T_G$ such that

$$\alpha(p)\beta(q) \geq 1 \Rightarrow \zeta(\delta_\lambda(Kp, Lq), S(p, q)) \geq C_G, \quad (1)$$

where $S(p, q) = \max \left\{ \delta_\lambda(p, q), \delta_\lambda(p, Kp), \delta_\lambda(q, Lq), \frac{\delta_\lambda(p, Lq) + \delta_\lambda(q, Kp)}{2} \right\}$, for all $p, q \in M_\delta$.

Theorem 2.2 Let M_δ be a δ -complete non-Archimedean modular metric space and K and L are generalized (α, β) -simulation contraction. Presume that the following circumstances hold:

- i. (K, L) is a cyclic (α, β) -admissible pair,
- ii. there exists $p_0 \in M_\delta$ such that $\alpha(p_0) \geq 1$,
- iii. K or L is a δ -continuous mapping,
- iv. if $\{p_n\}$ is a sequence in M_δ such that $p_n \rightarrow p$ and $\alpha(p_{2n}) \geq 1, \beta(p_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(p) \geq 1$ and $\beta(p) \geq 1$.

Then K and L hold a common fixed point. Additionally, if $\alpha(p)\beta(q) \geq 1$ for all $p, q \in \text{Fix}(K, L)$, then K and L hold a unique common fixed point.

Proof Let $p_0 \in M_\delta$ be such that $\alpha(p_0) \geq 1$. We will construct a sequence $\{p_n\}$ in M_δ by

$$\begin{aligned} p_{2n+2} &= Kp_{2n+1}, \\ p_{2n+1} &= Lp_{2n}, \end{aligned} \quad (2)$$

for all $n \in \mathbb{N}$. Also, as (K, L) is a cyclic (α, β) -admissible pair and $\alpha(p_0) \geq 1$, then

$$\beta(p_1) = \beta(Kp_0) \geq 1$$

which implies

$$\alpha(p_2) = \alpha(Lp_1) \geq 1.$$

By proceeding with this process, we get $\alpha(p_{2n}) \geq 1$ and $\beta(p_{2n+1}) \geq 1$ for all $n \in \mathbb{N}$. Thus, $\alpha(p_{2n})\beta(p_{2n+1}) \geq 1$ for all $n \in \mathbb{N}$. From (1), we have

$$\begin{aligned} C_G &\leq \zeta(\delta_\lambda(Kp_{2n}, Lp_{2n+1}), S(p_{2n}, p_{2n+1})) \\ &= \zeta(\delta_\lambda(p_{2n+1}, p_{2n+2}), S(p_{2n}, p_{2n+1})) \\ &< G(S(p_{2n}, p_{2n+1}), \delta_\lambda(p_{2n+1}, p_{2n+2})). \end{aligned}$$

Further, using (i) of Definition 1.10, we achieve

$$\delta_\lambda(p_{2n+1}, p_{2n+2}) < S(p_{2n}, p_{2n+1}), \quad (3)$$

where

$$\begin{aligned} S(p_{2n}, p_{2n+1}) &= \max \left\{ \delta_\lambda(p_{2n}, p_{2n+1}), \delta_\lambda(p_{2n}, Kp_{2n}), \delta_\lambda(p_{2n+1}, Lp_{2n+1}), \right. \\ &\quad \left. \frac{(p_{2n}, Lp_{2n+1}) + (p_{2n+1}, Kp_{2n})}{2} \right\} \\ &= \max \left\{ \delta_\lambda(p_{2n}, p_{2n+1}), \delta_\lambda(p_{2n}, p_{2n+1}), \delta_\lambda(p_{2n+1}, p_{2n+2}), \right. \\ &\quad \left. \frac{\delta_\lambda(p_{2n}, p_{2n+2}) + \delta_\lambda(p_{2n+1}, p_{2n+1})}{2} \right\} \\ &= \max \left\{ \delta_\lambda(p_{2n}, p_{2n+1}), \delta_\lambda(p_{2n}, p_{2n+1}), \delta_\lambda(p_{2n+1}, p_{2n+2}), \right. \\ &\quad \left. \frac{\delta_{\max\{\lambda, \lambda\}}(p_{2n}, p_{2n+2}) + \delta_\lambda(p_{2n+1}, p_{2n+1})}{2} \right\} \\ &\leq \max \left\{ \delta_\lambda(p_{2n}, p_{2n+1}), \delta_\lambda(p_{2n+1}, p_{2n+2}), \right. \\ &\quad \left. \frac{\delta_\lambda(p_{2n}, p_{2n+1}) + \delta_\lambda(p_{2n+1}, p_{2n+2})}{2} \right\} \\ &= \max \left\{ \delta_\lambda(p_{2n}, p_{2n+1}), \delta_\lambda(p_{2n+1}, p_{2n+2}) \right\}. \end{aligned}$$

