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Fixed Point Theorem for Multivalued Non-self Mappings in Partial Symmetric Spaces

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Abstract: In this paper, we proved a fixed point theorem for multi-valued non-self mappings in partial symmetric spaces. In doing so, we extended and generalized the results in literature by employing a convex structure for multi-valued non-self mappings using Rhoades type contractions. We also provided an illustrative example to support the results.

Keywords: Fixed point, partial symmetric spaces, multi-valued mapping, non-self mappings.

1 Introduction

In 1922, Banach [1] gave a contraction principle (BCP) for a self-mapping in metric space. Later on, Nadler [2] extended the Banach contraction principle from a single-valued map into the multi-valued map. Assad and Kirk [3] extended the results of Nadler to the subset of metrically convex metric space with Rothe's boundary condition to obtain the fixed point. Ćirić [4] gave a generalized Banach contraction principle and proved fixed point theorems for single-valued and multi-valued quasi-contractions.

Ćirić [5] proved a fixed point theorem for a class of non-self mappings that satisfy a generalized contraction condition. Several researchers proved different results for non-self contractive mappings. One can see, e.g., [6–10].

Fixed point theory for non-self multi-valued mappings was developed rapidly after the publication of Assad and Kirk's paper [3] in which they proved a non-self multi-valued version of Banach's contraction principle. Rhoades [11, 12] obtained the fixed point theorem for non-self set-valued mappings satisfying contractive type conditions. Du *et al.* [13] proved fixed point theorems for various multi-valued non-self maps by extending and generalizing self-maps to the class of non-self maps. For more results on multi-valued non-self maps, we refer the reader in [14–18] and the references therein.

In 2019, Asim *et al.* [19] initiated the study of Partial symmetric space by combining the concept of partial metric space due to Matthew [20] and symmetric space concept due to Wilson [21]. Using this space, they proved some related fixed point results for single-valued and multi-valued mappings. Since then, several researchers getting motivation to do their research in this direction. In 2021, Asim and Imdad [22] proved a common fixed point result in partial symmetric space. Furthermore, Asim *et al.* [23] proved a multi-valued result using Suzuki and Wardowski-type contraction mapping in partial symmetric space.

In this paper, we attempted to use the Rhoades contraction method to generalize the theorem due to Assad, and Kirk [3] for multi-valued non-self mappings on partial symmetric space. With this, we have denoted (X, p_s) as a partial symmetric space.

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2 Preliminaries

The following preliminaries and results will be helpful to develop the new theorem for this paper.

Definition 2.1. [19] Let X be a non-empty set. A mapping $p_s : X \times X \rightarrow \mathbb{R}_+$ is said to be partial symmetric if, for all $x, y, z \in X$, we have the following properties:

(p_{s1}) : $x = y$ if and only if $p_s(x, y) = p_s(x, x) = p_s(y, y)$,

(p_{s2}) : $p_s(x, x) \leq p_s(x, y)$,

(p_{s3}) : $p_s(x, y) = p_s(y, x)$.

Then the pair (X, p_s) is said to be partial symmetric space.

From p_{s1} and p_{s2} we have

$$p_s(x, y) = 0 \Rightarrow p_s(x, y) = p_s(x, x) = p_s(y, y) \Rightarrow x = y. \quad (1)$$

A partial symmetric space (X, p_s) reduces to a symmetric space if $p_s(x, x) = 0$, for all $x \in X$. Obviously, every symmetric space is partial symmetric space, but not conversely.

Let (X, p_s) be a partial symmetric space. Then, the p_s -open ball, with center $x \in X$ and radius $\varepsilon > 0$, is defined by: $B_{p_s}(x, \varepsilon) = \{y \in X : p_s(x, y) < p_s(x, x) + \varepsilon\}$,

Similarly, the p_s -closed ball, with center $x \in X$ and radius $\varepsilon > 0$, is defined by: $B_{p_s}[x, \varepsilon] = \{y \in X : p_s(x, y) \leq p_s(x, x) + \varepsilon\}$.

The family of p_s -open balls for all $x \in X$ and $\varepsilon > 0$, $U_{p_s} = \{B_{p_s}(x, \varepsilon) : x \in X, \varepsilon > 0\}$, forms basis of some topology τ_{p_s} on X .

Definition 2.2. [19] Let (X, p_s) be a partial symmetric space. Then,

(i) a sequence $\{x_n\}$ in (X, p_s) is said to be p_s -convergent to $x \in X$, with respect to τ_{p_s} if $p_s(x, x) = \lim_{n \rightarrow \infty} p_s(x_n, x)$.

(ii) a sequence $\{x_n\}$ in (X, p_s) is called a p_s -Cauchy sequence if only if $\lim_{n, m \rightarrow \infty} p_s(x_n, x_m)$ exists and is finite.

(iii) a partial symmetric space (X, p_s) is said to be p_s -complete if every p_s -Cauchy sequence $\{x_n\}$ in X is p_s -convergent, with respect to τ_{p_s} to a point $x \in X$, such that

$$p_s(x, x) = \lim_{n \rightarrow \infty} p_s(x_n, x) = \lim_{n, m \rightarrow \infty} p_s(x_n, x_m).$$

Definition 2.3. [19] Let (X, p_s) be a partial symmetric space. Then

(A_1) $\lim_{n \rightarrow \infty} p_s(x_n, x) = p_s(x, x)$ and $\lim_{n \rightarrow \infty} p_s(x_n, y) = p_s(x, y)$ imply that $x = y$, for a sequence $\{x_n\}$, $x, y \in X$.

(A_2) a partial symmetric p_s is said to be 1-continuous if $\lim_{n \rightarrow \infty} p_s(x_n, x) = p_s(x, x)$ implies that $\lim_{n \rightarrow \infty} p_s(x_n, y) = p_s(x, y)$, where $\{x_n\}$ is a sequence in X and $x, y \in X$.

(A_3) a partial symmetric p_s is said to be continuous if $\lim_{n \rightarrow \infty} p_s(x_n, x) = p_s(x, x)$ and $\lim_{n \rightarrow \infty} p_s(x_n, y) = p_s(x, y)$ imply that $\lim_{n \rightarrow \infty} p_s(x_n, y_n) = p_s(x, y)$ where $\{x_n\}$ and $\{y_n\}$ are sequences in X and $x, y \in X$.

(A_4) $\lim_{n \rightarrow \infty} p_s(x_n, x) = p_s(x, x)$ and $\lim_{n \rightarrow \infty} p_s(x_n, y_n) = p_s(x, x)$ imply $\lim_{n \rightarrow \infty} p_s(y_n, x) = p_s(x, x)$, for sequences (x_n) , (y_n) , and x in X .

(A_5) $\lim_{n \rightarrow \infty} p_s(x_n, y_n) = p_s(x, x)$ and $\lim_{n \rightarrow \infty} p_s(y_n, z_n) = p_s(x, x)$ imply $\lim_{n \rightarrow \infty} p_s(x_n, z_n) = p_s(x, x)$, for sequences (x_n) , (y_n) , (z_n) , and x in X .

