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Mohammad Asim, Samad Mujahid, and Izhar Uddin*

Meir-Keeler Contraction In Rectangular M -Metric Space

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Abstract: In this paper, we prove some fixed point theorems for a Meir-Keeler type Contraction in rectangular M -metric space. Thus, our results extend and improve very recent results in fixed point theory.

Keywords: Fixed point, Meir-Keeler contraction, M -metric space, rectangular M -metric space

MSC: 47H10, 54H25

1 Introduction

Fixed point theory provides essential tools for solving problems arising in various branches of mathematical analysis, such as split feasibility problems, variational inequality problems, nonlinear optimization problems, equilibrium problems, complementarity problems, selection and matching problems, and problems of proving the existence of solution of integral and differential equations. In Metric fixed point theory, the well-known Banach contraction principle [1] ensures the existence and uniqueness of fixed points of contraction maps in the setting of complete metric spaces. There are two ways to generalized Banach contraction principle [1]. The first one can change the active contraction and the second is to alter the underlying metric space. Many authors generalized the metric structure mainly: partial metric space [2], b -metric space [3], partial b -metric space [4], Branciari metric space [5, 6], partial rectangular metric space [7], M -metric space [8], rectangular M -metric space [9], rectangular M_b -metric space [10], extended rectangular $M_{r\zeta}$ -metric space [11], M_v -metric spaces [12] and some more [13]–[34].

In literature, there are many generalized contraction available. But due to their applicability and impact, we record the few names as: Meir-Keeler contraction [35], Kannan contraction [36], Chatterjea [37], Boyd and Wong contraction [38] etc. In 2000, Branciari [6] introduced rectangular metric space and obtained certain fixed point theorems. In 2014, Asadi et al. [8, 39] introduced the M -metric space, which extends the p -metric space [2] and established many fixed point theorems for Banach contraction principle and Meir-Keeler type contraction. In 2018, Özgür [9] extends both rectangular and M -metric space by introducing rectangular M -metric space and certain fixed point theorems obtained therein. Meir-Keeler has a very significant place among the contraction condition. The definition of Meir-Keeler contraction is as follows:

Definition 1.1. [35] Let M_r be a non-empty set. A Meir-Keeler mapping is a mapping $T : M_r \rightarrow M_r$ on an rectangular M -metric space (X, M_r) such that

$$\forall \epsilon > 0, \quad \exists \delta > 0 \text{ such that } \forall x, y \in X \quad \text{and} \quad \epsilon \leq m_r(x, y) < \epsilon + \delta \Rightarrow m_r(Tx, Ty) < \epsilon.$$

Mohammad Asim, Samad Mujahid: Department of Mathematics, Faculty of Science, Shree Guru Gobind Singh Tricentenary University, Gurugram, Haryana, India

Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India, E-mail: mailtoasim27@gmail.com, mujahidsamad721@gmail.com

***Corresponding Author: Izhar Uddin:** Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India, E-mail: izharuddin1@jmi.ac.in

Many authors studies Meir-Keeler contraction principle in different space. The aim of the paper is to study Meir-Keeler contraction in rectangular M -metric space.

2 Preliminaries

In this section, we collect some basic notions, definitions, examples, lemmas and auxiliary results.

Definition 2.1. [6] If X be a non-empty set. A function $r : X \times X \rightarrow R^+$ is said to be a rectangular metric on X if it satisfies the following (for all $x, y \in X$ and for all distinct point $u, v \in X \setminus \{x, y\}$):

- (i) $r(x, y) = 0$, if and only if $x = y$,
- (ii) $r(x, y) = r(y, x)$,
- (iii) $r(x, y) \leq r(x, u) + r(u, v) + r(v, y)$.

Then, the pair (X, r) is called a rectangular metric space. Also, called Branciari distance space or generalized metric space [6].

Definition 2.2. [7] If X be a non-empty set. A function $\rho : X \times X \rightarrow R^+$ is said to be a partial rectangular metric on X , if for any $x, y \in X$ and for all distinct point $u, v \in X \setminus \{x, y\}$ it satisfies the following conditions:

- (i) $x = y$ if and only if $\rho(x, y) = \rho(x, x) = \rho(y, y)$,
- (ii) $\rho(x, x) \leq \rho(x, y)$,
- (iii) $\rho(x, y) = \rho(y, x)$,
- (iv) $\rho(x, y) \leq \rho(x, u) + \rho(u, v) + \rho(v, y) - \rho(u, u) - \rho(v, v)$.

