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Approximate fixed points and fixed points for multi-valued almost E -contractions

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Abstract: In this paper, we introduce the concept of multi-valued almost E -contractions. We then present some approximate fixed point and fixed point results for such mappings in metric spaces. Our results generalize and improve several well-known results in literature. We also provide several illustrative examples to compare our findings with some earlier results. An application to homotopy theory is given.

Keywords: Approximate fixed points, Fixed points, Multi-valued mappings, E -contractions, Almost contractions, Homotopy theory

MSC 2020: 47H10, 54H25

1 Introduction

Let (X, d) be a metric space. We denote by $P(X)$ the set of all nonempty subsets of X , by $C(X)$ the set of all nonempty, closed subsets of X and by $CB(X)$ the set of all nonempty closed bounded subsets of X . For a subset A of X , we denote by $\bar{A}$ the closure of A and by ∂A the boundary of A . For $x \in X$ and $r > 0$, we denote by $B(x, r) := \{y \in X : d(y, x) < r\}$ the open ball at x with radius r . The distance from $x \in X$ to a subset A of X is defined by

$$D(x, A) := \inf\{d(x, a) : a \in A\}.$$

For two subsets A and B of X , the excess of A over B is defined by

$$e(A, B) := \sup\{d(a, B) : a \in A\}.$$

The generalized Hausdorff - Pompeiu distance between two subsets A and B of X is defined by

$$H(A, B) := \max\{e(A, B), e(B, A)\}.$$

Here, with convention, $D(x, \emptyset) = +\infty$ for $x \in X$ and $e(\emptyset, B) = 0$ for $B \subset X$. It is well-known that, for any $x, y \in X$ and $A \subset X$, it holds

$$D(x, A) \leq d(x, y) + D(y, A).$$

The following simple lemma may be known which will be used in the sequel. We present the proof for the reader's convenience.

Lemma 1. *Let (X, d) be a metric space, A be a nonempty subset of X , $x \in X$ and $\alpha \in [0, 1)$ and $\ell > D(x, A)$. If $D(x, A) > 0$, then there exists $a \in A$ such that*

$$d(x, a) < \ell \quad \text{and} \quad \alpha d(x, a) < D(x, A). \quad (1)$$

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Proof. It is obvious if $\alpha = 0$. If $\alpha > 0$, we set

$$\varepsilon = \min \left\{ \ell - D(x, A), \left(\frac{1}{\alpha} - 1 \right) D(x, A) \right\} > 0.$$

By the definition of $D(x, A)$, there exists $a \in A$ such that

$$d(x, a) < D(x, A) + \varepsilon.$$

By the definition of ε , we obtain (1). □

Let $T : X \rightarrow P(X)$ be a multi-valued mapping. We denote by

$$\text{gph}(T) := \{(x, y) \in X \times X : y \in T(x)\}$$

the graph of T . The multi-valued T mapping is said to be closed if $\text{gph}(T)$ is closed in $X \times X$, i.e., for all sequences $\{(x_n, y_n)\}$ in $\text{gph}(T)$ such that $\{(x_n, y_n)\}$ converges to some $(x, y) \in X \times X$, then $(x, y) \in \text{gph}(T)$. The map T is said to have closed values if $T(x)$ is closed for all $x \in X$. An element $x \in X$ is said to be a fixed point of T if $x \in T(x)$. An orbit of T at a point $x_0 \in X$ is a sequence $\{x_n\}$ in X such that $x_{n+1} \in T(x_n)$ for all $n \geq 0$. The set of all fixed points of T is denoted by $\text{Fix}(T)$.

A mapping $T : X \rightarrow P(X)$ is said to be a *multi-valued contraction* or *multi-valued α -contraction* if there exists a constant $\alpha \in [0, 1)$ such that

$$H(T(x), T(y)) \leq \alpha d(x, y), \quad \forall x, y \in X. \quad (2)$$

Nadler [28] (see also [12]) proved that every multi-valued contraction mapping with closed values in a complete metric space has at least one fixed point. This result is an extension of the famous Banach contraction principle to the context of set-valued mappings. After that, many authors have extended and generalized Nadler's fixed point theorem in different ways. One of the interesting results was given by M. Berinde and V. Berinde in [10] where they introduced the concept of multi-valued weak contractions and proved some fixed point theorems for multi-valued weak contractions.

Definition 1. ([10]) A multi-valued mapping $T : X \rightarrow P(X)$ is said to be *multi-valued weak contraction* or *multi-valued (θ, L) -weak contraction* if there exist $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(T(x), T(y)) \leq \theta d(x, y) + LD(y, T(x)), \quad \forall x, y \in X. \quad (3)$$

It is proved in [10] that every multi-valued weak contraction in a complete metric space has at least **one** fixed point. The results in [10] have been extended by many authors. For some extensions and generalizations of the results by Nadler as well as by Berinde - Berinde, we refer the reader to, e.g., [1, 9, 14, 24, 25, 29, 31–33] and references therein.

Recently, Fulga and Proca [15] introduced the notion of φ_E -Geraghty contractions and proved a fixed point theorem.

Definition 2. ([15]) A mapping $T : X \rightarrow X$ is said to be a φ_E -Geraghty contraction if there exists a function $\varphi : [0, \infty) \rightarrow [0, 1)$ which satisfies the condition

$$\lim_{n \rightarrow \infty} \varphi(t_n) = 1 \quad \text{implies that} \quad \lim_{n \rightarrow \infty} t_n = 0,$$

such that

$$d(Tx, Ty) \leq \varepsilon(E(x, y))E(x, y), \quad \forall x, y \in X,$$

where $E(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)|$.

Later, several researchers have investigated new fixed point results for mappings satisfying generalized conditions involving E -contractions (called P -contractions in some papers), i.e., conditions which involve the

expression $E(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)|$. For more details, we refer the reader to, e.g., [2–4, 6–8, 16–18, 21] and references therein. Fixed point results of multi-valued mappings concerning P -contractions can be found in [5, 19].

It is worth noting that in many practical situations the existence of an approximation solution or an approximate fixed point is more than enough, e.g., approximate Nash equilibria [35], approximate solutions of mathematical programming problems [26]. On the other hand, conditions for ensuring the existence of a fixed point are too strong and may not be satisfied in many real problems. Hence, it is natural to introduce and investigate the theory of ε -approximate fixed points. Starting from the paper by Tijs, Torre and Brânzei [34], many researchers have established various approximate fixed point results for single-valued and multi-valued mappings satisfying different contractive type conditions (see, e.g., [11, 20, 27, 30] and references therein).