Definition 2.4. [19] Let (X, p_s) be a partial symmetric space. A mapping $f : X \rightarrow X$ is said to be a Kannan-Ciric type k -contraction if, for all $x, y \in X$,

$$p_s(fx, fy) \leq k \max\{p_s(x, fx), p_s(y, fy)\} \quad (2)$$

where $k \in [0, 1)$.

In 2019 Asim *et al.* [19], proved a Kannan- Ćirić type k - contractions in the setting of partial symmetric space.

Theorem 2.1. [19] *Let (X, p_s) be a complete partial symmetric space and $f : X \rightarrow X$. Assume that the following conditions are satisfied:*

- (i) *f is a Kannan- Ćirić type k - contraction,*
- (ii) *f is continuous.*

Then T has a unique fixed point $x \in X$ such that $p_s(x, x) = 0$.

Example 1.1 [19] Let $X = \mathbb{R}$. Define a mapping $p_s : X \times X \rightarrow \mathbb{R}_+$ for all $x, y \in X$ and $p, q > 1$, as follows:

$$p_s(x, y) = |x - y|^p + |x - y|^q.$$

Then the pair (X, p_s) is a partial symmetric space.

Example 1.2 [19] Let $X = \mathbb{R}$. Define a mapping $p_s : X \times X \rightarrow \mathbb{R}_+$ for all $x, y \in X$ and $p, q > 1$, as below:

$$p_s(x, y) = (\max\{x, y\})^p + (\max\{x, y\})^q.$$

Then the pair (X, p_s) is a partial symmetric space.

Now, we are introducing some concepts of Hausdorff spaces as defined by Asim *et al.* [19] in partial symmetric spaces.

Let (X, p_s) be partial symmetric space and $CB^{p_s}(X)$ be the family of all nonempty, τ_{p_s} -closed and bounded subsets of (X, p_s) .

Observe that A is said to be bounded if there exists $x \in X$ and $N \geq 0$, such that for all $a \in A$,

$$p_s(x_0, a) \leq p_s(a, a) + N. \quad (3)$$

Furthermore, for $A, B \in CB^{p_s}(X)$ and $x \in X$, we define:

$$\begin{aligned} dist_{p_s}(x, A) &= \inf\{p_s(x, a) : a \in A\}; \\ \delta_{p_s}(A, B) &= \sup\{dist_{p_s}(a, B) : a \in A\}; \\ \delta_{p_s}(B, A) &= \sup\{dist_{p_s}(b, A) : b \in B\}. \end{aligned}$$

Lemma 2.1. [19] *Let (X, p_s) be a partial symmetric space and A a nonempty subset of X , then $a \in \bar{A}$ if and only if*

$$dist_{p_s}(a, A) = p_s(a, a),$$

where $\bar{A}$ is p - closed in (X, p_s) iff $A = \bar{A}$.

Proposition 2.1. [19] *Let (X, p_s) be a complete partial symmetric space, for $A, B, C \in CB^{p_s}(X)$, then we have the following;*

- (i) $\delta_{p_s}(A, A) = \sup\{dist_{p_s}(a, a) : a \in A\};$
- (ii) $\delta_{p_s}(A, A) < \delta_{p_s}(A, B);$
- (iii) $B \subset C \Rightarrow \delta_{p_s}(A, C) < \delta_{p_s}(A, B);$
- (iv) $\delta_{p_s}(A, B) = 0 \Rightarrow A \subseteq B;$ and
- (v) $\delta_{p_s}(A \cup B, C) = \max\{\delta_{p_s}(A, C), \delta_{p_s}(B, C)\}.$

Let (X, p_s) be a partial symmetric spaces. We define

$$H_{p_s}(A, B) = \max\{\delta_{p_s}(A, B), \delta_{p_s}(B, A)\}. \quad (4)$$

Proposition 2.2. [19] Let (X, p_s) be a complete partial symmetric space, for $A, B, C, D \in CB^{p_s}(X)$, the following condition holds:

- (1) $H_{p_s}(A, A) \leq H_{p_s}(A, B)$;
- (2) $H_{p_s}(A, B) = H_{p_s}(B, A)$;
- (3) $H_{p_s}(A \cup B, C \cup D) = \max\{H_{p_s}(A, C), H_{p_s}(B, D)\}$.

Proposition 2.3. [19] Let (X, p_s) be a complete partial symmetric space, for $A, B \in CB^{p_s}(X)$, we have

$$H_{p_s}(A, B) = 0 \Rightarrow A = B.$$

Lemma 2.2. [19] Let (X, p_s) be a partial symmetric space for $A, B \in CB^{p_s}(X)$. Then, for any $h < 1$ and $a \in A$, there exists $b \in B$ such that

$$p_s(a, b) = hH_{p_s}(A, B).$$

Definition 2.5. [2] A map $T : X \rightarrow CB(X)$ is said to be multi-valued contraction if there exist $\lambda \in [0, 1)$ such that

$$H(Tx, Ty) \leq \lambda d(x, y), \quad (5)$$

for all $x, y \in X$.

Assad and Kirk in [3], introduced the following lemma in convex metric space.

Lemma 2.3. [3] If C is a nonempty closed subset of a metrically convex metric space (X, d) , then for any $x \in C$, $y \notin C$, there exists a point $z \in \partial C$ (the boundary of C) such that

$$d(x, z) + d(z, y) = d(x, y).$$

Kumar and Rugumisa formulated the following lemma:

Lemma 2.4. [24] Let C be a non-empty subset of a metrically convex partial metric space (X, p) which is closed in (X, p^s) . If $x \in C$ and $y \in X \setminus C$, then there exists a point $z \in \partial C$ (the boundary of C) such that

$$p(x, z) + p(z, y) = p(x, y) + p(z, z). \quad (6)$$

Assad and Kirk [3] proved the following results in metrically convex metric spaces

Theorem 2.2. [3] Let (X, d) be a complete and metrically convex metric space, C a non empty closed subset of X and $T : C \rightarrow CB(X)$ be a mapping. Assume that the following conditions are holds:

- (i) $Tx \in C$ for each $x \in \partial C$,
- (ii) there exists $k \in (0, 1)$, $\forall x, y \in C$,

$$H(Tx, Ty) \leq k d(x, y). \quad (7)$$

Then T has a unique fixed point in C .

We shall use the following lemma, which is given by Kumar and Rugumisa [24] in metrically convex partial space and will be helpful in the proof of our main results.