Then, the pair (X, ρ) is called a partial rectangular metric space.

Notation 2.3 [8] The following notations are useful in the sequel:

- (i) $m_{xy} := m(x, x) \vee m(y, y) = \min\{m(x, x), m(y, y)\}$,
- (ii) $M_{xy} := m(x, x) \wedge m(y, y) = \max\{m(x, x), m(y, y)\}$.

Definition 2.3. [8] If X be a non-empty set. A function $m : X \times X \rightarrow R^+$ is called a m -metric if it satisfying the following conditions:

- (i) $m(x, x) = m(y, y) = m(x, y) \iff x = y$,
- (ii) $m_{xy} \leq m(x, y)$,
- (iii) $m(x, y) = m(y, x)$,
- (iv) $(m(x, y) - m_{xy}) \leq (m(x, z) - m_{xz}) + (m(z, y) - m_{zy})$.

Then, the pair (X, m) is called an M -metric space.

Notation 2.4 [9] The following notations are useful in the sequel:

- (i) $m_{r_{xy}} := \min\{m_r(x, x), m_r(y, y)\}$,
- (ii) $M_{r_{xy}} := \max\{m_r(x, x), m_r(y, y)\}$.

Definition 2.4. [9] If X be a non-empty set and $m_r : X \times X \rightarrow R^+$ is a mapping. If it satisfying the following conditions for all $x, y \in X$:

- (i) $m_r(x, y) = m_{r_{xy}} = M_{r_{xy}} \iff x = y$,
- (ii) $m_{r_{xy}} \leq m_r(x, y)$,
- (iii) $m_r(x, y) = m_r(y, x)$,
- (iv) $(m_r(x, y) - m_{r_{xy}}) \leq (m_r(x, u) - m_{r_{xu}}) + (m_r(u, v) - m_{r_{uv}}) + (m_r(v, y) - m_{r_{vy}})$ for all $u, v \in X \setminus \{x, y\}$.

Then, the pair (X, m_r) is called a rectangular M -metric space.

Notice that every m -metric is a rectangular m -metric.

Example 2.1. [9] Let (X, d) be a rectangular metric space and a function $\xi : [0, \infty) \rightarrow [0, \infty)$ be a one-to-one and nondecreasing function with $\xi(0) = \alpha$ such that $\xi(x + y + z) \leq \xi(x) + \xi(y) + \xi(z) - 2\alpha$ for all $x, y, z \geq 0$. Then, the function $m_r : X \times X \rightarrow [0, \infty)$ is defined as $m_r(x, y) = \xi(d(x, y))$, for all $x, y \in X$ is rectangular m -metric.

Example 2.2. [9] Let (X, d) be a rectangular metric space and a function $\xi : [0, \infty) \rightarrow [\alpha, \infty)$ be defined as $\xi(t) = mt + n$ with $\xi(0) = \alpha$ for all $t \in [0, \infty)$. From Example 1.1, the function $m_r(x, y) = md(x, y) + n$ is a rectangular m -metric.

It is clear that each rectangular M -metric on X generates a T_0 topology τ_{m_r} on X . The set

$$B_{m_r} = \{B(x, \epsilon) : x \in X, \epsilon > 0\},$$

where

$$B(x, \epsilon) := \{y \in X : m_r(x, y) - m_{r_{xy}} < \epsilon\},$$

for all $x \in X$ and $\epsilon > 0$ forms the base of τ_{m_r} .

Definition 2.5. [9] Let (X, m_r) be a rectangular M -metric space. Then

(1) A Sequence $\{x_n\}$ in X converges to a point x if and only if

$$\lim_{n \rightarrow \infty} (m_r(x_n, x) - m_{r_{x_n x}}) = 0. \quad (2.1)$$

(2) A Sequence $\{x_n\}$ in X is said to be m_r -Cauchy sequence if and only if

$$\lim_{n, m \rightarrow \infty} (m_r(x_n, x_m) - m_{r_{x_n x_m}}) \text{ and } \lim_{n, m \rightarrow \infty} (M_r(x_n, x_m) - m_{r_{x_n x_m}}) \quad (2.2)$$

exist and finite.