If

$$\max \{ \delta_\lambda(p_{2n}, p_{2n+1}), \delta_\lambda(p_{2n+1}, p_{2n+2}) \} = \delta_\lambda(p_{2n+1}, p_{2n+2})$$

for each $n \in \mathbb{N}$, then from (3), we have a contradiction. So, $S(p_{2n}, p_{2n+1}) = \delta_\lambda(p_{2n}, p_{2n+1})$ for all $n \in \mathbb{N}$. Consequently, (3) gives

$$\delta_\lambda(p_{2n+1}, p_{2n+2}) < \delta_\lambda(p_{2n}, p_{2n+1}). \quad (4)$$

Hence for all $n \in \mathbb{N}$, continuing this way we get

$$\delta_\lambda(p_{2n}, p_{2n+1}) < \delta_\lambda(p_{2n-1}, p_{2n-2}).$$

So $\{\delta_\lambda(p_{2n}, p_{2n+1})\}$ is a monotonically decreasing sequence of nonnegative real numbers, and hence there exists $l \geq 0$ such that

$$\lim_{n \rightarrow \infty} \delta_\lambda(p_{2n}, p_{2n+1}) = l. \quad (5)$$

Assume that $l > 0$. Then using (1) and (b) of the Definition 1.9, we have

$$\begin{aligned} C_G &\leq \limsup_{n \rightarrow \infty} \zeta(\delta_\lambda(Kp_{2n}, Lp_{2n+1}), S(p_{2n}, p_{2n+1})) \\ &= \limsup_{n \rightarrow \infty} \zeta(\delta_\lambda(p_{2n+1}, p_{2n+2}), \delta_\lambda(p_{2n}, p_{2n+1})) < C_G, \end{aligned}$$

which is a contradiction and so $l = 0$.

Now, we show that $\{p_n\}$ is a δ -Cauchy sequence. It is sufficient to show that $\{p_{2n}\}$ is a δ -Cauchy sequence. Assume on the contrary. There exists $\varepsilon > 0$ such that we can find two subsequences $\{m_k\}$ and $\{n_k\}$ of positive integers satisfying $n_k > m_k \geq k$ such the following inequalities hold:

$$\delta_\lambda(p_{2n_k}, p_{2m_k}) \geq \varepsilon, \text{ and } \delta_\lambda(p_{2n_k-1}, p_{2m_k}) < \varepsilon. \quad (6)$$

From (6) and (δ_7) , it follows that

$$\begin{aligned} \varepsilon &\leq \delta_\lambda(p_{2n_k}, p_{2m_k}) = \delta_{\max\{\lambda, \lambda\}}(p_{2n_k}, p_{2m_k}) \\ &\leq \delta_\lambda(p_{2n_k}, p_{2n_k-1}) + \delta_\lambda(p_{2n_k-1}, p_{2m_k}) \\ &< \varepsilon + \delta_\lambda(p_{2n_k}, p_{2n_k-1}). \end{aligned} \quad (7)$$

On taking limit as $k \rightarrow \infty$ in above relation, we obtain that

$$\lim_{k \rightarrow \infty} \delta_\lambda(p_{2n_k}, p_{2m_k}) = \varepsilon. \quad (8)$$

Also, we have $\alpha(p_{2n_k})\beta(p_{2m_k-1}) \geq 1$ and by (1),

$$\begin{aligned} C_G &\leq \zeta(\delta_\lambda(Kp_{2n_k}, Lp_{2m_k-1}), S(p_{2n_k}, p_{2m_k-1})) \\ &= \zeta(\delta_\lambda(p_{2n_k+1}, p_{2m_k}), S(p_{2n_k}, p_{2m_k-1})), \end{aligned} \quad (9)$$

where

$$\begin{aligned} S(p_{2n_k}, p_{2m_k-1}) &= \max\{\delta_\lambda(p_{2n_k}, p_{2m_k-1}), \delta_\lambda(p_{2n_k}, Kp_{2n_k}), \\ &\quad \delta_\lambda(p_{2m_k-1}, Lp_{2m_k-1}), \frac{\delta_\lambda(p_{2n_k}, Lp_{2m_k-1}) + \delta_\lambda(p_{2m_k-1}, Kp_{2n_k})}{2}\} \\ &= \max\{\delta_\lambda(p_{2n_k}, p_{2m_k-1}), \delta_\lambda(p_{2n_k}, p_{2n_k+1}), \\ &\quad \delta_\lambda(p_{2m_k-1}, p_{2m_k}), \frac{\delta_\lambda(p_{2n_k}, p_{2m_k}) + \delta_\lambda(p_{2m_k-1}, p_{2n_k+1})}{2}\}. \end{aligned} \quad (10)$$