Lemma 2.5. [24] If C is a nonempty closed subset of a metrically convex partial symmetric space (X, p_s) , then for any $x \in C$, $y \notin C$, there exists a point $z \in \partial C$ (the boundary of C) such that

$$p_s(x, z) + p_s(z, y) = p_s(x, y) + p_s(z, z).$$

3 Main Results

In this section, we prove our main theorem as given below:

Theorem 3.1. *Let (X, p_s) be a complete metrically convex partial symmetric spaces, C a non empty closed subset of X and $T : C \longrightarrow CB^{p_s}(X)$ be a multivalued mapping. Assume that the following conditions are holds:*

- (i) $Tx \in C$ for each $x \in \partial C$,
- (ii) T is continuous,
- (iii) There exists $k \in (0, 1)$, $\forall x, y \in C$ with $H_{p_s}(Tx, Ty) > 0$; such that

$$H_{p_s}(Tx, Ty) \leq k \max \left\{ p_s(x, y), p_s(x, Tx), p_s(y, Ty), \frac{p_s(x, Ty) + p_s(y, Tx)}{2} \right\}. \quad (8)$$

Then, T has a unique fixed point, if there exists $x \in X$ such that $p_s(x, x) = 0$.

Proof. Suppose that T has no fixed points then $p_s(x, Tx) > 0$ for all $x \in C$. We start by constructing two sequences $\{x_n\}$ and $\{y_n\}$, respectively in the following way.

Let $x_0 \in \partial C$ and $y_1 \in Tx_0$, if $y_1 \in C$, let $x_1 = y_1$, then our proof would be completed. Since T is a multivalued mapping $x_1 \in Tx_0$. If $y_1 \notin C$, then, by Lemma 2.5 there exists $x_1 \in \partial C$ such that

$$p_s(x_0, x_1) + p_s(x_1, y_1) = p_s(x_0, y_1) + p_s(x_1, x_1). \quad (9)$$

Thus for $x_1 \in C$ and using Lemma 2.2, we can choose $y_2 \in Tx_1$ such that

$$p_s(y_1, y_2) \leq h H_{p_s}(Tx_0, Tx_1). \quad (10)$$

For $h = \frac{1}{\sqrt{k}}$, Equation (10) becomes

$$p_s(y_1, y_2) \leq \frac{1}{\sqrt{k}} H_{p_s}(Tx_0, Tx_1).$$

However, if $y_2 \in C$, let $x_2 = y_2$. If $y_2 \notin C$. Then, using Lemma 2.5 there exists $x_2 \in \partial C$ such that

$$p_s(x_1, x_2) + p_s(x_2, y_2) = p_s(x_1, y_2) + p_s(x_2, x_2).$$

Thus $x_2 \in C$ and using Lemma 2.2, we can choose $y_3 \in Tx_2$ such that

$$p_s(y_2, y_3) \leq h H_{p_s}(Tx_1, Tx_2). \quad (11)$$

Again for $h = \frac{1}{\sqrt{k}}$ in (11) we have

$$p_s(y_2, y_3) \leq \frac{1}{\sqrt{k}} H_{p_s}(Tx_1, Tx_2).$$

Continuing the arguments and following the iterative process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$, such that for $n \in \mathbb{N}$, we have

- (i) $y_{n+1} \in Tx_n$;
- (ii) $p_s(y_n, y_{n+1}) \leq (\frac{1}{\sqrt{k}}) H_{p_s}(Tx_{n-1}, Tx_n)$;
- (iii) $y_n \in C \Rightarrow y_n = x_n$;
- (iv) $y_n \neq x_n$, whenever $y_n \notin C$ and $x_n \in \partial C$ implies that

$$p_s(x_{n-1}, x_n) + p_s(x_n, y_n) = p_s(x_{n-1}, y_n) + p_s(x_n, x_n).$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Assume that P and Q are two sets in C such that

$$P = \{x_i \in \{x_n\} : x_i = y_i, i = 1, 2, \dots\},$$

$$Q = \{x_i \in \{x_n\} : x_i \neq y_i, i = 1, 2, \dots\}.$$

Obviously, if $x_n \in Q$, for some n , then x_n and x_{n+1} belong to P ;

Thus, we have three possible cases.

Case I

If $x_n, x_{n+1} \in P$, then $y_n = x_n, y_{n+1} = x_{n+1}$. Assume that $x_n = y_n = Tx_{n-1}$

$x_{n+1} = y_{n+1} = Tx_n$. Then, we have

$$p_s(x_n, x_{n+1}) = p_s(y_n, y_{n+1}) = p_s(Tx_{n-1}, Tx_n).$$

Since $\{x_n\} \in C$ for all $n \in \mathbb{N}$, by using (8), we shows that $x_n, x_{n+1} \in \partial C$.

Now,

$$p_s(x_n, x_{n+1}) = p_s(Tx_{n-1}, Tx_n).$$

$$\begin{aligned} H_{p_s}(Tx_{n-1}, Tx_n) &\leq k \max \left\{ p_s(x, y), p_s(x, Tx), p_s(y, Ty), \right. \\ &\quad \left. \frac{p_s(x, Ty) + p_s(y, Tx)}{2} \right\} \end{aligned} \quad (12)$$

Thus, letting $x = x_{n-1}$ and $y = x_n$ in (12), we have

$$\begin{aligned} H_{p_s}(Tx_{n-1}, Tx_n) &\leq k \max \left\{ p_s(x_{n-1}, x_n), p_s(x_{n-1}, Tx_{n-1}), p_s(x_n, Tx_n), \right. \\ &\quad \left. \frac{p_s(x_{n-1}, Tx_n) + p_s(x_n, Tx_{n-1})}{2} \right\} \\ &\leq k \max \left\{ p_s(x_{n-1}, x_n), p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1}), \right. \\ &\quad \left. \frac{p_s(x_{n-1}, x_{n+1}) + p_s(x_n, x_n)}{2} \right\}. \end{aligned} \quad (13)$$

By Lemma 2.5

$$p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1}) = p_s(x_{n-1}, x_{n+1}) + p_s(x_n, x_n). \quad (14)$$

Using (14) in (13), we obtain

$$\begin{aligned} H_{p_s}(Tx_{n-1}, Tx_n) &\leq k \max \left\{ p_s(x_{n-1}, x_n), p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1}), \right. \\ &\quad \left. \frac{p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1}) - p_s(x_n, x_n) + p_s(x_n, x_n)}{2} \right\} \\ &\leq k \max \left\{ p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1}), \frac{p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})}{2} \right\} \\ &\leq k \max \left\{ p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1}) \right\} \end{aligned}$$

By using Lemma 2.2 and taking $h = \frac{1}{\sqrt{k}}$, we have

$$\begin{aligned} p_s(x, y) &= hH_{p_s}(Tx, Ty), \\ &= \frac{1}{\sqrt{k}} H_{p_s}(Tx_{n-1}, Tx_n), \\ p_s(x_n, x_{n+1}) &\leq (\sqrt{k})^n \max(p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1})). \end{aligned} \quad (15)$$

Which shows that, either

$$p_s(x_n, x_{n+1}) \leq (\sqrt{k})^n p_s(x_{n-1}, x_n), \quad (16)$$

or

$$p_s(x_n, x_{n+1}) \leq (\sqrt{k})^n p_s(x_n, x_{n+1}), \quad (17)$$

is the maximum of (15). Inductively, using (16) we proceed as follows:

Let $x_n \in Tx_n$ and $p_s(x_n, x_{n+1}) \leq (\sqrt{k})^n p_s(x_{n-1}, x_n)$. For all $n \geq 1$ and $n, m \in \mathbb{N}$, we have

$$p_s(x_n, x_m) \leq p_s(x_n, x_{n+1}) + p_s(x_{n+1}, x_{n+2}) + \cdots + p_s(x_{m-1}, x_m).$$

$$p_s(x_n, x_m) \leq \left\{ (\sqrt{k})^n + (\sqrt{k})^{n+1} + \cdots + (\sqrt{k})^{n+m-1} \right\} p_s(x_0, x_1).$$

Applying the sum of G.P, we obtain

$$p_s(x_n, x_m) \leq \frac{(\sqrt{k})^n}{1 - \sqrt{k}} p_s(x_0, x_1).$$

$$\lim_{n, m \rightarrow \infty} p_s(x_n, x_m) = 0. \quad (18)$$

Similarly, using Equation (17). Assume that

$$(\sqrt{k})^n \max(p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1})) = (\sqrt{k})^n p_s(x_n, x_{n+1}).$$

So, we have

$$p_s(x_n, x_{n+1}) \leq (\sqrt{k})^n p_s(x_n, x_{n+1}).$$

For continuity of T in the above inequality, we obtain

$$p_s(x_n, x_{n+1}) \leq (\sqrt{k})^n p_s(x_n, x_{n+1}) \leq p_s(x_n, x_{n+1}),$$

which is a contradiction and so

$$p_s(x_n, x_{n+1}) \leq (\sqrt{k})^n p_s(x_{n-1}, x_n). \quad (19)$$

Cosequently, for $y_n, y_{n+1} \in P$ we have

$$p_s(y_n, y_{n+1}) = p_s(x_n, x_{n+1}) = p_s(Ty_{n-1}, Ty_n).$$

Since $\{y_n\} \in C$ for all $n \in \mathbb{N}$, by using (8), we shows that $y_n, y_{n+1} \in \partial C$.

Now,

$$p_s(y_n, y_{n+1}) = p_s(Ty_{n-1}, Ty_n).$$

$$H_{p_s}(Ty_{n-1}, Ty_n) \leq k \max \left\{ p_s(x, y), p_s(x, Tx), p_s(y, Ty), \frac{p_s(x, Ty) + p_s(y, Tx)}{2} \right\} \quad (20)$$

Thus, letting $x = y_{n-1}$ and $y = y_n$ in (20), we have

$$\begin{aligned} H_{p_s}(Ty_{n-1}, Ty_n) &\leq k \max \left\{ p_s(y_{n-1}, y_n), p_s(y_{n-1}, Ty_{n-1}), p_s(y_n, Ty_n), \right. \\ &\quad \left. \frac{p_s(y_{n-1}, Ty_n) + p_s(y_n, Ty_{n-1})}{2} \right\} \\ &\leq k \max \left\{ p_s(y_{n-1}, y_n), p_s(y_{n-1}, y_n), p_s(y_n, y_{n+1}), \right. \\ &\quad \left. \frac{p_s(y_{n-1}, y_{n+1}) + p_s(y_n, y_n)}{2} \right\}. \end{aligned} \quad (21)$$

By Lemma 2.5

$$p_s(y_{n-1}, y_n) + p_s(y_n, y_{n+1}) = p_s(y_{n-1}, y_{n+1}) + p_s(y_n, x_n). \quad (22)$$

Applying (22) in (21), one obtains

$$\begin{aligned} H_{p_s}(Ty_{n-1}, Ty_n) &\leq k \max \left\{ p_s(y_{n-1}, y_n), p_s(y_{n-1}, y_n), p_s(y_n, y_{n+1}), \right. \\ &\quad \left. \frac{p_s(y_{n-1}, y_n) + p_s(y_n, y_{n+1}) - p_s(y_n, y_n) + p_s(y_n, y_n)}{2} \right\} \\ &\leq k \max \left\{ p_s(y_{n-1}, y_n), p_s(y_n, y_{n+1}), \frac{p_s(y_{n-1}, y_n) + p_s(y_n, y_{n+1})}{2} \right\} \\ &\leq k \max \left\{ p_s(y_{n-1}, y_n), p_s(y_n, x_{n+1}) \right\} \end{aligned}$$

By using Lemma 2.2 and taking $h = \frac{1}{\sqrt{k}}$, we have

$$\begin{aligned} p_s(x, y) &= hH_{p_s}(Tx, Ty), \\ &= \frac{1}{\sqrt{k}}H_{p_s}(Ty_{n-1}, Ty_n), \\ p_s(y_n, y_{n+1}) &\leq (\sqrt{k})^n \max(p_s(y_{n-1}, y_n), p_s(y_n, y_{n+1})). \end{aligned} \quad (23)$$

Which shows that, either

$$p_s(y_n, y_{n+1}) \leq (\sqrt{k})^n p_s(y_{n-1}, y_n), \quad (24)$$

or

$$p_s(y_n, y_{n+1}) \leq (\sqrt{k})^n p_s(y_n, y_{n+1}), \quad (25)$$

is the maximum of (23). By following similar steps from above, we reach the same results as in equation (19).

Case II

If $x_n \in P$, $x_{n+1} \in Q$. Assume that $x_n = y_n = Tx_{n-1}$ and $x_{n+1} \neq y_{n+1} = Tx_n$. Then, we have

$$\begin{aligned} p_s(x_n, x_{n+1}) &\leq p_s(x_n, y_{n+1}), \\ &= p_s(y_n, y_{n+1}), . \end{aligned}$$

Since $\{x_n\} \in C$ for all $n \in \mathbb{N}$, using contraction condition we shows that $x_n, x_{n-1} \in \partial C$.