Definition 3. ([11, 34]) Let (X, d) be a metric space, $T : X \rightarrow P(X)$ be a multi-valued mapping and $\varepsilon \geq 0$. An element $x^* \in X$ is said to be an ε -approximate fixed point of T if $D(x^*, T(x^*)) \leq \varepsilon$. The set of all ε -approximate fixed points of T is denoted by $\varepsilon\text{-Fix}(T)$, i.e.,

$$\varepsilon\text{-Fix}(T) = \{x \in X : D(x, T(x)) \leq \varepsilon\}.$$

It is evident that $\text{Fix}(T) \subset 0\text{-Fix}(T)$ and $\text{Fix}(T) = 0\text{-Fix}(T)$ if T has closed values. Moreover, approximate fixed points may exist under mild conditions in which exact fixed points do not exist. For example, if T is a multi-valued contraction mapping from a metric space X to $P(X)$, then $\varepsilon\text{-Fix}(T) \neq \emptyset$ for every $\varepsilon > 0$ (see, e.g., [27]). However, the set of fixed points of T may be empty when X is not complete. We recall here the following interesting result which is presented in [27].

Theorem 1. ([27]) Let (X, d) be a metric space, $\kappa \in (0, 1)$ and let $T : X \rightarrow P(X)$ be a multi-valued κ -contraction. Then, for every $\varepsilon > 0$, the set of ε -approximate fixed points of T is nonempty, i.e., $\varepsilon\text{-Fix}(T) \neq \emptyset$ and the following estimate is satisfied:

$$D(x, \varepsilon\text{-Fix}(T)) \leq \frac{1}{1-\kappa} D(x, T(x)), \quad \forall x \in X. \quad (4)$$

Noting that, by using the results of Theorem 1 and some results about set-convergence in metric spaces, the authors in [27] derive several new results concerning approximate versions of Lim's Lemma, and the celebrated global Lyusternik-Graves Theorem (see, e.g., [13] for more details about these results and their applications).

Inspired by above mentioned results, in this paper, we first introduce the concept of multi-valued almost E -contractions and prove an approximate fixed point theorem for such mappings on incomplete metric spaces. This result extends and generalizes several approximate fixed point results in the literature. An approximation of Lim's type result and a data dependence result for approximate fixed point sets of multi-valued almost E -contractions are also presented. As corollaries, we also present some fixed point results for multi-valued almost E -contractions in complete metric spaces. An application to homotopy theory is given as well.

2 Approximate fixed points for multi-valued almost E - contractions

We begin this section by introducing the concept of multi-valued almost E -contractions.

Definition 4. Let (X, d) be a metric space and $T : X \rightarrow P(X)$ be a multi-valued map. We say that T is a *multi-valued almost E -contraction* or *multi-valued (α, β, L) -almost E -contraction* if there exist constants $\alpha \in [0, 1)$, $\beta \in [0, 1]$ and $L \geq 0$ such that

$$e(T(x), T(y)) \leq \alpha d(x, y) + \beta |D(x, T(x)) - D(y, T(y))| + LD(y, T(x)), \quad \forall x, y \in X. \quad (5)$$

Remark 1. The condition (5) includes several well-known contractive conditions in the literature. In fact, if we let $\beta = 0$ in (5), then we get the relation

$$e(T(x), T(y)) \leq \alpha d(x, y) + LD(y, T(x)), \quad \forall x, y \in X. \quad (6)$$

Since $e(T(x), T(y)) \leq H(T(x), T(y))$ for all $x, y \in X$, if T is a multi-valued weak contraction, then T satisfies (6). Thus, a multi-valued weak contraction is a special case of multi-valued almost E -contractions.

If $L = 0$, then (5) reduces to

$$e(T(x), T(y)) \leq \alpha d(x, y) + \beta |D(x, T(x)) - D(y, T(y))|, \quad \forall x, y \in X.$$

By the symmetry, we have

$$e(T(y), T(x)) \leq \alpha d(x, y) + \beta |D(x, T(x)) - D(y, T(y))|, \quad \forall x, y \in X.$$

These relations imply that

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta |D(x, T(x)) - D(y, T(y))|, \quad \forall x, y \in X.$$

If $\alpha = \beta \in (0, 1)$ in the latter inequality, i.e.,

$$H(T(x), T(y)) \leq \alpha [d(x, y) + |D(x, T(x)) - D(y, T(y))|], \quad \forall x, y \in X,$$

then we call T a *multi-valued E -contraction* with constant α .

We next give an example of a multi-valued mapping which is a multi-valued (α, β, L) -almost E -contraction but it is not a multi-valued weak contraction.

Example 1. Let $X = \mathbb{R}$ with the usual metric. Define the mapping $T : X \rightarrow P(X)$ by

$$T(x) = \begin{cases} \{0\} & \text{if } x \leq 0, \\ [-2x, -x] & \text{if } x > 0. \end{cases}$$

For any $\alpha \in (0, 1)$ and $L \geq 0$, let $x = -y > 0$, we have

$$\alpha d(x, y) + LD(y, T(x)) = \alpha(x - y) + L \cdot 0 = 2\alpha x < 2x = H(T(x), T(y)).$$

Thus, T is not a multi-valued weak contraction and so T is not a multi-valued contraction.

We next show that T is a multi-valued (α, β, L) almost E -contraction with $\alpha = \beta = 2/3$ and $L = 1$, i.e.,

$$e(T(x), T(y)) \leq \frac{2}{3} d(x, y) + \frac{2}{3} |D(x, T(x)) - D(y, T(y))| + D(y, T(x)), \quad (7)$$

for all $x, y \in X$. It is evident that (7) holds for $x, y \in (-\infty, 0]$. We now consider three cases:

Case 1. $x, y \in (0, \infty)$. We may assume $x > y > 0$. In this case

$$e(T(x), T(y)) \leq H(T(x), T(y)) = 2(x - y),$$

and the right-hand side of (7) equal to

$$\frac{2}{3}(x - y) + \frac{2}{3}|2x - 2y| + D(y, T(x)) = 2(x - y) + D(y, T(x)) \geq 2(x - y).$$

Thus, (7) holds.

Case 2. $x \leq 0$ and $y > 0$. In this case, we have $e(T(x), T(y)) = y$ and $D(y, T(x)) = y$. Thus, (7) holds.

Case 3. $x > 0$ and $y \leq 0$. In this case, $e(T(x), T(y)) = 2x$ and the right-hand side of (7) equal to

$$RHS := \frac{2}{3}(x - y) + \frac{2}{3}|2x + y| + D(y, T(x)).$$

If $y \geq -2x$, then $RHS = 2x + D(y, T(x)) \geq 2x = e(T(x), T(y))$. Thus, (7) holds. If $y < -2x$, then

$$RHS \geq \frac{2}{3}(x - y) > \frac{2}{3}(x + 2x) = 2x = e(T(x), T(y)).$$

Thus, (7) holds.

The following is the first main result of this paper. It concerns with approximate fixed points of multi-valued almost E -contraction mappings.

Theorem 2. *Let (X, d) be a metric space and $T : X \rightarrow P(X)$ be a multi-valued (α, β, L) -almost E -contraction. Then, for every $\varepsilon > 0$, the set of ε -approximate fixed points of T is nonempty, i.e., $\varepsilon\text{-Fix}(T) \neq \emptyset$ and the following estimate holds true:*

$$D(x, \varepsilon\text{-Fix}(T)) \leq \frac{1+\beta}{1-\alpha} D(x, T(x)), \quad (8)$$

for all $x \in X$.