Also, from (6) and (δ_7) , it follows that

$$\begin{aligned} \delta_\lambda(p_{2n_k+1}, p_{2m_k}) &= \delta_{\max\{\lambda, \lambda\}}(p_{2n_k+1}, p_{2m_k}) \\ &\leq \delta_\lambda(p_{2n_k+1}, p_{2n_k}) + \delta_\lambda(p_{2n_k}, p_{2m_k}) \\ &= \delta_\lambda(p_{2n_k+1}, p_{2n_k}) + \delta_{\max\{\lambda, \lambda\}}(p_{2n_k}, p_{2m_k}) \\ &\leq \delta_\lambda(p_{2n_k+1}, p_{2n_k}) + \delta_\lambda(p_{2n_k}, p_{2n_k-1}) \\ &\quad + \delta_\lambda(p_{2n_k-1}, p_{2m_k}) \\ &< \delta_\lambda(p_{2n_k+1}, p_{2n_k}) + \delta_\lambda(p_{2n_k}, p_{2n_k-1}) + \varepsilon, \end{aligned} \quad (11)$$

$$\begin{aligned} \delta_\lambda(p_{2m_k-1}, p_{2n_k+1}) &= \delta_{\max\{\lambda, \lambda\}}(p_{2m_k-1}, p_{2n_k+1}) \\ &\leq \delta_\lambda(p_{2m_k-1}, p_{2m_k}) + \delta_\lambda(p_{2m_k}, p_{2n_k+1}) \\ &= \delta_\lambda(p_{2m_k-1}, p_{2m_k}) + \delta_{\max\{\lambda, \lambda\}}(p_{2m_k}, p_{2n_k+1}) \\ &\leq \delta_\lambda(p_{2m_k-1}, p_{2m_k}) + \delta_\lambda(p_{2m_k}, p_{2n_k-1}) \\ &\leq \delta_\lambda(p_{2n_k-1}, p_{2n_k+1}) \\ &= \delta_\lambda(p_{2m_k-1}, p_{2m_k}) + \delta_\lambda(p_{2m_k}, p_{2n_k-1}) \\ &\quad + \delta_{\max\{\lambda, \lambda\}}(p_{2n_k-1}, p_{2n_k+1}) \\ &\leq \delta_\lambda(p_{2m_k-1}, p_{2m_k}) + \delta_\lambda(p_{2n_k-1}, p_{2m_k}) \\ &\quad + \delta_\lambda(p_{2n_k-1}, p_{2n_k}) + \delta_\lambda(p_{2n_k}, p_{2n_k+1}) \\ &< \varepsilon + \delta_\lambda(p_{2m_k-1}, p_{2m_k}) + \delta_\lambda(p_{2n_k-1}, p_{2n_k}) \\ &\quad + \delta_\lambda(p_{2n_k}, p_{2n_k+1}), \end{aligned} \quad (12)$$

and

$$\begin{aligned}
 \delta_{\lambda}(p_{2n_k}, p_{2m_k-1}) &= \delta_{\max\{\lambda, \lambda\}}(p_{2n_k}, p_{2m_k-1}) \\
 &\leq \delta_{\lambda}(p_{2n_k}, p_{2n_k-1}) + \delta_{\lambda}(p_{2n_k-1}, p_{2m_k-1}) \\
 &= \delta_{\lambda}(p_{2n_k}, p_{2n_k-1}) + \delta_{\max\{\lambda, \lambda\}}(p_{2n_k-1}, p_{2m_k-1}) \\
 &\leq \delta_{\lambda}(p_{2n_k}, p_{2n_k-1}) + \delta_{\lambda}(p_{2n_k-1}, p_{2m_k}) \\
 &\quad + \delta_{\lambda}(p_{2m_k}, p_{2m_k-1}) \\
 &< \varepsilon + \delta_{\lambda}(p_{2n_k}, p_{2n_k-1}) + \delta_{\lambda}(p_{2m_k}, p_{2m_k-1}).
 \end{aligned} \tag{13}$$

Moreover, because (K, L) is a cyclic (α, β) -admissible pair we attain $\alpha(p_{2n_k})\beta(p_{2m_k-1}) \geq 1$. Then by (1),

$$\begin{aligned}
 C_G &\leq \zeta(\delta_{\lambda}(Kp_{2n_k}, Lp_{2m_k-1}), S(p_{2n_k}, p_{2m_k-1})) \\
 &= \zeta(\delta_{\lambda}(p_{2n_k+1}, p_{2m_k}), S(p_{2n_k}, p_{2m_k-1})) \\
 &< G(S(p_{2n_k}, p_{2m_k-1}), \delta_{\lambda}(p_{2n_k+1}, p_{2m_k})).
 \end{aligned} \tag{14}$$

Thence by (i) of Definition 1.10, we gain

$$\delta_{\lambda}(p_{2n_k+1}, p_{2m_k}) < S(p_{2n_k}, p_{2m_k-1}). \tag{15}$$

Taking the limit as $k \rightarrow \infty$ and by using (6), (8), (10), (11), (12), (13), we procure

$$\lim_{k \rightarrow \infty} \delta_{\lambda}(p_{2n_k+1}, p_{2m_k}) = \lim_{k \rightarrow \infty} S(p_{2n_k}, p_{2m_k-1}) = \varepsilon. \tag{16}$$

Taking $s_n = \delta_{\lambda}(p_{2n_k+1}, p_{2m_k})$, $t_n = \lim_{k \rightarrow \infty} S(p_{2n_k}, p_{2m_k-1})$ and using (13), (15) and (b) of Definition 1.9, we obtain

$$\begin{aligned}
 C_G &\leq \limsup_{k \rightarrow \infty} \zeta(\delta_{\lambda}(Kp_{2n_k}, Lp_{2m_k-1}), S(p_{2n_k}, p_{2m_k-1})) \\
 &= \limsup_{k \rightarrow \infty} \zeta(\delta_{\lambda}(p_{2n_k+1}, p_{2m_k}), S(p_{2n_k}, p_{2m_k-1})) < C_G.
 \end{aligned}$$

The attained last inequality causes a contradiction. Accordingly $\{p_{2n}\}$ is a δ -Cauchy sequence.