Now,

$$p_s(x_n, x_{n+1}) \leq hH_{p_s}(Tx_{n-1}, Tx_n).$$

By using $x = x_{n-1}$ and $y = x_n$ in (8), we have

$$\begin{aligned} H_{p_s}(Tx_{n-1}, Tx_n) &\leq k \max \left\{ p_s(x_{n-1}, x_n), p_s(x_{n-1}, Tx_{n-1}), p_s(x_n, Tx_n), \right. \\ &\quad \left. \frac{p_s(x_{n-1}, Tx_n) + p_s(x_n, Tx_{n-1})}{2} \right\}, \\ &\leq k \max \left\{ p_s(x_{n-1}, x_n), p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1}), \right. \\ &\quad \left. \frac{p_s(x_{n-1}, x_{n+1}) + p_s(x_n, x_n)}{2} \right\}. \end{aligned} \quad (26)$$

By Lemma 2.5, we have

$$p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1}) = p_s(x_{n-1}, x_{n+1}) + p_s(x_n, x_n). \quad (27)$$

Apply Equation (27) in (26), we obtain

$$\begin{aligned} H_{p_s}(Tx_{n-1}, Tx_n) &\leq k \max \left\{ p_s(x_{n-1}, x_n), p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1}), \right. \\ &\quad \left. \frac{p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1}) - p_s(x_n, x_n) + p_s(x_n, x_n)}{2} \right\} \\ &\leq k \max \left\{ p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1}), \right. \\ &\quad \left. \frac{p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})}{2} \right\} \\ &\leq k \max \left\{ p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1}) \right\} \end{aligned}$$

By using Lemma 2.2 and taking $h = \frac{1}{\sqrt{k}}$, we have

$$\begin{aligned} p_s(x, y) &= hH_{p_s}(Tx, Ty), \\ &= \frac{1}{\sqrt{k}}H_{p_s}(Tx_{n-1}, Tx_n), \\ p_s(x_n, x_{n+1}) &\leq (\sqrt{k})^n \max\{p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1})\}. \end{aligned} \quad (28)$$

Thus

$$p_s(x_n, x_{n+1}) \leq (\sqrt{k})^n p_s(x_{n-1}, x_n).$$

We have the same result as in the case I.

Case III

If $x_n \in Q$, $x_{n+1} \in P$, $y_n \neq x_n$, $x_{n-1} \in P$, $x_{n+1} \in P$. Assume that $x_{n+1} = Tx_n$, $x_n \neq y_n = Tx_{n-1}$. Then, we have

$$\begin{aligned} p_s(x_n, x_{n+1}) &\leq p_s(x_n, y_n) + p_s(y_n, x_{n+1}), \\ p_s(x_n, x_{n+1}) &\leq p_s(x_n, y_n) + p_s(y_n, y_{n+1}) \end{aligned} \quad (29)$$

From Case I, if $y_n, y_{n+1} \in P \times Q$, then (29) becomes

$$\begin{aligned} p_s(x_n, x_{n+1}) &\leq p_s(x_n, y_n) + H_{p_s}(Tx_{n-1}, Tx_n), \\ p_s(x_n, x_{n+1}) &\leq p_s(x_n, y_n) + (\sqrt{k})^n (p_s(x_{n-1}, x_n)). \end{aligned} \quad (30)$$

Using (iv) we have

$$p_s(x_{n-1}, x_n) + p_s(x_n, y_n) = p_s(x_{n-1}, y_n) + p_s(x_n, x_n).$$

Since, $p_s(x_n, x_n) \Rightarrow p_s(x, x) = 0$, the above equality becomes

$$\begin{aligned} p_s(x_{n-1}, x_n) + p_s(x_n, y_n) &= p_s(x_{n-1}, y_n), \\ p_s(x_n, y_n) &= p_s(x_{n-1}, y_n) - p_s(x_{n-1}, x_n). \end{aligned} \quad (31)$$

Applying (31) in (30), we get

$$p_s(x_n, x_{n+1}) \leq p_s(x_{n-1}, y_n) - p_s(x_{n-1}, x_n) + (\sqrt{k})^n (p_s(x_{n-1}, x_n)).$$

Since, $k < 1$ we obtain

$$\begin{aligned} p_s(x_n, x_{n+1}) &\leq p_s(x_{n-1}, y_n), \\ &\leq p_s(y_{n-1}, y_n) \end{aligned} \quad (32)$$

$$p_s(y_{n-1}, y_n) = p_s(y_{n-1}, Tx_{n-1}) \leq hH_{p_s}(Tx_{n-2}, Tx_{n-1}),$$

Since, $x_n \in Q \Rightarrow x_{n-1} \in P$. Thus,

$$p_s(x_n, x_{n+1}) = hH_{p_s}(Tx_{n-2}, Tx_{n-1}).$$

Since $x_{n-1} \in C$ for all $n \in \mathbb{N}$, by using (8) we can show that $x_{n-2}, x_{n-1} \in \partial C$.

By substituting $x = x_{n-2}$ and $y = x_{n-1}$ in (8), we have

$$\begin{aligned} H_{p_s}(Tx_{n-2}, Tx_{n-1}) &\leq k \max\left\{p_s(x_{n-2}, x_{n-1}), p_s(x_{n-2}, Tx_{n-2}), p_s(x_{n-1}, Tx_{n-1}), \right. \\ &\quad \left. \frac{p_s(x_{n-2}, Tx_{n-1}) + p_s(x_{n-1}, Tx_{n-2})}{2}\right\}. \\ &\leq k \max\left\{p_s(x_{n-2}, x_{n-1}), p_s(x_{n-2}, x_{n-1}), p_s(x_{n-1}, x_n), \right. \\ &\quad \left. \frac{p_s(x_{n-2}, x_n) + p_s(x_{n-1}, x_{n-1})}{2}\right\}. \end{aligned}$$

By using Lemma 2.5, we have

$$\begin{aligned}
 &\leq k \max \left\{ p_s(x_{n-2}, x_{n-1}), p_s(x_{n-2}, x_{n-1}), p_s(x_{n-1}, x_n), \right. \\
 &\quad \left. \frac{p_s(x_{n-2}, x_{n-1}) + p_s(x_{n-1}, x_n) - p_s(x_{n-1}, x_{n-1}) + p_s(x_{n-1}, x_{n-1})}{2} \right\} \\
 &\leq k \max \left\{ p_s(x_{n-2}, x_{n-1}), p_s(x_{n-2}, x_{n-1}), p_s(x_{n-1}, x_n), \right. \\
 &\quad \left. \frac{p_s(x_{n-2}, x_{n-1}) + p_s(x_{n-1}, x_n)}{2} \right\} \\
 &\leq k \max \{ p_s(x_{n-2}, x_{n-1}), p_s(x_{n-1}, x_n) \}
 \end{aligned} \tag{33}$$

Using Lemma 2.2 and taking $h = \frac{1}{\sqrt{k}}$ in (33), we get

$$\begin{aligned}
 p_s(x, y) &= h H_{p_s}(Tx, Ty), \\
 p_s(x_n, x_{n+1}) &\leq \frac{1}{\sqrt{k}} H_{p_s}(Tx_{n-2}, Tx_{n-1}), \\
 &\leq \sqrt{k} \max \{ p_s(x_{n-2}, x_{n-1}), p_s(x_{n-1}, x_n) \},
 \end{aligned}$$

which implies that, either

$$p_s(x_n, x_{n+1}) \leq \sqrt{k} \max \{ p_s(x_{n-1}, x_n) \}.$$

or

$$p_s(x_n, x_{n+1}) \leq \sqrt{k} \max \{ p_s(x_{n-2}, x_{n-1}) \}.$$

Thus, for $n \geq 2$, we have

$$p_s(x_n, x_{n+1}) = \begin{cases} (\sqrt{k})^n p_s(x_{n-2}, x_{n-1}), \\ (\sqrt{k})^n p_s(x_{n-1}, x_n). \end{cases}$$