(3) A rectangular M -metric space is said to be m_r complete if every m_r -Cauchy sequence $\{x_n\}$ converges to a point x such that

$$\lim_{n \rightarrow \infty} (m_r(x_n, x) - m_{r_{x_n x}}) = 0. \text{ and } \lim_{n \rightarrow \infty} (M_r(x_n, x) - m_{r_{x_n x}}) = 0. \quad (2.3)$$

Lemma 2.1. [9] Assume that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$ in a rectangular M -metric space (X, m_r) . Then,

$$\lim_{n \rightarrow \infty} (m_r(x_n, y_n) - m_{r_{x_n y_n}}) = m_r(x, y) - m_{r_{xy}}.$$

Lemma 2.2. [9] Assume that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$ in a rectangular M -metric space (X, m_r) . Then,

$$\lim_{n \rightarrow \infty} (m_r(x_n, y) - m_{r_{x_n y}}) = m_r(x, y) - m_{r_{xy}}, \forall y \in X.$$

Lemma 2.3. [9] Assume that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$ in a rectangular M -metric space (X, m_r) . Then,

$$\lim_{n \rightarrow \infty} (m_r(x_n, y_n) - m_{r_{x_n y_n}}) = m_r(x, y) - m_{r_{xy}}.$$

Lemma 2.4. [9] If $x_n \rightarrow x$ and $y_n \rightarrow y$ in a rectangular M -metric space (X, m_r) , then, $m_r(x, y) = m_{r_{xy}}$. Further, if $m_r(x, x) = m_r(y, y)$, then $x = y$.

Lemma 2.5. [9] Let $\{x_n\}$ be a sequence in a rectangular M -metric space (X, m_r) , such that there exists $r \in [0, 1)$ such that

$$m_r(x_{n+1}, x_n) \leq r m_r(x_n, x_{n-1}) \text{ for all } n \in \mathbb{N}. \quad (2.4)$$

Then

$$(A) \lim_{n \rightarrow \infty} m_r(x_n, x_{n-1}) = 0,$$

- (B) $\lim_{n \rightarrow \infty} m_r(x_n, x_n) = 0$,
 (C) $\lim_{n, m \rightarrow \infty} m_{r_{x_n x_m}} = 0$,
 (D) $\{x_n\}$ is an m_r -Cauchy sequence.

Proof. [9] Using the definition of convergence and inequality (2.4), the proof of the condition (A) follows easily. From the Condition $(m_{r_{xy}} \leq m_r(x, y))$ and the Condition (A), we get

$$\lim_{n \rightarrow \infty} \min\{m_r(x_n, x_n), m_r(x_{n-1}, x_{n-1})\} = \lim_{n \rightarrow \infty} m_{r_{x_n x_{n-1}}} \\ \leq \lim_{n \rightarrow \infty} m_r(x_n, x_{n-1}) = 0.$$

Therefore, the Condition (B) holds. Since $\lim_{n \rightarrow \infty} m_r(x_n, x_n) = 0$, the Condition (C) holds. Using the previous conditions and the definition 2.5, we see that the Condition (D) holds. $\square$

Theorem 2.1 [9] Let (X, m_r) be a rectangular M -metric space and T be a self-mapping on X . If there exists $k \in [0, 1)$ such that

$$m_r(Tx, Ty) \leq km_r(x, y) \text{ for all } x, y \in X \quad (2.5)$$

and consider the sequence $\{x_n\}_{n \geq 0}$ defined by $x_{n+1} = Tx_n$. If $x_n \rightarrow u$ as $n \rightarrow \infty$.

Then, $Tx_n \rightarrow Tu$ as $n \rightarrow \infty$.

Theorem 2.2 [9] Let (X, m_r) be a complete rectangular M -metric space and T be a self-mapping on X . If there exists $k \in (0, 1)$ such that

$$m_r(Tx, Ty) \leq km_r(x, y) \text{ for all } x, y \in X. \quad (2.6)$$

Then, T has a unique fixed point $u \in X$, where $m_r(u, u) = 0$.

Theorem 2.3 [9] Let (X, m_r) be a complete rectangular M -metric space and T be a self-mapping on X . If there exists $k \in [0, \frac{1}{2})$ such that

$$m_r(Tx, Ty) \leq k[m_r(x, Tx) + m_r(y, Ty)] \text{ for all } x, y \in X. \quad (2.7)$$

Then, T has a unique fixed point $u \in X$, where $m_r(u, u) = 0$.

3 Main results

The Following definition is new version of the definition in [35] for an M_r -Metric space.