As M_{δ} is a δ -complete non-Archimedean modular metric space, there exists $u \in M_{\delta}$ such that $\delta_{\lambda}(p_n, u) \rightarrow 0$ as $n \rightarrow \infty$.

Now we present u is a common fixed point of K and L . From (iii), we will presume that K is a δ -continuous mapping. Since $\delta_{\lambda}(p_{2n}, j) \rightarrow 0$ as $n \rightarrow \infty$ and K is a δ -continuous mapping, we get

$$\delta_{\lambda}(p_{2n+1}, j) = \delta_{\lambda}(Kp_{2n+1}, Kj) \rightarrow 0,$$

as $n \rightarrow \infty$. Hence $p_{2n+1} \rightarrow Kj$ as $n \rightarrow \infty$. But $p_{2n+1} \rightarrow j$ as $n \rightarrow \infty$. By the uniqueness of the limit, we achieve $Kj = j$.

From (iv), we have $\beta(j) \geq 1$. This implies that $\alpha(p_{2n})\beta(j) \geq 1$ for all $n \in \mathbb{N}$. Next, we will demonstrate that j is a fixed point of L . We presume that $j \neq Kj$, that is $\delta_{\lambda}(j, Kj) > 0$. By K and L are generalized (α, β) -simulation contraction, we attain

$$C_G \leq \zeta(\delta_{\lambda}(Kp_{2n}, Lj), S(p_{2n}, j)) < G(S(p_{2n}, j), \delta_{\lambda}(Kp_{2n}, Lj)) \tag{17}$$

and then by (i) of Definition 1.10, we procure

$$\delta_{\lambda}(Kp_{2n}, Lj) < S(p_{2n}, j). \tag{18}$$

Taking limit as $n \rightarrow \infty$ in (18), we have

$$\delta_{\lambda}(j, Lj) < \delta_{\lambda}(j, Lj),$$

which is a contradiction. Thus $j = Lj$ and hence j is a common fixed point K and L .

Ultimately, we will demonstrate the uniqueness of a common fixed point of K and L . We assume that s is another common fixed point, that is, $\delta_{\lambda}(j, s) \neq 0$. From hypothesis, we gain $\alpha(j) \geq 1$ and $\beta(s) \geq 1$ and thus $\alpha(j)\beta(s) \geq 1$. By using (1),

$$C_G \leq \zeta(\delta_{\lambda}(Kj, Ls), S(j, s)) < G(S(j, s), \delta_{\lambda}(Kj, Ls)) \tag{19}$$

and from (i) of Definition 1.10, we have

$$\begin{aligned}\delta_\lambda(Kj, Ls) &< S(j, s) \\ &= \max \left\{ \delta_\lambda(j, s), \delta_\lambda(j, Kj), \delta_\lambda(s, Ls), \frac{\delta_\lambda(j, Ls) + \delta_\lambda(s, Kj)}{2} \right\} \\ &= \max \left\{ \delta_\lambda(j, s), \delta_\lambda(j, Kj), \delta_\lambda(s, Ls), \frac{\delta_\lambda(j, s) + \delta_\lambda(s, j)}{2} \right\} \\ &= \max \{ \delta_\lambda(j, s), 0 \} \\ &= \delta_\lambda(j, s),\end{aligned}$$

which is a contradiction. Hence $j = s$.

3 Consequences

This section aims to acquaint some consequences from the main theorem.

Corollary 3.1 Let M_δ be a δ -complete non-Archimedean modular metric space and $K, L : M_\kappa \rightarrow M_\delta$ be two mappings. Presume that there exists $k \in (0, 1)$ such that

$$\begin{aligned}p, q \in M_\kappa \text{ with } \alpha(p)\beta(q) \geq 1 &\Rightarrow \\ \delta_\lambda(Kp, Lq) \leq k \max \left\{ \delta_\lambda(p, q), \delta_\lambda(p, Kp), \delta_\lambda(q, Lq), \frac{\delta_\lambda(p, Lq) + \delta_\lambda(q, Kp)}{2} \right\}.\end{aligned}$$

Suppose that the following circumstances hold:

- (K, L) is a cyclic (α, β) -admissible pair,
- there exists $p_0 \in M_\delta$ such that $\alpha(p_0) \geq 1$,
- K or L is a δ -continuous mapping,
- if $\{p_n\}$ is a sequence in M_δ such that $p_n \rightarrow \xi$ and $\alpha(p_{2n}) \geq 1$ and $\beta(p_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(p) \geq 1$ and $\beta(p) \geq 1$.