Letting,

$$\delta = \max(p_s(x_{n-2}, x_{n-1}), p_s(x_{n-1}, x_n)).$$

Thus in all three cases, Case I, Case II, Case III, we have

$$\begin{aligned}
 p_s(x_n, x_{n+1}) &\leq (\sqrt{k})^n \max \{ p_s(x_{n-2}, x_{n-1}), p_s(x_{n-1}, x_n) \}. \\
 &\leq (\sqrt{k})^n \delta.
 \end{aligned}$$

Inductively for all cases, there exists a sequence $P_n \in C$ such that

$$x_{n+1} \in Tx_n \text{ and } p_s(x_n, x_{n+1}) \leq (\sqrt{k})^n \max(p_s(x_0, x_1), p_s(x_1, x_2)).$$

For $n > 1$, since $n, m \in \mathbb{N}$, $n > m$, we obtain

$$\begin{aligned}
 p_s(x_n, x_m) &\leq \sum_{i=n}^{m-1} p_s(x_i, x_{i+1}) \\
 &\leq \sum_{i=n}^{m-1} (\sqrt{k})^i \delta \\
 &\leq \sum_{i=n}^{\infty} (\sqrt{k})^i \delta \\
 &\leq \delta \frac{(\sqrt{k})^n}{1 - \sqrt{k}}.
 \end{aligned} \tag{34}$$

As $n, m \rightarrow \infty$, we obtain

$$\lim_{m, n \rightarrow \infty} p_s(x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a p_s -Cauchy sequence in C . There exists $x \in C$ such that $\{x_n\}$ p_s -converges to x .

$$\begin{aligned}\lim_{n,m \rightarrow \infty} p_s(x_n, x_m) &= \lim_{n \rightarrow \infty} p_s(x_n, x), \\ &= p_s(x, x) = 0.\end{aligned}$$

We proved that $\{x_n\}$ is Cauchy sequence. Also, since (X, p_s) is a complete convex partial symmetric space, there is $x \in X$ such that $x_n \rightarrow x$, and $p(x, x) = 0$.

Now we will show that x is a fixed point of T . We do this by showing that $p_s(x, Tx) = p_s(x, x) = 0$. Suppose $p_s(x, Tx) > 0$. Then there exists some $n_0 \in \mathbb{N}$ such that $p_s(x_n, Tx) > 0$ for all $n > n_0$.

By substituting $x = x_n$ and $y = x$ in (8), we have

$$H_{p_s}(Tx_n, Tx) \leq k \max \left\{ p_s(x_n, x) p_s(x_n, Tx_n), p_s(x, Tx), \frac{p_s(x_n, Tx) + p_s(x, Tx_n)}{2} \right\}. \quad (35)$$

Taking $n \rightarrow \infty$ in (35) and applying the fact that T is continuous, we get

$$\begin{aligned}H_{p_s}(Tx_n, Tx) &\leq k \max \left\{ p_s(x, x), p_s(x, Tx), p_s(x, Tx), \frac{p_s(x, Tx) + p_s(x, Tx)}{2} \right\}, \\ H_{p_s}(Tx_n, Tx) &\leq k p_s(x, Tx)\end{aligned} \quad (36)$$

By Lemma 2.2 implies that

$$p_s(x_n, x) = \frac{1}{\sqrt{k}} H_{p_s}(Tx_n, Tx).$$

Using (36) and $n \rightarrow \infty$ in above inequality we obtain

$$p_s(x, x) \leq \frac{1}{\sqrt{k}} k p_s(x, Tx).$$

By the continuity of T , we get

$$0 \leq p_s(x, Tx).$$

Therefore,

$$p_s(x, x) = p_s(x, Tx) = 0,$$

which is a contradiction. Hence, x is a fixed point of T .

Next, we claim that x is unique. Suppose that x^* is another element in C such that $Tx^* = x^*$, $Tx = x$. Then

$$\begin{aligned}H_{p_s}(Tx, Tx^*) &\leq k \max \left\{ p_s(x, x^*), p_s(x, Tx), p_s(x^*, Tx^*), \frac{p_s(x, Tx^*) + p_s(x^*, Tx)}{2} \right\}, \\ H_{p_s}(Tx, Tx^*) &\leq k p_s(x, x^*)\end{aligned} \quad (37)$$

By Lemma 2.2 implies that

$$p_s(x, x^*) = \frac{1}{\sqrt{k}} H_{p_s}(Tx, Tx^*).$$

Using (37) and in above inequality we obtain

$$p_s(x, x^*) \leq \frac{1}{\sqrt{k}} k p_s(x, x^*).$$

By the continuity of T , we get

$$0 \leq p_s(x, x^*).$$

Hence

$$p_s(x, x^*) = 0,$$

implying that $x = x^*$. Which is a contradiction. Hence, the fixed point is not unique. $\square$

We give the following corollary, which is motivated from Ali *et al.* [16].

Corollary 3.1. Let (X, p_s) be a complete metrically convex partial symmetric space, suppose D is a nonempty subset of X . Let $T : D \rightarrow CB^{p_s}(X)$ be a mapping such that $Tx \neq 0$ for each $x, y \in D$, $q, r \geq 2$ with $H_{p_s}(Tx, Ty) \geq 0$, we have

$$H_{p_s}(Tx, Ty) = \psi(M(x, y)),$$

where

$$M(x, y) = \max \left\{ p(x, y), \frac{p(x, Tx) + p(y, Ty)}{q}, \frac{p(x, Ty) + p(y, Tx)}{r} \right\},$$

and ψ is an increasing function in Ψ . Also assume that the following conditions hold:

- (i) there exist $x_0 \in \partial D$ and $x_1 \in Tx_0$ such that $Tx_0 \in D$,
- (ii) T is continuous,
- (iii) for any sequence $\{x_n\}$ in D such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, $p_s(x_n, x) \rightarrow p_s(x, x)$ as $n \rightarrow \infty$.

Then T has a fixed point.

Proof. Suppose that T has no fixed points then $p_s(x_0, Tx_0) > 0$ for all $x \in D$. We start by constructing two sequences $\{x_n\}$ and $\{y_n\}$, respectively, in the following way.