Definition 3.1 Let (X, M_r) be a rectangular M -metric space. A mapping $T : M_r \rightarrow M_r$ is said to be Meir-Keeler contraction if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in X \quad \epsilon \leq m_r(x, y) < \epsilon + \delta \Rightarrow m_r(Tx, Ty) < \epsilon. \quad (3.1)$$

Theorem 3.1. Let (X, M_r) be a m_r -complete rectangular metric space and let T a Meir-Keeler contraction. Then, T has a unique fixed point $z \in X$. Moreover, for all $x \in X$, the sequence $\{T_n(x)\}$ converges to z .

Proof. Let $x_0 \in X$ and $x_n = Tx_{n-1}$ for all $n = 1, 2, \dots$. Hence, by Condition (3.1), we have

$$m_r(x_n, x_{n+1}) = m_r(Tx_{n-1}, Tx_{n-2}) \leq m_r(x_{n-1}, x_{n-2}).$$

So the sequence $\{m_r(x_n, x_{n+1})\}$ is bounded below and decreasing. Thus, $m_r(x_n, x_{n+1}) \rightarrow m$ for some $m \in \mathbb{R}^+$. Let $m > 0$, therefore $m_r(x_n, x_{n+1}) \geq m$. On the other hand for $m > 0$ there exists $\delta(m) > 0$ such that

$$m \leq m_r(x_{n+1}, x_{n+2}) < m + \delta(m) \Rightarrow m_r(Tx_{n+1}, Tx_{n+2}) = m_r(x_n, x_{n+1}) < m.$$

Which implies that it is contradiction; so $m = 0$, i.e.,

$$\lim_{n \rightarrow \infty} m_r(x_n, x_{n+1}) = 0 \quad (3.2)$$

and

$$\lim_{n \rightarrow \infty} \min\{m_r(x_n, x_n), m_r(x_{n-1}, x_{n-1})\} = 0 = \lim_{n \rightarrow \infty} m_{r_{x_n, x_{n-1}}} \leq \lim_{n \rightarrow \infty} m_r(x_n, x_{n-1}).$$

Similarly,

$$\lim_{m, n \rightarrow \infty} m_{r_{x_m, x_n}} = 0 \text{ and } \lim_{m, n \rightarrow \infty} M_{r_{x_m, x_n}} = 0, \quad (3.3)$$

since, $\lim_{n \rightarrow \infty} m_r(x_n, x_n) = 0$.

Now, we show that $T_n(x)$ is m_r -Cauchy sequence. For this, we have to show that $\lim_{n, m \rightarrow \infty} m_r(x_m, x_n) = 0$. Let on contrary that $\lim_{n, m \rightarrow \infty} m_r(x_m, x_n) \neq 0$. So, for some $\epsilon > 0$, we have $\limsup_{m, n \rightarrow \infty} m_r(x_m, x_n) > 2\epsilon$. Also, by hypothesis, $\exists \delta > 0$ such that

$$\epsilon \leq m_r(x, y) < \epsilon + \delta \Rightarrow m_r(Tx, Ty) < \epsilon$$

which remains true with δ replaced by $\delta' = \min\{\delta, \epsilon\}$. By employing 3.2, we have

$$\exists N > 0, \quad \forall n \left(n > N \Rightarrow m_r(x_n, x_{n+1}) < \frac{\delta'}{3} \right)$$

and for $m, n > N$, $m_r(x_m, x_n) > 2\epsilon \Rightarrow m_r(x_n, x_{n+1}) < \epsilon$ also $\epsilon + \delta' < 2\epsilon < m_r(x_m, x_n)$, that there exist i with $m < i < n$ with

$$\epsilon + \frac{2\delta'}{3} < m_r(x_m, x_i) - m_{r_{x_m, x_i}} < \epsilon + \delta'. \quad (3.4)$$

However, for all $m, i \in \mathbb{N}$, we obtain

$$\begin{aligned} (m_r(x_m, x_i) - m_{r_{x_m, x_i}}) &\leq (m_r(x_m, x_{m+1}) - m_{r_{x_m, x_{m+1}}}) + (m_r(x_{m+1}, x_{i+1}) \\ &\quad - m_{r_{x_{m+1}, x_{i+1}}}) + (m_r(x_{i+1}, x_i) - m_{r_{x_{i+1}, x_i}}) \\ &\leq \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} \end{aligned}$$

which contradicts (3.4). Thus, the sequence $\{x_n\}$ is a m_r -Cauchy sequence (by 3.4 and $\lim_{m, n \rightarrow \infty} m_r(x_m, x_n) = 0$).