Then, K and L admit a common fixed point.

Corollary 3.2 Let M_δ be a complete non-Archimedean modular metric space and $K, L : M_\delta \rightarrow M_\delta$ be two mappings. Assume that there exists a continuous function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(t) = 0$ if and only if $t = 0$ such that

$$\begin{aligned}p, q \in M_\delta \text{ with } \alpha(p)\beta(q) \geq 1 &\Rightarrow \\ \delta_\lambda(Kp, Lq) \leq \max \left\{ \delta_\lambda(p, q), \delta_\lambda(p, Kp), \delta_\lambda(q, Lq), \frac{\delta_\lambda(p, Lq) + \delta_\lambda(q, Kp)}{2} \right\} \\ - \varphi \left(\max \left\{ \delta_\lambda(p, q), \delta_\lambda(p, Kp), \delta_\lambda(q, Lq), \frac{\delta_\lambda(p, Lq) + \delta_\lambda(q, Kp)}{2} \right\} \right).\end{aligned}$$

Suppose that the following conditions hold:

- (K, L) is a cyclic (α, β) -admissible pair,
- there exists $p_0 \in M_\delta$ such that $\alpha(p_0) \geq 1$,
- K or L is a δ -continuous mapping,
- if $\{p_n\}$ is a sequence in M_δ such that $p_n \rightarrow p$ and $\alpha(p_{2n}) \geq 1$ and $\beta(p_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(p) \geq 1$ and $\beta(p) \geq 1$.

Then, K and L admit a common fixed point.

We will replace the condition of cyclic (α, β) -admissible pair with cyclic (α, β) -admissible mapping in our main result, that is, we will take $K = L$. Then, we have the following results.

Corollary 3.3 Let M_δ be a complete non-Archimedean modular metric space and $K : M_\delta \rightarrow M_\delta$ be a given mapping. Assume that there exists $\zeta \in T_G$ such that

$$\begin{aligned}p, q \in M_\delta \text{ with } \alpha(p)\beta(q) \geq 1 &\Rightarrow \\ \zeta \left(\delta_\lambda(Kp, Lq), \max \left\{ \delta_\lambda(p, q), \delta_\lambda(p, Kp), \delta_\lambda(q, Lq), \frac{\delta_\lambda(p, Kq) + \delta_\lambda(q, Kp)}{2} \right\} \right) &\geq C_G.\end{aligned}$$

Suppose that the following conditions hold:

- K is a cyclic (α, β) -admissible mapping;

- ii. there exists $p_0 \in M_\delta$ such that $\alpha(p_0) \geq 1$ and $\beta(p_0) \geq 1$;
 - iii. V is a δ -continuous mapping, or;
 - iv. if $\{p_n\}$ is a sequence in M_δ such that $p_n \rightarrow p$ and $\beta(p_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(p) \geq 1$.
- Then, K admits a fixed point.

Corollary 3.4 Let M_δ be a complete non-Archimedean modular metric space and $K : M_\delta \rightarrow M_\delta$ be a given mapping. Assume that there exists $\zeta \in T_G$ such that

$$\alpha(p)\beta(q) \geq 1 \Rightarrow \zeta(\delta_\lambda(Kp, Kq), \delta_\lambda(p, q)) \geq C_G,$$

$p, q \in M_\delta$. Suppose that the following conditions hold:

- i. K is a cyclic (α, β) -admissible mapping;
 - ii. there exists $p_0 \in M_\delta$ such that $\alpha(p_0) \geq 1$ and $\beta(p_0) \geq 1$;
 - iii. K is a δ -continuous mapping, or;
 - iv. if $\{p_n\}$ is a sequence in M_δ such that $p_n \rightarrow \xi$ and $\beta(p_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(p) \geq 1$.
- Then, K admits a fixed point.

Example 3.5 Let $M_\delta = \mathbb{R}$, $\delta_\lambda(p, q) = \frac{1}{\lambda} |p - q|$ for all $p \in M_\delta$. Define the mapping $K : M_\delta \rightarrow M_\delta$ as follows:

$$Kp = \begin{cases} \frac{p}{4}, & \text{if } p \in [0, 1] \\ p^2, & \text{if } p > 1. \end{cases}$$

and $\alpha, \beta : M_\delta \rightarrow [0, +\infty)$ are defined by

$$\alpha(p) = \beta(p) = \begin{cases} 1, & p \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

If we consider the functions $\zeta(p, q) = \frac{1}{3}q - p$, $G(q, p) = q - p$ and $C_G = \frac{p \cdot q}{12^a}$, $a > 2$, then all the hypotheses of Corollary 3.4 are satisfied and $p = 0$ is a unique fixed point of K .

On the other hand, $\delta_\lambda(K\sqrt{6}, K\sqrt{5}) = \frac{1}{\lambda} > \frac{\sqrt{6}-\sqrt{5}}{\lambda} = \delta_\lambda(\sqrt{6}, \sqrt{5})$, $\lambda > 0$. Thus K is not a Banach contraction with respect to M_δ .