Proof. Let $\varepsilon > 0$, $x \in X$ and $\delta > 0$ be arbitrary. Set $\ell = D(x, T(x)) + \delta$. If $D(x, T(x)) = 0$, then $x \in \varepsilon\text{-Fix}(T)$. Moreover, $D(x, \varepsilon\text{-Fix}(T)) = 0 = D(x, T(x))$, which implies that (8) holds. Assume that $D(x, T(x)) > 0$. Then, by Lemma 1, there exists $x_1 \in T(x)$ such that

$$d(x, x_1) < \ell \quad \text{and} \quad \alpha d(x, x_1) < D(x, T(x)).$$

By (5), we have

$$\begin{aligned} D(x_1, T(x_1)) &\leq e(T(x), T(x_1)) \\ &\leq \alpha d(x, x_1) + \beta |D(x, T(x)) - D(x_1, T(x_1))| + LD(x_1, T(x)) \\ &= \alpha d(x, x_1) + \beta |D(x, T(x)) - D(x_1, T(x_1))|. \end{aligned} \quad (9)$$

Assume that $D(x, T(x)) \leq D(x_1, T(x_1))$. Then, it follows from (9) that

$$D(x_1, T(x_1)) \leq \alpha d(x, x_1) + \beta D(x_1, T(x_1)) - \beta D(x, T(x))$$

or, equivalently,

$$(1 - \beta)D(x_1, T(x_1)) \leq \alpha d(x, x_1) - \beta D(x, T(x)).$$

Since $D(x, T(x)) < D(x_1, T(x_1))$ and $\alpha d(x, x_1) < D(x, T(x))$, we have

$$(1 - \beta)D(x, T(x)) < (1 - \beta)D(x_1, T(x_1)).$$

This is a contradiction. Thus, $D(x_1, T(x_1)) < D(x, T(x))$. Then, one has from (9) that

$$D(x_1, T(x_1)) \leq \alpha d(x, x_1) + \beta D(x, T(x)) - \beta D(x_1, T(x_1)).$$

This implies that

$$D(x_1, T(x_1)) \leq \frac{\alpha}{1+\beta} d(x, x_1) + \frac{\beta}{1+\beta} D(x, T(x)) < \frac{\alpha+\beta}{1+\beta} \ell = \theta \ell,$$

where $\theta = \frac{\alpha+\beta}{1+\beta}$.

If $D(x_1, T(x_1)) = 0$, then $x_1 \in \varepsilon\text{-Fix}(T)$. In this case,

$$D(x, \varepsilon\text{-Fix}(T)) \leq d(x, x_1) < \ell = D(x, T(x)) + \delta < \frac{1}{1-\theta} D(x, T(x)) + \delta.$$

Letting $\delta \rightarrow 0^+$, we get

$$D(x, \varepsilon\text{-Fix}(T)) \leq \frac{1}{1-\theta} D(x, T(x)),$$

i.e., (8) holds. Assume now that $D(x_1, T(x_1)) > 0$. Then, again by Lemma 1, there exists $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) < \theta \ell \quad \text{and} \quad \alpha d(x_1, x_2) < D(x_1, T(x_1)).$$

Again, by (5), we have

$$\begin{aligned} D(x_2, T(x_2)) &\leq e(T(x_1), T(x_2)) \\ &= \alpha d(x_1, x_2) + \beta |D(x_1, T(x_1)) - D(x_2, T(x_2))|. \end{aligned} \quad (10)$$

If $D(x_1, T(x_1)) \leq D(x_2, T(x_2))$, then

$$\begin{aligned}(1 - \beta)D(x_1, T(x_1)) &\leq (1 - \beta)D(x_2, T(x_2)) \leq \alpha d(x_1, x_2) - \beta D(x_1, T(x_1)) \\ &< D(x_1, T(x_1)) - \beta D(x_1, T(x_1)) = (1 - \beta)D(x_1, T(x_1)),\end{aligned}$$

which is a contradiction. Thus, $D(x_2, T(x_2)) < D(x_1, T(x_1))$. By (10),

$$\begin{aligned}D(x_2, T(x_2)) &\leq \frac{\alpha}{1 + \beta}d(x_1, x_2) + \frac{\beta}{1 + \beta}D(x_1, T(x_1)) \\ &\leq \frac{\alpha}{1 + \beta}\theta\ell + \frac{\beta}{1 + \beta}\theta\ell = \theta^2\ell.\end{aligned}$$

Continuing this progress, we can construct a sequence $\{x_n\}$ in X such that

- (a) there exists $k \in \mathbb{N}$ such that $D(x_k, T(x_k)) = 0$ and $d(x_i, x_{i+1}) < \theta^i\ell$ for all $i = 1, \dots, k - 1$, or
(b) for all $n \geq 1$

$$x_{n+1} \in T(x_n), \quad D(x_n, T(x_n)) < \theta^n\ell \quad \text{and} \quad d(x_n, x_{n+1}) < \theta^n\ell \quad (11)$$

If (a) holds, then $x_k \in \varepsilon\text{-Fix}(T)$ and

$$\begin{aligned}D(x, \varepsilon\text{-Fix}(T)) &\leq d(x, x_k) \\ &\leq d(x, x_1) + d(x_1, x_2) + \dots + d(x_{k-1}, x_k) \\ &< \ell + \theta\ell + \dots + \theta^{k-1}\ell \\ &< \frac{1}{1 - \theta}\ell = \frac{1 + \beta}{1 - \alpha}[D(x, T(x)) + \delta].\end{aligned}$$

Letting $\delta \rightarrow 0^+$, we get (8).

Assume that (b) happens. Since $\theta \in [0, 1)$, $\lim_{n \rightarrow \infty} \theta^n\ell = 0$. Hence, there exists $n_0 \in \mathbb{N}$ such that $\theta^n\ell < \varepsilon$ for all $n \geq n_0$. This means that $D(x_n, T(x_n)) < \varepsilon$ for all $n \geq n_0$. Thus, all elements of the sequence $\{x_n\}_{n \geq n_0}$ are ε -approximate fixed points of T . Moreover, for all $n \in \mathbb{N}$ with $n \geq 2$, we have

$$\begin{aligned}d(x, x_n) &\leq d(x, x_1) + d(x_1, x_2) + \dots + d(x_{n-1}, x_n) \\ &< \ell + \theta\ell + \dots + \theta^{n-1}\ell \\ &\leq \ell(1 + \theta + \dots + \theta^{n-1}) \\ &= \frac{\ell}{1 - \theta} = \frac{1 + \beta}{1 - \alpha}[D(x, T(x)) + \delta].\end{aligned}$$

Since $x_n \in \varepsilon\text{-Fix}(T)$ for n large enough (i.e., $n \geq n_0$), we have

$$D(x, \varepsilon\text{-Fix}(T)) \leq d(x, x_n) < \frac{1 + \beta}{1 - \alpha}[D(x, T(x)) + \delta].$$

Letting $\delta \rightarrow 0^+$, we get (8). This ends the proof. $\square$

In Theorem 2, if we let $\beta = 0$ and $L = 0$, we obtain Theorem 1 - one of the main results of [27]. If we let $\beta = 0$, then we get the following ε -approximate fixed point result for multi-valued weak contraction mappings.