Since by m_r -completeness of X , there exists $x^* \in X$ such that $x_n \rightarrow x^*$, i.e.,

$$\lim_{n \rightarrow \infty} (m_r(x_n, x^*) - m_{r_{x_n, x^*}}) = 0.$$

Since $m_{r_{x_n, x_n}} \rightarrow 0$, hence $m_{r_{x_n, x^*}} \rightarrow 0$ and $m_r(x_n, x^*) \rightarrow 0$. Thus, by hypothesis

$$m_r(Tx_n, Tx^*) \leq m_r(x_n, x^*) \rightarrow 0.$$

Hence, by definition of rectangular M -metric space, we have

$$m_{r_{Tx_n, Tx^*}} \leq m_r(Tx_n, Tx^*).$$

So, $Tx_n \rightarrow Tx^*$. Equation 3.2 implies that $m_r(x_n, Tx_n) \rightarrow 0$. Since $m_{r_{x_n, Tx_n}} \rightarrow 0$, we obtain

$$m_r(x^*, Tx^*) = m_{r_{x^*, Tx^*}}.$$

On other hand, $Tx_{n-1} = x_n \rightarrow x^*$ and also $x_{n+1} = Tx_n \rightarrow Tx^*$, we have

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} (m_r(x_n, Tx_n) - m_{r_{x_n, Tx_n}}) \\ &= \lim_{n \rightarrow \infty} (m_r(x_n, x_{n-1}) - m_{r_{x_n, Tx_n}}) \\ &= (m_r(x^*, x^*) - m_{r_{x^*, Tx^*}}) \end{aligned}$$

$$(m_r(Tx^*, Tx^*) - m_{r_{x^*, Tx^*}}).$$

Thus, $m_r(x^*, x^*) = m_{r_{x^*, Tx^*}} = m_r(Tx^*, Tx^*)$ and since

$$m_r(x^*, Tx^*) = m_{r_{x^*, Tx^*}} = m_r(Tx^*, Tx^*) = m_r(x^*, x^*).$$

Hence, $x^* = Tx^*$, that is, T has a fixed point in X . Finally, we show the uniqueness of a fixed point of T . Assume that T has two distinct fixed points $x^*, x^{**} \in X$ such that $Tx^* = x^*$ and $Tx^{**} = x^{**}$. Then, by the definition of Meir-Keeler contraction if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq m_r(x^*, x^{**}) < \epsilon + \delta \Rightarrow m_r(Tx^*, Tx^{**}) = m_r(x^*, x^{**}) < \epsilon,$$

which is a contradiction. Hence, $x^* = x^{**}$. This concludes the proof. $\square$

$$\text{Put } C_r(x, y) = m_r(x, y) + \frac{(1+m_r(x, Tx))m_r(y, Ty)}{1+m_r(x, y)} + \frac{m_r(x, Tx)m_r(y, Ty)}{m_r(x, y)}.$$

Theorem 3.2. Let (X, m_r) be a complete rectangular M -metric space and let T be a continuous mapping from X into itself satisfying the following condition:

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in X \quad \epsilon \leq KC_r(x, y) < \epsilon + \delta \Rightarrow m_r(Tx, Ty) < \epsilon, \quad (3.5)$$

for some $K \in]0, \frac{1}{3}[$. Then, T has a unique fixed point $u \in X$. Moreover, for all $x \in X$, the sequence $\{T_n(x)\}$ converges to u .

Proof. We observe that (1) trivially implies that T is strict contraction. i.e,

$$x \neq y \Rightarrow m_r(Tx, Ty) < KC_r(x, y). \quad (3.6)$$

Let $x_0 \in X$ and $x_n = Tx_{n-1}$. So, we have

$$\begin{aligned} C_r(x_{n-1}, x_n) &= m_r(x_{n-1}, x_n) + \frac{(1 + m_r(x_{n-1}, x_n))m_r(x_n, x_{n+1})}{1 + m_r(x_{n-1}, x_n)} + \frac{(m_r(x_{n-1}, x_n)m_r(x_n, x_{n+1}))}{m_r(x_{n-1}, x_n)} \\ &\leq K(m_r(x_{n-1}, x_n) + 2m_r(x_n, x_{n+1})) \\ m_r(x_n, x_{n+1}) &= m_r(Tx_{n-1}, Tx_n) \\ &\leq KC_r(x_{n-1}, x_n) \\ &\leq K(m_r(x_{n-1}, x_n) + 2m_r(x_n, x_{n+1})) \end{aligned}$$