Let $x_0 \in \partial D$ and $y_1 \in Tx_0$, if $y_1 \in D$, let $x_1 = y_1$, then our proof would be completed. Since T is a multivalued mapping $x_1 \in Tx_0$. If $y_1 \notin D$, then, by Lemma 2.5 there exists $x_1 \in \partial D$ such that

$$p_s(x_0, x_1) + p_s(x_1, y_1) = p_s(x_0, y_1) + p_s(x_1, x_1). \quad (38)$$

Thus for $x_1 \in D$ and using Lemma 2.2, we can choose $y_2 \in Tx_1$ such that

$$p_s(y_1, y_2) \leq h H_{p_s}(Tx_0, Tx_1). \quad (39)$$

For $h = \frac{1}{\sqrt{k}}$, equation (39) becomes

$$p_s(y_1, y_2) \leq \frac{1}{\sqrt{k}} H_{p_s}(Tx_0, Tx_1).$$

However, if $y_2 \in D$, let $x_2 = y_2$. If $y_2 \notin D$. Then, using Lemma 2.5 there exists $x_2 \in \partial D$ such that

$$p_s(x_1, x_2) + p_s(x_2, y_2) = p_s(x_1, y_2) + p_s(x_2, x_2).$$

Thus $x_2 \in D$ and using Lemma 2.2, we can choose $y_3 \in Tx_2$ such that

$$p_s(y_2, y_3) \leq h H_{p_s}(Tx_1, Tx_2). \quad (40)$$

Again for $h = \frac{1}{\sqrt{k}}$ in (40) we have

$$p_s(y_2, y_3) \leq \frac{1}{\sqrt{k}} H_{p_s}(Tx_1, Tx_2).$$

The other steps follow the similar proof of Theorem (3.1). This complete the proof. $\square$

Now, we give an example to illustrate the use of Theorem 3.1.

Example 3.1

Let $X = [0, 1]$. Denote the unit interval of real numbers, with the usual convex partial symmetric $p_s(x, y) = \|x - y\|$, and a multivalued mapping $T : C \rightarrow CB^{p_s}(X)$ given by

$$Tx = \begin{cases} \frac{x}{2}, & 0 \leq x \leq \frac{2}{3}, \\ -\frac{x}{2} + 1, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

Define $T : C \rightarrow 2^{[0,1]}$, $C = [0, \frac{2}{3}] \cup [\frac{2}{3}, 1]$.

Proof. Since ∂C (boundary of C), $\partial C = \{0, \frac{2}{3}, 1\}$. Which shows that the fixed point of T are 0 and $\frac{2}{3}$.

By using (i) we have

$$0 \in \partial C \Rightarrow T0 = 0.$$

$$\frac{2}{3} \in \partial C \Rightarrow T\frac{2}{3} = \frac{1}{3}.$$

Also,

$$1 \in \partial C \Rightarrow T1 = \frac{1}{2}.$$

$$\frac{2}{3} \in \partial C \Rightarrow T\frac{2}{3} = \frac{2}{3}.$$

Using (ii), we note that $\{0\}$ and $\{0, \frac{2}{3}\}$ are bounded sets in (X, p_s) . By Lemma 2.1, if $x \in \{0, 1\}$, then

$$\begin{aligned} x \in \overline{\{0\}} &\Leftrightarrow \text{dist}_{p_s}(x, \{0\}) = p_s(x, x) \\ &\Leftrightarrow p_s(x, 0) = |x - 0| \\ &\Leftrightarrow x = 0 \Leftrightarrow x \in \{0\}. \end{aligned}$$

Next,

$$\begin{aligned} x \in \left\{0, \frac{2}{3}\right\} &\Leftrightarrow \text{dist}_{p_s}\left(x, \left\{0, \frac{2}{3}\right\}\right) = p_s(x, x) \\ &\Leftrightarrow \min\left\{p_s(x, 0), \left\{0, \frac{2}{3}\right\}\right\} \\ &\Leftrightarrow \min\left\{|x - 0|, |x - \frac{2}{3}|\right\} \\ &\Leftrightarrow x \in \left\{0, \frac{2}{3}\right\}. \end{aligned}$$

Hence, $\left\{0, \frac{2}{3}\right\}$ is also closed with respect to partial convex symmetric p_s .

Note that T is continuous under the partial symmetric p_s .

Now, to show that the contractive condition of Theorem 3.1 is satisfied, we have the following cases:

Case 1. We now calculate $H_{p_s}(Tx, Ty)$ for $x, y \in \left\{0, \frac{2}{3}\right\}$. Then,

$$Tx = \left\{0, \frac{1}{3}\right\}, Ty = \left\{0, \frac{1}{3}\right\}.$$

$$\begin{aligned} H_{p_s}(Tx, Ty) &= H_{p_s}\left(\left\{0, \frac{1}{3}\right\}, \left\{0, \frac{1}{3}\right\}\right) \\ &= \max\{\delta_{p_s}(Tx, Ty), \delta_{p_s}(Ty, Tx)\} \end{aligned}$$

$$\begin{aligned} \delta_{p_s}(Tx, Ty) &= \max\{p_s(a, Ty); a \in Tx\} \\ &= \max\left\{p_s(0, Ty), p_s\left(\frac{1}{3}, Ty\right)\right\} \end{aligned}$$

$$\begin{aligned} p_s(0, Ty) &= \min\left\{p_s(0, 0), p_s\left(0, \frac{1}{3}\right)\right\} \\ &= \min\left\{0, \frac{1}{3}\right\} = 0. \end{aligned}$$

$$\begin{aligned} p_s\left(\frac{1}{3}, Ty\right) &= \min\left\{p_s\left(\frac{1}{3}, 0\right), p_s\left(\frac{1}{3}, \frac{1}{3}\right)\right\} \\ &= \min\left\{\frac{1}{3}, 0\right\} = 0. \end{aligned}$$

$$\delta_{p_s}(Tx, Ty) = \max\{0, 0\} = 0.$$

$$\begin{aligned}
\delta_{p_s}(Ty, Tx) &= \max\{p_s(a, Tx); a \in Ty\} \\
&= \max\left\{p_s(0, Tx), p_s\left(\frac{1}{3}, Tx\right)\right\} \\
\delta_{p_s}(Ty, Tx) &= \max\left\{0, \frac{1}{3}\right\} = 0. \\
H_{p_s}(Tx, Ty) &= \max\{\delta_p(Tx, Ty), \delta_p(Ty, Tx)\}, \\
&= \max\{0, 0\} = 0.
\end{aligned}$$

In a similar manner we calculate

$$\begin{aligned}
p_s(x, y) &= p_s\left(0, \frac{2}{3}\right) = \frac{2}{3}, \\
p_s(x, Tx) &= p_s\left(0, \left\{0, \frac{1}{3}\right\}\right) = 0 \\
p_s(y, Ty) &= p_s\left(\frac{2}{3}, \left\{0, \frac{1}{3}\right\}\right) = \frac{1}{3} \\
p_s(x, Ty) &= p_s\left(0, \left\{0, \frac{1}{3}\right\}\right) = 0 \\
p_s(y, Tx) &= p_s\left(\frac{2}{3}, \left\{0, \frac{1}{3}\right\}\right) = \frac{1}{3}
\end{aligned}$$

Using the above equations in (8) we obtain

$$\begin{aligned}
0 &\leq k \max\left\{\frac{2}{3}, 0, \frac{1}{3}, \frac{0 + \frac{1}{3}}{2}\right\} \\
&\leq k \max\left\{\frac{2}{3}, 0, \frac{1}{3}, \frac{1}{6}\right\} \\
&\leq \frac{2}{3}k.
\end{aligned}$$

Equation 8 satisfies with $k = \frac{1}{2}$.