Corollary 1. *Let (X, d) be a metric space and $T : X \rightarrow P(X)$ be a multi-valued (α, L) -weak contraction. Then, for every $\varepsilon > 0$, the set of ε -approximate fixed points of T is nonempty, i.e., $\varepsilon\text{-Fix}(T) \neq \emptyset$ and the following estimate holds:*

$$D(x, \varepsilon\text{-Fix}(T)) \leq \frac{1}{1 - \alpha}D(x, T(x)), \quad \forall x \in X. \quad (12)$$

We remark that the estimate (12) is new for the case of multi-valued weak contraction mappings. Similarly, if we let $L = 0$ in Theorem 2, we obtain the following ε -approximate fixed point result for multi-valued E -contraction mappings.

Corollary 2. *Let (X, d) be a metric space and $T : X \rightarrow P(X)$ be a multi-valued E -contraction with constant $\alpha \in (0, 1)$. Then, for every $\varepsilon > 0$, the set of ε -approximate fixed points of T is nonempty, i.e., $\varepsilon\text{-Fix}(T) \neq \emptyset$ and the following estimate holds:*

$$D(x, \varepsilon\text{-Fix}(T)) \leq \frac{1 + \alpha}{1 - \alpha}D(x, T(x)), \quad \forall x \in X. \quad (13)$$

Example 2. Let (X, d) and T be as in Example 1. For $\varepsilon > 0$, we have $\varepsilon\text{-Fix}(T) = [-\varepsilon, \varepsilon/2]$. Thus, for all $x \in X$, we have

$$D(x, \varepsilon\text{-Fix}(T)) = \begin{cases} -x - \varepsilon & \text{if } x < -\varepsilon, \\ 0 & \text{if } x \in [-\varepsilon, \varepsilon/2], \\ x - \varepsilon/2 & \text{if } x > \varepsilon/2, \end{cases}$$

and

$$D(x, \varepsilon\text{-Fix}(T)) \leq 5D(x, T(x)) = \frac{1+\beta}{1-\beta}D(x, T(x)).$$

Therefore, (8) is verified for this example.

We next provide an example to illustrate Corollary 1.

Example 3. Let $X = \left\{ \frac{1}{2^n} : n = 0, 1, 2, \dots \right\}$ with the usual metric d . It is evident that (X, d) is not a complete metric space. We define the mapping $T : X \rightarrow P(X)$ as

$$T(x) = \begin{cases} \{1\} & \text{if } x = 1, \\ \left\{ \frac{1}{2^{n+1}}, \frac{1}{2^{n+2}} \right\} & \text{if } x = \frac{1}{2^n}, \quad n = 1, 2, \dots \end{cases}$$

For $n \geq 1$, we have

$$H\left(T(1), T\left(\frac{1}{2^n}\right)\right) = 1 - \frac{1}{2^{n+2}}, \quad \text{and} \quad d\left(1, \frac{1}{2^n}\right) = 1 - \frac{1}{2^n}.$$

If there exists $\alpha \in (0, 1)$ such that

$$H\left(T(1), T\left(\frac{1}{2^n}\right)\right) \leq \alpha d\left(1, \frac{1}{2^n}\right),$$

then

$$1 - \frac{1}{2^{n+2}} \leq \alpha \left(1 - \frac{1}{2^n}\right).$$

Letting $n \rightarrow \infty$, we have $1 \leq \alpha$. This contradicts to $\alpha < 1$. Thus, T is not a multi-valued contraction.

We next show that T is a multi-valued (α, β, L) almost E -contraction with $\alpha = \frac{1}{2}$, $\beta = 0$ and $L = 2$, i.e.,

$$e(T(x), T(y)) \leq \frac{1}{2}d(x, y) + 2D(y, T(x)), \quad (14)$$

for all $x, y \in X$. There are four cases to consider.

Case 1. $x = 1$ and $y = \frac{1}{2^n}$ with $n \geq 1$. Since

$$e\left(T(1), T\left(\frac{1}{2^n}\right)\right) = 1 - \frac{1}{2^{n+1}} \quad \text{and} \quad D\left(\frac{1}{2^n}, T(1)\right) = 1 - \frac{1}{2^n},$$

the inequality (14) holds for this case.

Case 2. $x = \frac{1}{2^n}$ and $y = 1$ with $n \geq 1$. In this case,

$$e\left(T\left(\frac{1}{2^n}\right), T(1)\right) = 1 - \frac{1}{2^{n+2}} \quad \text{and} \quad D\left(1, T\left(\frac{1}{2^n}\right)\right) = 1 - \frac{1}{2^{n+1}}.$$

Thus, (14) holds.

Case 3. $x = \frac{1}{2^n}$ and $y = \frac{1}{2^{n+k}}$ with $n \geq 1$ and $k \geq 1$. In this case,

$$e\left(T\left(\frac{1}{2^n}\right), T\left(\frac{1}{2^{n+k}}\right)\right) = \frac{1}{2^{n+1}} - \frac{1}{2^{n+k+1}} = \left(1 - \frac{1}{2^k}\right) \frac{1}{2^{n+1}},$$

and

$$d\left(\frac{1}{2^n}, \frac{1}{2^{n+k}}\right) = \frac{1}{2^n} - \frac{1}{2^{n+k}} = 2\left(1 - \frac{1}{2^k}\right) \frac{1}{2^{n+1}}.$$

Thus, (14) holds.

Case 4. $x = \frac{1}{2^{n+k}}$ and $y = \frac{1}{2^n}$ with $n \geq 1$ and $k \geq 1$. Since

$$e\left(T\left(\frac{1}{2^{n+k}}\right), T\left(\frac{1}{2^n}\right)\right) = \frac{1}{2^{n+2}} - \frac{1}{2^{n+k+2}} = \left(1 - \frac{1}{2^k}\right) \frac{1}{2^{n+2}},$$

and

$$d\left(\frac{1}{2^{n+k}}, \frac{1}{2^n}\right) = 4\left(1 - \frac{1}{2^k}\right) \frac{1}{2^{n+2}},$$

(14) holds for this case.

For every $\varepsilon > 0$, we have $\frac{1}{2^n} \in \varepsilon\text{-Fix}(T)$ for $n = 0$ and for n large enough. For $m \geq 1$, we have

$$\frac{1}{1-\alpha} D\left(\frac{1}{2^m}, T\left(\frac{1}{2^m}\right)\right) = \frac{1}{2^m}.$$

For n large enough,

$$D\left(\frac{1}{2^m}, \varepsilon\text{-Fix}(T)\right) \leq \left|\frac{1}{2^m} - \frac{1}{2^n}\right| \leq \frac{1}{2^m} = \frac{1}{1-\alpha} D\left(\frac{1}{2^m}, T\left(\frac{1}{2^m}\right)\right).$$

Therefore, (12) is verified for this example.