4 Common fixed points on non-Archimedean modular metric spaces with a directed graph

Let M_δ be a non-Archimedean modular metric space and $\Delta = \{(p, p) : p \in M\}$ denotes the diagonal of $M_\delta \times M_\delta$. Let Z be a directed graph such that the set $V(Z)$ of its vertices coincides with M and $J(Z)$ be the set of edges of the graph such that $\Delta \subseteq J(Z)$.

Let M_δ be a non-Archimedean modular metric space endowed with a graph Z and $K, L : M_\delta \rightarrow M_\delta$. We set

$$M_{KL} = \{p \in M : (p, Kp) \in J(Z) \text{ and } (Kp, LKp) \in J(Z)\}.$$

Definition 4.1 Let M_δ be a non-Archimedean modular metric space endowed with a directed graph Z and (K, L) be a cyclic (α, β) admissible pair. We say that K and L are generalized (α, β) - Z -simulation contraction if there exists $\zeta \in T_G$ such that

$$\alpha(p)\beta(\eta) \geq 1 \Rightarrow \zeta(\delta_\lambda(Kp, Lq), S(p, q)) \geq C_G, \quad (20)$$

where $S(p, q) = \max \left\{ \delta_\lambda(p, q), \delta_\lambda(p, Kp), \delta_\lambda(q, Lq), \frac{\kappa_\lambda(p, Lq) + \delta_\lambda(q, Lp)}{2} \right\}$, for all $(p, q) \in J(Z)$.

Theorem 4.2 Let M_δ be a δ -complete non-Archimedean modular metric space endowed with a directed graph Z . K and L are generalized (α, β) - Z -simulation contraction. Suppose that the following conditions hold:

- i. there exists $p_0 \in M_{KL}$ such that $\alpha(p_0) \geq 1$;
- ii. if $\{p_n\}$ is a sequence in M such that $p_n \rightarrow p$, $(p_{2n}, p_{2n+1}) \in J(Z)$ for all $n \geq 0$;

- ii_a. L is δ -continuous, $(p, p_{2n_k+1}) \in J(Z)$ for all $k \geq 0$ and $\alpha(p_{2n}) \geq 1$ then $\alpha(p) \geq 1$, or;
- ii_b. K is δ -continuous, $(p_{2n_k}, p) \in J(Z)$ for all $k \geq 0$ and $\beta(p_{2n+1}) \geq 1$, then $\beta(p) \geq 1$;
- iii. there is a sequence $\{p_n\}_{n \in \mathbb{N}}$ in M_δ , such that

$$(p_{2n}, Kp_{2n}) \in J(Z) \Rightarrow (p_{2n+2}, Kp_{2n+2}) \in J(Z)$$

and

$$(p_{2n+1}, Lp_{2n+1}) \in J(Z) \Rightarrow (p_{2n+3}, Lp_{2n+3}) \in J(Z);$$

Then K and L hold a common fixed point.

Proof Let p_0 be a given point in M_{KL} , then $(p_0, Kp_0) \in J(Z)$ and $(Kp_0, LKp_0) \in J(Z)$, that is, $(p_0, p_1) \in J(Z)$ and $(p_1, Lp_1) = (p_1, p_2) \in J(Z)$. From (iii), it follows that $(p_2, Kp_2) \in J(Z)$ and $(p_3, Lp_3) \in J(Z)$. That is, $(p_2, p_3) \in J(Z)$ and $(p_3, p_4) \in J(Z)$. Continuing this way, we can obtain a sequence $\{p_n\}$ in M_δ such that $(p_{2n}, Kp_{2n}) \in J(Z)$ and $(p_{2n+1}, Lp_{2n+1}) \in J(Z)$ for all $n \in \mathbb{N}$. Also, $(p_{2n}, p_{2n+1}) \in J(Z)$ and $(p_{2n+1}, p_{2n+2}) \in J(Z)$ for all $n \in \mathbb{N}$. Using arguments as in the proof of the Theorem 2.2, we obtain that

$$\delta_\lambda(p_n, p_{n+1}) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (21)$$

and $\{p_n\}$ is δ -Cauchy sequence. Since M_δ is a δ -complete, there exists a $p^* \in M_\delta$ such that

$$\delta_\lambda(p_n, p^*) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now we show that p^* is a common fixed point. As $\delta_\lambda(p_n, p^*) \rightarrow 0$ and $(p_{2n}, p_{2n+1}) \in J(Z)$, so there exists a subsequence $\{p_{2n_k}\}$ of $\{p_{2n}\}$ such that from (ii) either L is δ -continuous, $(p^*, p_{2n_k+1}) \in J(Z)$ and $\alpha(p^*) \geq 1$ or K is δ -continuous, $(p_{2n_k}, p^*) \in J(Z)$ and $\beta(p^*) \geq 1$. Assume that L is δ -continuous, $(p^*, p_{2n_k+1}) \in J(Z)$ and $\alpha(p^*) \geq 1$. Thus