Case 2.

Next, we calculate $H_{p_s}(Tx, Ty)$ for $x \in \left\{0, \frac{2}{3}\right\}$, $y \in \left\{\frac{2}{3}, 1\right\}$. Then,

$$Tx = \left\{0, \frac{1}{3}\right\}, Ty = \left\{\frac{2}{3}, \frac{1}{2}\right\}.$$

$$\begin{aligned}
H_{p_s}(Tx, Ty) &= H_{p_s}\left(\left\{0, \frac{1}{3}\right\}, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right) \\
&= \max\{\delta_{p_s}(Tx, Ty), \delta_{p_s}(Ty, Tx)\}
\end{aligned}$$

$$\begin{aligned}
\delta_{p_s}(Tx, Ty) &= \max\{p_s(a, Ty); a \in Tx\} \\
&= \max\left\{p_s(0, Ty), p_s\left(\frac{1}{3}, Ty\right)\right\} \\
&= \max\left\{p_s\left(0, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right), p_s\left(\frac{1}{3}, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right)\right\} \\
\delta_{p_s}(Tx, Ty) &= \max\left\{\frac{1}{2}, \frac{1}{6}\right\} = \frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
\delta_{p_s}(Ty, Tx) &= \max\{p_s(a, Tx); a \in Ty\} \\
&= \max\left\{p_s\left(\frac{2}{3}, Tx\right), p_s\left(\frac{1}{3}, Tx\right)\right\} \\
&= \max\left\{p_s\left(\frac{2}{3}, \left\{0, \frac{1}{3}\right\}\right), p_s\left(\frac{1}{3}, \left\{0, \frac{1}{3}\right\}\right)\right\}
\end{aligned}$$

$$\begin{aligned}
\delta_{p_s}(Ty, Tx) &= \max\left\{\frac{1}{3}, 0\right\} = \frac{1}{3}. \\
H_{p_s}(Tx, Ty) &= \max\{\delta_p(Tx, Ty), \delta_p(Ty, Tx)\}, \\
&= \max\left\{\frac{1}{2}, \frac{1}{3}\right\} = \frac{1}{2}
\end{aligned}$$

In a similar manner we calculate

$$\begin{aligned} p_s(x, y) &= p_s(0, 1) = 1, \\ p_s(x, Tx) &= p_s\left(0, \left\{0, \frac{1}{3}\right\}\right) = 0 \\ p_s(y, Ty) &= p_s\left(1, \left\{0, \frac{1}{3}\right\}\right) = 0 \\ p_s(x, Ty) &= p_s\left(0, \left\{\frac{2}{3}, 1\right\}\right) = \frac{2}{3} \\ p_s(y, Tx) &= p_s\left(1, \left\{0, \frac{1}{3}\right\}\right) = \frac{2}{3} \end{aligned}$$

Using the above equations in (8) we obtain

$$\begin{aligned} \frac{1}{2} &\leq k \max\left\{1, 0, 0, \frac{\frac{2}{3} + \frac{2}{3}}{2}\right\} \\ &\leq k \max\left\{1, 0, 0, \frac{2}{3}\right\} \\ &\leq k. \end{aligned}$$

Equation 8 satisfies with $k = \frac{2}{3}$.

Case 3.

Furthermore, we calculate $H_{p_s}(Tx, Ty)$ for $x, y \in \left\{\frac{2}{3}, 1\right\}$. Let,

$$Tx = \left\{\frac{2}{3}, \frac{1}{2}\right\}, Ty = \left\{\frac{2}{3}, \frac{1}{2}\right\}.$$

$$\begin{aligned} H_{p_s}(Tx, Ty) &= H_{p_s}\left(\left\{\frac{2}{3}, \frac{1}{2}\right\}, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right) \\ &= \max\{\delta_{p_s}(Tx, Ty), \delta_{p_s}(Ty, Tx)\} \end{aligned}$$

$$\begin{aligned} \delta_{p_s}(Tx, Ty) &= \max\{p_s(a, Ty); a \in Tx\} \\ &= \max\left\{p_s\left(\frac{2}{3}, Ty\right), p_s\left(\frac{1}{2}, Ty\right)\right\} \\ &= \max\left\{p_s\left(\frac{2}{3}, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right), p_s\left(\frac{1}{2}, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right)\right\} \\ \delta_{p_s}(Tx, Ty) &= \max\{0, 0\} = 0. \end{aligned}$$

$$\begin{aligned} \delta_{p_s}(Ty, Tx) &= \max\{p_s(a, Tx); a \in Ty\} \\ &= \max\left\{p_s\left(\frac{2}{3}, Tx\right), p_s\left(\frac{1}{2}, Tx\right)\right\} \\ &= \max\left\{p_s\left(\frac{2}{3}, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right), p_s\left(\frac{1}{2}, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right)\right\} \end{aligned}$$

$$\delta_{p_s}(Ty, Tx) = \max\{0, 0\} = 0.$$

$$\begin{aligned} H_{p_s}(Tx, Ty) &= \max\{\delta_{p_s}(Tx, Ty), \delta_{p_s}(Ty, Tx)\}, \\ &= \max\{0, 0\} = 0 \end{aligned}$$

In a similar manner we calculate

$$\begin{aligned} p_s(x, y) &= p_s(0, 1) = 1, \\ p_s(x, Tx) &= p_s\left(0, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right) = \frac{1}{2} \\ p_s(y, Ty) &= p_s\left(1, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right) = \frac{1}{3} \\ p_s(x, Ty) &= p_s\left(0, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right) = \frac{1}{2} \\ p_s(y, Tx) &= p_s\left(1, \left\{\frac{2}{3}, \frac{1}{2}\right\}\right) = \frac{1}{3} \end{aligned}$$

Using the above equations in (8) we get

$$\begin{aligned} 0 &\leq k \max \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{\frac{1}{2} + \frac{1}{3}}{2} \right\} \\ &\leq k \max \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{5}{12} \right\} \\ 0 &\leq k. \end{aligned}$$

Equation 8 satisfies with $k = \frac{1}{2}$.

Which shows that the contraction (8) is satisfied. Therefore, the condition of the Theorem 3.1 is correct; thus, 0 and $\frac{2}{3}$ are two fixed points of T . $\square$

4 Conclusions

The main contribution of this study to fixed point theory is the fixed point result given in Theorem 3.1. This theorem provides the fixed point conditions for a substantial class of non-self mappings on various abstract spaces. This paper, Motivated by the results obtained by Asim *et al* [16] in partial symmetric metric space and Assad and Kirk in metrically convex metric space. We proved a fixed point theorem for multi-valued non-self mappings in partial symmetric spaces by employing a convex structure using Rhoades type contractions [11], which generalizes some well-known results in the literature. We also provided an illustrative example to support the results.

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