The next theorem is an approximate fixed point result of Lim's type for multi-valued almost E -contractions. Theorem 3 and its corollary extend [27, Theorem 15] where an estimate similar to (17) for multi-valued contraction mappings was investigated.

Theorem 3. Let X be a metric space and $T_1, T_2 : X \rightarrow P(X)$ be multi-valued (α, β, L) -almost E -contractions. Then, for every $\varepsilon > 0$, the set of ε -approximate fixed points of T_i , $i = 1, 2$, is nonempty. Moreover,

$$e(\varepsilon\text{-Fix}(T_1), \varepsilon\text{-Fix}(T_2)) \leq \frac{1+\beta}{1-\alpha} \varepsilon + \frac{1+\beta}{1-\alpha} \sup_{x \in X} e(T_1(x), T_2(x)). \quad (15)$$

Proof. Let $\varepsilon > 0$. By Theorem 2, both $\varepsilon\text{-Fix}(T_1)$ and $\varepsilon\text{-Fix}(T_2)$ are nonempty. We now prove (15). Let $x \in X$. By the estimate (8), we have

$$D(x, \varepsilon\text{-Fix}(T_2)) \leq \frac{1}{1-\theta} D(x, T_2(x)), \quad (16)$$

where $\theta = \frac{\alpha+\beta}{1+\beta}$.

For any $\delta > 0$, there exists $y \in T_1(x)$ such that $d(x, y) < D(x, T_1(x)) + \delta$. From (16),

$$\begin{aligned} D(x, \varepsilon\text{-Fix}(T_2)) &\leq \frac{1}{1-\theta} D(x, T_2(x)) \\ &\leq \frac{1}{1-\theta} [d(x, y) + D(y, T_2(x))] \\ &= \frac{1}{1-\theta} d(x, y) + \frac{1}{1-\theta} D(y, T_2(x)) \\ &< \frac{1}{1-\theta} [D(x, T_1(x)) + \delta] + \frac{1}{1-\theta} D(y, T_2(x)) \\ &\leq \frac{1}{1-\theta} D(x, T_1(x)) + \frac{\delta}{1-\theta} + \frac{1}{1-\theta} e(T_1(x), T_2(x)). \end{aligned}$$

Letting $\delta \rightarrow 0^+$, we get

$$D(x, \varepsilon\text{-Fix}(T_2)) \leq \frac{1}{1-\theta} [D(x, T_1(x)) + e(T_1(x), T_2(x))].$$

Taking supremum in both sides of the latter inequality over $x \in \varepsilon\text{-Fix}(T_1)$, one has

$$\begin{aligned} e(\varepsilon\text{-Fix}(T_1), \varepsilon\text{-Fix}(T_2)) &\leq \sup_{x \in \varepsilon\text{-Fix}(T_1)} \frac{1}{1-\theta} [D(x, T_1(x)) + e(T_1(x), T_2(x))] \\ &\leq \frac{1}{1-\theta} \sup_{x \in \varepsilon\text{-Fix}(T_1)} D(x, T_1(x)) \\ &\quad + \frac{1}{1-\theta} \sup_{x \in \varepsilon\text{-Fix}(T_1)} e(T_1(x), T_2(x)) \\ &\leq \frac{1}{1-\theta} \varepsilon + \frac{1}{1-\theta} \sup_{x \in X} e(T_1(x), T_2(x)) \\ &= \frac{1+\beta}{1-\alpha} \varepsilon + \frac{1+\beta}{1-\alpha} \sup_{x \in X} e(T_1(x), T_2(x)). \end{aligned}$$

This ends the proof. $\square$

Corollary 3. Let X be a metric space and $T_1, T_2 : X \rightarrow P(X)$ be multi-valued (α, β, L) -almost E -contractions. Then, for every $\varepsilon > 0$, the set of ε -approximate fixed points of T_i , $i = 1, 2$, is nonempty and

$$H(\varepsilon\text{-Fix}(T_1), \varepsilon\text{-Fix}(T_2)) \leq \frac{1+\beta}{1-\alpha} \varepsilon + \frac{1+\beta}{1-\alpha} \sup_{x \in X} H(T_1(x), T_2(x)). \quad (17)$$

Proof. Let $\varepsilon > 0$. By Theorem 3, we have

$$e(\varepsilon\text{-Fix}(T_1), \varepsilon\text{-Fix}(T_2)) \leq \frac{1+\beta}{1-\alpha} \varepsilon + \frac{1+\beta}{1-\alpha} \sup_{x \in X} e(T_1(x), T_2(x)).$$

Since

$$e(T_1(x), T_2(x)) \leq H(T_1(x), T_2(x)), \quad \forall x \in X,$$

we have

$$e(\varepsilon\text{-Fix}(T_1), \varepsilon\text{-Fix}(T_2)) \leq \frac{1+\beta}{1-\alpha} \varepsilon + \frac{1+\beta}{1-\alpha} \sup_{x \in X} H(T_1(x), T_2(x)). \quad (18)$$

Similarly, one has

$$e(\varepsilon\text{-Fix}(T_2), \varepsilon\text{-Fix}(T_1)) \leq \frac{1+\beta}{1-\alpha} \varepsilon + \frac{1+\beta}{1-\alpha} \sup_{x \in X} H(T_1(x), T_2(x)). \quad (19)$$

It follows from (18) and (19) that

$$H(\varepsilon\text{-Fix}(T_2), \varepsilon\text{-Fix}(T_1)) \leq \frac{1+\beta}{1-\alpha} \varepsilon + \frac{1+\beta}{1-\alpha} \sup_{x \in X} H(T_1(x), T_2(x)).$$

The proof is complete. $\square$

From the proof of Theorem 3, we can also derive a data dependence result for the ε -approximate fixed point set of a multi-valued almost E -contraction.

Theorem 4. Let (X, d) be a metric space, $\varepsilon > 0$ and $T, S : X \rightarrow P(X)$ be such that

- (a) T is a multi-valued (α, β, L) -almost E -contraction;
- (b) $\varepsilon\text{-Fix}(S) \neq \emptyset$;
- (c) there exists $\eta > 0$ such that $e(T(x), S(x)) \leq \eta$ for all $x \in X$.

Then,

$$e(\varepsilon\text{-Fix}(S), \varepsilon\text{-Fix}(T)) \leq \frac{1+\beta}{1-\alpha} (\varepsilon + \eta).$$

Proof. As in the proof of Theorem 3, replacing $\varepsilon\text{-Fix}(T_2)$ by $\varepsilon\text{-Fix}(T)$, T_1 by S and taking supremum over $x \in \varepsilon\text{-Fix}(S)$, we get

$$e(\varepsilon\text{-Fix}(S), \varepsilon\text{-Fix}(T)) \leq \frac{1+\beta}{1-\alpha} \varepsilon + \frac{1+\beta}{1-\alpha} \sup_{x \in X} e(S(x), T(x)).$$

Using (c), we obtain the desired inequality. The proof is complete. $\square$

3 Fixed points results for multi-valued almost E -contractions and application

In this section, we present some fixed point results for multi-valued almost E -contraction mappings. An application to homotopy theory is also given.