$$\delta_\lambda(Lp_{2n_k+1}, Lp^*) = \delta_\lambda(p_{2n_k+2}, Lp^*) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This implies that $Lp^* = p^*$. From Definition 4.1, we have $\alpha(p^*)\beta(p_{2n_k+1}) \geq 1$

$$\begin{aligned} C_G &\leq \zeta(\delta_\lambda(Kp^*, Lp_{2n_k+1}), S(p^*, p_{2n_k+1})) \\ &< G(S(p^*, p_{2n_k+1}), \delta_\lambda(Kp^*, Lp_{2n_k+1})) \end{aligned} \quad (22)$$

and then by (i) of Definition 1.10, we get

$$\delta_\lambda(Kp^*, Lp_{2n_k+1}) < S(p^*, p_{2n_k+1}). \quad (23)$$

where

$$\begin{aligned} S(p^*, p_{2n_k+1}) &= \max\{\delta_\lambda(p^*, p_{2n_k+1}), \delta_\lambda(p^*, Kp^*), \delta_\lambda(p_{2n_k+1}, Lp_{2n_k+1}), \\ &\quad \delta_\lambda(p^*, Lp_{2n_k+1}), \delta_\lambda(p_{2n_k+1}, Kp^*)\}. \end{aligned}$$

On taking the limit as $k \rightarrow \infty$ on both sides of inequality (23), we have

$$\delta_\lambda(p^*, Kp^*) < \delta_\lambda(p^*, Kp^*), \quad (24)$$

a contradiction. Thus $Kp^* = p^*$, and hence p^* is a common fixed point. Similarly, the result follows if we suppose that K is δ -continuous, $(p_{2n_k}, p^*) \in J(Z)$ and $\beta(p^*) \geq 1$.

We note that Theorem 4.2 does not guarantee the uniqueness of a common fixed point.

To obtain the uniqueness, additional assumptions as given in the following theorem are needed.

Theorem 4.3 Let M_δ be a δ -complete non-Archimedean modular metric space endowed with a directed graph Z . K and L are generalized (α, β) - Z -simulation contraction. Suppose that the following conditions hold:

- i. there exists $p_0 \in M_{KL}$ such that $\alpha(p_0) \geq 1$;
- ii. if $\{p_n\}$ is a sequence in M_δ such that $p_n \rightarrow p$, $(p_{2n}, p_{2n+1}) \in J(Z)$ for all $n \geq 0$;
- ii_a. L is δ -continuous, $(p, p_{2n_k+1}) \in J(Z)$ for all $k \geq 0$ and $\alpha(p_{2n}) \geq 1$ then $\alpha(p) \geq 1$, or;
- ii_b. K is δ -continuous, $(p_{2n_k}, p) \in J(Z)$ for all $k \geq 0$ and $\beta(p_{2n+1}) \geq 1$, then $\beta(p) \geq 1$;

iii. there is a sequence $\{p_n\}_{n \in \mathbb{N}}$ in M_δ , such that

$$(p_{2n}, Kp_{2n}) \in J(Z) \Rightarrow (p_{2n+2}, Kp_{2n+2}) \in J(Z)$$

and

$$(p_{2n+1}, Lp_{2n+1}) \in J(Z) \Rightarrow (p_{2n+3}, Lp_{2n+3}) \in J(Z);$$

iv. there exist $p^*, q^* \in \text{Fix}(K, L)$ such that $(p^*, q^*) \in J(Z)$, $\alpha(p^*) \geq 1$ and $\beta(q^*) \geq 1$.

Then K and L hold a unique common fixed point.

Proof Let p^*, q^* be common fixed points of K and L , then $(p^*, q^*) \in J(Z)$. Also, since $\alpha(p^*) \geq 1$ and $\beta(q^*) \geq 1$. Thus, by Definition 4.1, we get that

$$C_G \leq \zeta \left(\delta_\lambda \left(Kp^*, Lq^* \right), S \left(p^*, q^* \right) \right) < G \left(S \left(p^*, q^* \right), \delta_\lambda \left(Kp^*, Lq^* \right) \right) \quad (25)$$

and from (i) of the Definition 1.10, we have

$$\begin{aligned} \delta_\lambda \left(Kp^*, Lq^* \right) &< S \left(p^*, q^* \right) \\ &= \max \left\{ \delta_\lambda \left(p^*, q^* \right), 0 \right\} \\ &= \delta_\lambda \left(p^*, q^* \right). \end{aligned}$$

which is a contradiction. Hence $p^* = q^*$.

Example 4.4 Let $M_\delta = [0, 1]$, $\delta_\lambda(p, q) = \frac{1}{\lambda} |p - q|$ for all $p \in M_\delta$ and $E(G) = \{(p, q) : p, q \in [0, 1]\}$. Define the mappings $K, L : M \rightarrow M$ as follows:

$$Kp = \frac{p}{2}, \quad Lp = \frac{p}{3}, \quad p \in M_\delta.$$

and $\alpha, \beta : M_\delta \rightarrow [0, +\infty)$ are defined by

$$\alpha(p) = \beta(p) = \begin{cases} 1, & p \in (0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

If we consider the functions $\zeta(p, q) = \frac{3}{4}q - p$, $G(q, p) = q - p$ and $C_G = \frac{p \cdot q}{2^{4a}}$, $a > 2$, then all the hypotheses of Theorem 4.3 are satisfied and $p = 0$ is a unique common fixed point.