Therefore,

$$m_r(x_n, x_{n+1}) \leq m_r(x_{n-1}, x_n)$$

where $r = \frac{K}{1-K} < 1$. Now, by Lemma (2.5), $\{x_n\}$ is a m_r -Cauchy sequence and by completeness of X , $Tx_{n-1} = x_n \rightarrow x^*$ in m_r for some $x^* \in X$. Since T is a continuous mapping, so $x_n = Tx_{n-1} \rightarrow Tx^*$ in m_r . Now, by Lemma (2.5), we find

$$\begin{aligned} m_r(x^*, Tx^*) &= m_{r_{x^*, Tx^*}} \\ 0 &= \lim_{n \rightarrow \infty} (m_r(x_n, Tx_n) - m_{r_{x_n, Tx_n}}) \\ &= m_r(x^*, x^*) - m_{r_{x^*, Tx^*}} \\ &= m_r(Tx^*, Tx^*) - m_{r_{x^*, Tx^*}}. \end{aligned}$$

By Lemma (2.1) and

$$m_r(x^*, Tx^*) = m_{r_{x^*, Tx^*}} = m_r(Tx^*, Tx^*) = m_r(x^*, x^*)$$

So $x^* = Tx^*$. Hence, by contraction 3.6 the uniqueness part is clear. $\square$

Corollary 3.1 [13] Let (X, d) be a complete metric space and let T be a continuous mapping from X into itself satisfying the following condition:

$$\forall x, y \in X, x \neq y \quad d(Tx, Ty) \leq KC(x, y),$$

for some $K \in (0, \frac{1}{3})$. Then, T has a unique fixed point $u \in X$. Moreover, for all $x \in X$, the sequence $\{T_n(x)\}$ converges to u .

4 Applications

In this section, take an idea of Samet et al. [40], we shall state an integral version of the Gupta-Saxena result.

Theorem 4.1. Let (X, m_r) be an rectangular M -metric space and T be a self mapping defined on X . Assume that there exists a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following:

- (i) $\varphi(0) = 0$ and $t > 0 \Rightarrow \varphi(t) > 0$;
- (ii) φ is nondecreasing and right continuous;
- (iii) for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \varphi(KC_r(x, y)) < \epsilon + \delta \Rightarrow \varphi(m_r(Tx, Ty)) < \epsilon, \quad (4.1)$$

for some $K \in]0, \frac{1}{3}[$ and for all $x, y \in X$ with $x \neq y$.
Then 3.5 is satisfied.

Proof. Fix $\epsilon > 0$, so $\varphi(\epsilon) > 0$. Hence by 4.1 there exists $\delta_1 > 0$ such that

$$\forall x, y \in X, x \neq y, \quad \varphi(\epsilon) \leq \varphi(KC_r(x, y)) < \varphi(\epsilon) + \delta_1 \Rightarrow \varphi(m_r(Tx, Ty)) < \varphi(\epsilon). \quad (4.2)$$

According to the right continuity of φ

$$\exists \delta > 0 \quad \varphi(\epsilon + \delta_1) < \varphi(\epsilon) + \delta.$$

Now, for $x, y \in X$ with $x \neq y$ and fixed

$$\epsilon \leq KC_r(x, y) < \epsilon + \delta. \quad (4.3)$$

Since φ is a nondecreasing mapping, we have

$$\varphi(\epsilon) \leq \varphi(KC_r(x, y)) < \varphi(\epsilon + \delta_1) < \varphi(\epsilon) + \delta.$$

So, we get

$$\varphi(m_r(Tx, Ty)) < \varphi(\epsilon) \Rightarrow \varphi(m_r(Tx, Ty)) < \epsilon$$

.

□

Corollary 4.1 Let (X, m_r) be a rectangular M -metric space and let T be a self-mapping defined on X . Let $h : [0, \infty) \rightarrow [0, \infty)$ be a locally integrable function such that

- (1) $t > 0 \Rightarrow \int_0^t h(s)ds > 0$;

(2) for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{1}{K}\epsilon \leq \int_0^{C_r(x,y)} h(s)ds < \frac{1}{K}\epsilon + \delta \Rightarrow \int_0^{\frac{1}{K}m_r(Tx,Ty)} h(s)ds < \frac{1}{K}\epsilon, \quad (4.4)$$

for some $0 < K < \frac{1}{3}$ and for all $x, y \in X$ with $x \neq y$. Then, 3.5 is satisfied.

5 Conclusion

As the rectangular m -metric is relatively new addition to the existing literature, therefore in this article, we established Meir-keeler contraction in rectangular M -metric space. As an application we derived some fixed points of mappings of integral type.

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