3.1 Fixed points results for multi-valued almost E -contractions

Our first fixed point theorem is stated as follows.

Theorem 5. Let (X, d) be a metric space and $T : X \rightarrow C(X)$ be a multi-valued (α, β, L) -almost E -contraction. If one of the following condition holds:

- (i) the graph of T is complete;
- (ii) X is complete, T has closed values and $x \rightarrow D(x, T(x))$ is lower semi-continuous;
- (iii) X is complete, T has closed values and $\beta < 1$;

then

- (a) $\text{Fix}(T)$ is nonempty.
- (b) For any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^\infty$ of T at x_0 converging to a fixed point $\bar{x}$ of T such that

$$d(x_n, \bar{x}) \leq \left(\frac{\alpha + \beta}{1 + \beta} \right)^n \frac{1 + \beta}{1 - \alpha} d(x_0, x_1), \quad n = 0, 1, 2, \dots \quad (20)$$

- (c) The following estimate is satisfied:

$$D(x_0, \text{Fix}(T)) \leq \frac{1 + \beta}{1 - \alpha} D(x_0, T(x_0)), \quad \forall x_0 \in X. \quad (21)$$

Proof. Let $x \in X$ and ℓ be such that $D(x, T(x)) < \ell < D(x, T(x)) + 1$. Set

$$\theta = \frac{\alpha + \beta}{1 + \beta}.$$

As in the proof of Theorem 2, we can construct a sequence $\{x_n\}$ in X with $x_0 = x$ satisfying either

- (I) there exists a nonnegative integer n_0 such that $D(x_{n_0}, T(x_{n_0})) = 0$ and

$$x_{n+1} \in T(x_n), D(x_n, T(x_n)) < \theta^n \ell, \text{ and } d(x_n, x_{n+1}) < \theta^n \ell,$$

for all $n = 0, \dots, n_0 - 1$; or

- (II) for all $n \geq 0$

$$x_{n+1} \in T(x_n), D(x_n, T(x_n)) < \theta^n \ell, \text{ and } d(x_n, x_{n+1}) < \theta^n \ell. \quad (22)$$

Assume that (I) holds. Since $D(x_{n_0}, T(x_{n_0})) = 0$ and T has closed values, $x_{n_0} \in T(x_{n_0})$. In this case, we can put $x_n = x_{n_0}$ for all $n \geq n_0$ and the sequence $\{x_n\}$ satisfies (II). That is, we can always construct an orbit $\{x_n\}$ of T at x_0 satisfies (22) for all $n \geq 0$. For $m > n$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) < \sum_{i=n}^{m-1} \theta^i \ell \leq \frac{\theta^n}{1 - \theta} \ell. \quad (23)$$

Since $\theta \in (0, 1)$, it follows from (23) that $\{x_n\}$ is a Cauchy sequence.

Assume that (i) holds. Since the graph of T is complete and $x_{n+1} \in T(x_n)$, $\{x_n\}$ converges to some $\bar{x} \in X$ with $(\bar{x}, \bar{x}) \in \text{gph}(T)$. Thus, $\bar{x}$ is a fixed point of T and so $\text{Fix}(T) \neq \emptyset$.

Suppose now that (ii) holds. Since X is complete, there is some $\bar{x} \in X$ such that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. By the lower semicontinuity of $x \rightarrow D(x, T(x))$, we have

$$0 \leq D(\bar{x}, T(\bar{x})) \leq \liminf_{n \rightarrow \infty} D(x_n, T(x_n)) \leq \liminf_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

Thus, $D(\bar{x}, T(\bar{x})) = 0$. The closedness of $T(\bar{x})$ implies that $\bar{x} \in T(\bar{x})$, i.e., $\text{Fix}(T) \neq \emptyset$.

Let assume now that (iii) holds. Since X is complete, there exists $\bar{x} \in X$ such that $\lim_{n \rightarrow \infty} x_n = \bar{x}$. Since $x_{n+1} \in T(x_n)$, we have $\lim_{n \rightarrow \infty} D(x_n, T(x_n)) = 0$. Moreover,

$$\begin{aligned} D(\bar{x}, T(\bar{x})) &\leq d(\bar{x}, x_{n+1}) + D(x_{n+1}, T(\bar{x})) \\ &\leq d(\bar{x}, x_{n+1}) + e(T(x_n), T(\bar{x})) \\ &\leq d(\bar{x}, x_{n+1}) + \alpha d(x_n, \bar{x}) \\ &\quad + \beta |D(x_n, T(x_n)) - D(\bar{x}, T(\bar{x}))| + LD(\bar{x}, T(x_n)) \\ &\leq d(\bar{x}, x_{n+1}) + \alpha d(x_n, \bar{x}) + \beta |D(x_n, T(x_n)) - D(\bar{x}, T(\bar{x}))| \\ &\quad + LD(\bar{x}, x_n) + D(x_n, T(x_n)). \end{aligned}$$

Letting $n \rightarrow \infty$, we get $D(\bar{x}, T(\bar{x})) \leq \beta D(\bar{x}, T(\bar{x}))$ which implies $D(\bar{x}, T(\bar{x})) = 0$. By the closedness of $T(\bar{x})$, we obtain $\bar{x} \in T(\bar{x})$, i.e., $\text{Fix}(T) \neq \emptyset$.

Since (23) holds for any $\ell > D(x_0, T(x_0))$, we have, for all $m > n$ that

$$d(x_n, x_m) \leq \frac{\theta^n}{1 - \theta} D(x_0, T(x_0)). \quad (24)$$

Letting $m \rightarrow \infty$, we get

$$d(x_n, \bar{x}) \leq \frac{\theta^n}{1 - \theta} D(x_0, T(x_0)) \leq \left(\frac{\alpha + \beta}{1 + \beta} \right)^n \frac{1 + \beta}{1 - \alpha} d(x_0, x_1).$$

Thus, (20) holds. Now, in (24), letting $n = 0$, we have

$$d(x, x_m) \leq \frac{1}{1 - \theta} D(x_0, T(x_0)) = \frac{1 + \beta}{1 - \alpha} D(x_0, T(x_0)).$$

Letting $m \rightarrow \infty$, we get

$$D(x, \text{Fix}(T)) \leq d(x, \bar{x}) \leq \frac{1 + \beta}{1 - \alpha} D(x_0, T(x_0)).$$

Thus, (21) holds. This ends the proof. $\square$

Remark 2. (i) Theorem 5 covers [27, Corollary 3], [10, Theorem 3] and Nadler's theorem [28].

(ii) The estimate (21) can be obtained from the estimate (8) by using some results about set-convergence as in [27].