5 Application to dynamic programming

The purpose of this section is to present an application on dynamic programming.

Throughout this section, we assume that A and B are Banach spaces, $W \subseteq A$ and $D \subseteq B$ is a decision space. Now, consider the following system of functional equations:

$$P(p) = \sup_{q \in D} \{s(p, q) + \chi(p, q, P(\tau(p, q)))\}, \quad (26)$$

$$Q(p) = \sup_{q \in D} \{s(p, q) + \varsigma(p, q, Q(\tau(p, q)))\}, \quad (27)$$

where $\tau : W \times D \rightarrow W$, $r : W \times D \rightarrow \mathbb{R}$, $\chi : W \times D \times \mathbb{R} \rightarrow \mathbb{R}$, $\varsigma : W \times D \times \mathbb{R} \rightarrow \mathbb{R}$.

Let $M = B(W)$ denotes the space of all bounded real-valued function on W . Consider the metric given by

$$\delta_\lambda(m, n) = \frac{1}{\lambda} \sup_{p \in W} |m(p) - n(p)|, \quad \text{for all } m, n \in A \text{ and } \lambda > 0.$$

Then M_δ is a δ -complete non-Archimedean modular metric space.

Now, take the mappings $E, F : M_\delta \rightarrow M_\delta$ as

$$Em(p) = \sup_{q \in D} \{s(p, q) + \chi(p, q, m(\tau(p, q)))\}, \quad (28)$$

and

$$Fm(p) = \sup_{q \in D} \{s(p, q) + \zeta(p, q, m(\tau(p, q)))\}, \quad (29)$$

where $p \in W$ and $m \in M_\delta$. If the functions s, χ, τ are bounded, then E and F are well-defined.

Theorem 5.1 Suppose that there exists $k \in [0, 1)$ such that for each $(p, q) \in W \times D$ and $m_1, m_2 \in M_\delta$, the inequality

$$|\chi(p, q, m_1(\tau(p, q))) - \zeta(p, q, m_2(\tau(p, q)))| \leq k|m_1 - m_2|,$$

holds. Then, E and F admit a common fixed point in M_δ .

Proof Let $\varepsilon > 0$ be an arbitrary real number, $p \in W$ and $m_1, m_2 \in M_\delta$. Then by (28) and (29), there exist $q_1, q_2 \in D$ such that

$$Em_1(p) < s(p, q_1) + \chi(p, q_1, m_1(\tau(p, q_1))) + \varepsilon, \quad (30)$$

$$Fm_2(p) < s(p, q_2) + \zeta(p, q_2, m_2(\tau(p, q_2))) + \varepsilon, \quad (31)$$

$$Em_1(p) \geq s(p, q_2) + \chi(p, q_2, m_1(\tau(p, q_2))), \quad (32)$$

and

$$Fm_2(p) \geq s(p, q_1) + \zeta(p, q_1, m_2(\tau(p, q_1))). \quad (33)$$

Then from (30) and (33), it follows easily that

$$\begin{aligned} Em_1(p) - Fm_2(p) &\leq \chi(p, q_1, m_1(\tau(p, q_1))) + \varepsilon - \zeta(p, q_1, m_2(\tau(p, q_1))) \\ &\leq |\chi(p, q_1, m_1(\tau(p, q_1))) - \zeta(p, q_1, m_2(\tau(p, q_1)))| + \varepsilon \\ &\leq k|m_1 - m_2| + \varepsilon. \end{aligned}$$

Similarly, from (31) and (32), we have

$$Fm_2(p) - Em_1(p) \leq k|m_1 - m_2| + \varepsilon.$$

Thus, we deduce from above inequalities that

$$|Em_1(p) - Fm_2(p)| \leq k|m_1 - m_2| + \varepsilon. \quad (34)$$

Since the inequality (34) is valid for any $p \in W$, then

$$\delta_\lambda(Em_1, Fm_2) = \frac{1}{\delta} |Em_1 - Fm_2| \leq \frac{1}{\delta} k|m_1 - m_2| + \varepsilon = k\delta_\lambda(m_1, m_2) + \varepsilon,$$

also $\varepsilon > 0$ is arbitrary, so

$$\delta_\lambda(Em_1, Fm_2) \leq k\delta_\lambda(m_1, m_2).$$

Now, taking $\zeta(a, b) = kb - a$, $G(b, a) = b - a$, then for $C_G = 0$, $S(p, q) = \delta_\lambda(p, q)$, and $\alpha(p)\beta(q) = 1$ in Theorem 2.2. Hence all conditions of Theorem 2.2 are satisfied. Therefore E and F have a common fixed point, that is, the functional equation (26) and (27) have a common solution in M_δ .

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