Example 4. Let (X, d) and T be as in Example 1. Then, all conditions of Theorem 5 are satisfied. We have $\text{Fix}(T) = \{0\}$. For all $x \in X$, we have $D(x, \text{Fix}(T)) = |x|$ and

$$D(x, T(x)) = \begin{cases} -x & \text{if } x \leq 0, \\ 2x & \text{if } x > 0. \end{cases}$$

Thus,

$$D(x, \text{Fix}(T)) \leq 2D(x, T(x)) \leq 5D(x, T(x)) = \frac{1 + \beta}{1 - \beta} D(x, T(x))$$

for all $x \in X$, i.e., (21) is verified.

Example 5. Let (X, d) and T be as in Example 3. That is, T is a multi-valued (α, β, L) -almost E -contraction with $\alpha = \frac{1}{2}$, $\beta = 0$ and $L = 2$. We have $\text{Fix}(T) = \{1\}$ and for $n \geq 1$,

$$D\left(\frac{1}{2^n}, \text{Fix}(T)\right) = 1 - \frac{1}{2^n} \rightarrow 1 \quad \text{as } n \rightarrow \infty,$$

whereas

$$D\left(\frac{1}{2^n}, T\left(\frac{1}{2^n}\right)\right) = \frac{1}{2^{n+1}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, for n large enough, we have

$$D\left(\frac{1}{2^n}, \text{Fix}(T)\right) > \frac{1+\beta}{1-\alpha} D\left(\frac{1}{2^n}, T\left(\frac{1}{2^n}\right)\right) = 2D\left(\frac{1}{2^n}, T\left(\frac{1}{2^n}\right)\right).$$

So (21) does not hold for this example. This happens since (X, d) and the graph of T are not complete.

Using the estimate (21) and arguing as in the proof of Theorem 3 and Corollary 3, one can prove the following result.

Theorem 6. Let X be a metric space and $T_1, T_2 : X \rightarrow C(X)$ be multi-valued (α, β, L) -almost E -contractions. Assume that one of the following conditions holds

- (a) the graph of T_i , $i = 1, 2$, is complete;
- (b) X is complete and $x \rightarrow D(x, T_i(x))$, $i = 1, 2$, is lower semi-continuous;
- (c) X is complete and $\beta < 1$.

Then, $\text{Fix}(T_1) \neq \emptyset$, $\text{Fix}(T_2) \neq \emptyset$ and

$$e(\text{Fix}(T_1), \text{Fix}(T_2)) \leq \frac{1+\beta}{1-\alpha} \sup_{x \in X} e(T_1(x), T_2(x)) \quad (25)$$

and

$$H(\text{Fix}(T_1), \text{Fix}(T_2)) \leq \frac{1+\beta}{1-\alpha} \sup_{x \in X} H(T_1(x), T_2(x)). \quad (26)$$

The following corollary covers [25, Theorem 1].

Corollary 4. Let (X, d) be a metric space and $\{T_n\}_{n \geq 0}$ be a sequence of multi-valued (α, β, L) -almost E -contractions from X into $C(X)$. Assume that one of the following conditions holds:

- (a) the graph of T_n , $n = 0, 1, 2, \dots$, is complete;
- (b) X is complete and $x \rightarrow D(x, T_n(x))$ is lower semi-continuous for all $n = 0, 1, 2, \dots$;
- (c) X is complete and $\beta < 1$.

If $\lim_{n \rightarrow \infty} H(T_n(x), T_0(x)) = 0$ uniformly for all $x \in X$, then

$$\lim_{n \rightarrow \infty} H(\text{Fix}(T_n), \text{Fix}(T_0)) = 0.$$

Proof. By Theorem 5, $\text{Fix}(T_n) \neq \emptyset$ for all $n = 0, 1, 2, \dots$. Since

$$\lim_{n \rightarrow \infty} H(T_n(x), T_0(x)) = 0$$

uniformly for all $x \in X$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{x \in X} H(T_n(x), T_0(x)) < (1 - \theta)\varepsilon,$$

for all $n \geq N$, where $\theta = (\alpha + \beta)/(1 + \beta)$. Using the estimate (26), we have

$$H(\text{Fix}(T_n), \text{Fix}(T_0)) < \varepsilon,$$

for all $n \geq N$. This implies that $\lim_{n \rightarrow \infty} H(\text{Fix}(T_n), \text{Fix}(T_0)) = 0$. The proof is complete. $\square$

One can also give the local versions of above results. For instance, we have the following local fixed point theorem for multi-valued almost E -contractions.

Theorem 7. Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : X \rightarrow C(X)$. Assume that there exist $\alpha \in [0, 1)$, $\beta \in [0, 1)$ and $L \geq 0$ such that

$$e(T(x), T(y)) \leq \alpha d(x, y) + \beta |D(x, T(x)) - D(y, T(y))| + LD(y, T(x)), \quad (27)$$

for all $x, y \in B(x_0, r)$. If

$$D(x_0, T(x_0)) < \frac{(1-\alpha)r}{1+\beta},$$

then T has a fixed point in $B(x_0, r)$ and

$$D(x_0, \text{Fix}(T)) \leq \frac{1+\beta}{1-\alpha} D(x_0, T(x_0)). \quad (28)$$

Proof. Set $\theta = \frac{\alpha+\beta}{1+\beta} \in [0, 1)$. Let ℓ be such that

$$D(x_0, T(x_0)) < \ell < \frac{(1-\alpha)r}{1+\beta} = (1-\theta)r.$$

If $D(x_0, T(x_0)) = 0$, then x_0 is a fixed point of T in $B(x_0, r)$ and the proof is finished. If $D(x_0, T(x_0)) > 0$, then there exists $x_1 \in T(x_0)$ such that $d(x_0, x_1) < \ell$ and $\alpha d(x_0, x_1) < D(x_0, T(x_0))$. Since $d(x_0, x_1) < \ell < r$, $x_1 \in B(x_0, r)$. Using (27) and arguing as in the proof of Theorem 2, we have $D(x_1, T(x_1)) < \theta\ell$. Again, if $D(x_1, T(x_1)) = 0$, then $x_1 \in \text{Fix}(T)$ and $D(x_0, \text{Fix}(T)) \leq d(x_0, x_1) < \ell$. Since $\ell > D(x_0, T(x_0))$ is arbitrary, the latter inequality implies that

$$D(x_0, \text{Fix}(T)) \leq D(x_0, T(x_0)) \leq \frac{1+\beta}{1-\alpha} D(x_0, T(x_0)),$$

i.e., (28) holds. Otherwise, there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) < \theta\ell$ and $\alpha d(x_1, x_2) < D(x_1, T(x_1))$. In this case,

$$d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) < \ell + \theta\ell < (1-\theta^2)r \leq r,$$

i.e., $x_2 \in B(x_0, r)$. Continuing this progress, as arguing in the proof of Theorem 5, we can construct an orbit $\{x_n\}$ of T at x_0 such that

$$x_{n+1} \in T(x_n) \quad \text{and} \quad d(x_n, x_{n+1}) < \theta^n \ell$$

and moreover $x_n \in B(x_0, r)$ for all n . Again, as in the proof of Theorem 5, $\{x_n\}$ is a Cauchy sequence and it converges to some $\bar{x} \in X$ satisfying

$$d(x_n, \bar{x}) \leq \frac{\theta^n}{1-\theta} D(x_0, T(x_0)),$$

for all $n \geq 0$. In particular,

$$d(x_0, \bar{x}) \leq \frac{1}{1-\theta} D(x_0, T(x_0)) < \frac{1}{1-\theta} (1-\theta)r = r,$$

i.e., $\bar{x} \in B(x_0, r)$. The latter inequality also implies that (28) holds. This ends the proof. $\square$

3.2 An application to homotopy theory

To conclude this paper, we give an application of our results to homotopy theory. Similar results for other general contractive multi-valued mappings can be found in, e.g., [1, 9, 24, 29] and references therein.

Theorem 8. Let (X, d) be a complete metric space, Ω be an open subset of X and $F : \overline{\Omega} \times [0, 1] \rightarrow C(X)$. Assume that the following conditions are satisfied:

- (i) $x \notin F(x, t)$ for $x \in \partial\Omega$ and $t \in [0, 1]$;
- (ii) there exist $\alpha, \beta \in [0, 1)$ and $L \geq 0$ such that

$$e(F(x, t), F(y, t)) \leq \alpha d(x, y) + \beta |D(x, F(x, t)) - D(y, F(y, t))| + LD(y, F(x, t))$$

for all $x, y \in \overline{\Omega}$, $t \in [0, 1]$;

- (iii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $t, s \in [0, 1]$ with $|t - s| < \delta$, then $e(F(x, t), F(x, s)) < \varepsilon$ for all $x \in \overline{\Omega}$;

- (iv) $\inf\{D(x, F(x, t)) : x \in \overline{\Omega}, t \in [0, 1]\} > 0$.

If $F(\cdot, 0)$ has a fixed point, then $F(\cdot, t)$ has a fixed point in Ω for all $t \in [0, 1]$.

Proof. Assume that $F(\cdot, 0)$ has a fixed point. We set

$$\Lambda = \{t \in [0, 1] : x \in F(x, t) \text{ for some } x \in \Omega\}.$$

Since $F(\cdot, 0)$ has a fixed point and $x \notin F(x, t)$ for $x \in \partial\Omega$ and $t \in [0, 1]$, $F(\cdot, 0)$ has a fixed point in Ω . Thus, $0 \in \Lambda$, i.e., Λ is nonempty. We shall show that Λ is both closed and open in $[0, 1]$.

We first show that Λ is closed in $[0, 1]$. Let $\{t_k\}$ be a sequence in Λ and it converges to some $\lambda \in [0, 1]$. We are going to show that $\lambda \in \Lambda$. Since $\{t_k\} \subset \Lambda$, there exists a sequence $\{x_k\}$ in Ω such that $x_k \in F(x_k, t_k)$ for all k . We claim that

$$\inf\{D(x_k, \partial\Omega) : k \geq 1\} > 0. \quad (29)$$

Assume that the claim is not true. Then, we may assume (after taking subsequence of $\{x_k\}$) that there exists a sequence $\{y_k\}$ in $\partial\Omega$ such that

$$d(x_k, y_k) < \frac{1}{k}, \quad \forall k \geq 1.$$

This together with (iv) yields

$$0 < \inf\{D(x, F(x, t)) : x \in \partial\Omega, t \in [0, 1]\} \leq \liminf_{k \rightarrow \infty} D(y_k, F(y_k, t_k)). \quad (30)$$

Now, using (ii), we have for all k that

$$\begin{aligned} D(y_k, F(y_k, t_k)) &\leq d(y_k, x_k) + D(x_k, F(y_k, t_k)) \\ &< \frac{1}{k} + e(F(x_k, t_k), F(y_k, t_k)) \\ &\leq \frac{1}{k} + \alpha d(x_k, y_k) + \beta |D(x_k, F(x_k, t_k)) - D(y_k, F(y_k, t_k))| \\ &\quad + LD(y_k, F(x_k, t_k)) \\ &\leq \frac{1+\alpha}{k} + \beta D(y_k, F(y_k, t_k)) + LD(y_k, F(x_k, t_k)) \\ &\leq \frac{1+\alpha}{k} + \beta D(y_k, F(y_k, t_k)) + Ld(y_k, x_k) + LD(x_k, F(x_k, t_k)) \\ &\leq \frac{1+\alpha+L}{k} + \beta D(y_k, F(y_k, t_k)). \end{aligned}$$

This implies that

$$D(y_k, F(y_k, t_k)) \leq \frac{1+\alpha+L}{(1-\beta)k}.$$

Taking the limit as $k \rightarrow \infty$, we get

$$\liminf_{k \rightarrow \infty} D(y_k, F(y_k, t_k)) \leq 0$$

which contradicts (30). Thus, (29) holds. As a results, there exists $r > 0$ such that $B(x_k, r) \subset \Omega$ for all k . Moreover, since $t_k \rightarrow \lambda$ as $k \rightarrow \infty$, by (iii), there exists a natural number k_0 such that

$$D(x_{k_0}, F(x_{k_0}, \lambda)) \leq e(F(x_{k_0}, t_{k_0}), F(x_{k_0}, \lambda)) < \frac{(1-\alpha)r}{1+\beta}.$$

Applying Theorem 7, we conclude that $F(\cdot, \lambda)$ has a fixed point $\bar{x} \in B(x_{k_0}, r) \subset \Omega$, i.e., there exists $\bar{x} \in \Omega$ such that $\bar{x} \in F(\bar{x}, \lambda)$. This implies that $\lambda \in \Lambda$. Thus, Λ is closed in $[0, 1]$.

We now show that Λ is open in $[0, 1]$. Let $t_0 \in \Lambda$. Then there exists $x_0 \in \Omega$ such that $x_0 \in F(x_0, t_0)$. Since Ω is open, there exists $r > 0$ such that $B(x_0, r) \subset \Omega$. By (iii), for $\varepsilon = \frac{(1-\alpha)r}{1+\beta}$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$D(x_0, F(x_0, t)) \leq e(F(x_0, t_0), F(x_0, t)) < \frac{(1-\alpha)r}{1+\beta},$$

for all $t \in [0, 1]$ satisfying $|t - t_0| < \delta$. Hence, by Theorem 7, for each $t \in [0, 1]$ with $|t - t_0| < \delta$, $F(\cdot, t)$ has a fixed point in $B(x_0, r) \subset \Omega$. That is, for each $t \in [0, 1]$ with $|t - t_0| < \delta$, there exists $x_t \in \Omega$ such that $x_t \in F(x_t, t)$. This means that

$$\{t \in [0, 1] : |t - t_0| < \delta\} \subset \Lambda.$$

Thus, Λ is open in $[0, 1]$. Since Λ is both closed and open in $[0, 1]$ and $[0, 1]$ is connected, we conclude that $\Lambda = [0, 1]$. This ends the proof. $\square$

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