Research Article

Prashant Patel* and Rahul Shukla

Mann-Dotson's algorithm for a countable family of non-self Lipschitz mappings in hyperbolic metric space

https://doi.org/10.1515/taa-2022-0134
received September 11, 2022; accepted March 23, 2023

Abstract: The aim of this article is to present some $\Delta$-convergence and strong convergence results for a countable family of non-self mappings. More precisely, we employ Mann-Dotson’s algorithm to approximate, common fixed points of a countable family of non-self $L_n$-Lipschitz mappings in hyperbolic metric spaces.

Keywords: Lipschitz mapping, inward condition, hyperbolic space

MSC 2020: Primary 47H10, 47H09

1 Introduction

The geometric properties of a space are very effective and important in fixed point theory to produce useful results. In particular, geometric properties of a space play an important role in metric fixed point theory. Giving a convex structure to a Banach space is easier since a Banach space is a vector space. Because of its convex properties, the geometry of Banach spaces has undergone many studies. However, gaining convex structures in metric spaces is more difficult. To overcome this problem, Takahashi [38] described convex structure in metric spaces. After that, many researchers obtained fixed point results in convex metric spaces (see [9,10] and the references therein). Because many real-world problems can be expressed in nonlinear form, studying these problems in nonlinear structures rather than linear structures such as Banach spaces will be a more realistic approach. At this stage, metric fixed point theory can be studied using a good mathematical framework called hyperbolic space because of its rich geometrical characteristics and nonlinear structure.

Finding fixed points of nonexpansive mappings and $L$-Lipschitz mappings by Krasnosel’skii-Mann algorithm [1,2,19,22] has been extensively studied in the last few decades for self-mapping (see also [7,32]). If the $L$-Lipschitz mapping is non-self, then most of the Krasnosel’skii-Mann type algorithms are based on the nearest point projection technique. But in many applications, calculating the nearest point projection is not easy, and it may require an approximation algorithm by itself, even in the case of Hilbert spaces [8,23]. To overcome this problems Colao and Marino [8] used inwardness condition on non-self mapping, introduced a new line of research, proved some fixed point results for non-self nonexpansive mapping using the Mann-Dotson algorithm, and established weak and strong convergence results in real Hilbert space. Tufa and Zegeye [40] extended the results of Colao and Marino [8] to multivalued mappings

* Corresponding author: Prashant Patel, Department of Mathematics and Applied Mathematics, University of Johannesburg, Kingsway Campus, Auckland Park 2006, South Africa, e-mail: prashant.patel9999@gmail.com
Rahul Shukla: Department of Mathematical Sciences and Computing, Walter Sisulu University, Mthatha 5117, South Africa, e-mail: rshukla@wsu.ac.za

Open Access. © 2023 the author(s), published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 International License.
in CAT(0) spaces. Recently, Suanoom and Klin-eam [37] presented some fixed point results for generalized nonexpansive mappings in hyperbolic spaces.

Pant and Shukla [31] used the Krasnosel’skiĭ-Mann algorithm to approximate common fixed points of a countable family of Lipschitzian non-self mappings in Banach spaces. Motivated by the above work, in this article we extend results from Banach space to more general space, i.e. hyperbolic space using inwardness condition, and we prove some Δ-convergence and strong convergence results using Mann-Dotson’s algorithm for a countable family of non-self \( L_n \)-Lipschitz mappings. We ensure that Mann-Dotson’s algorithm converges to common fixed points under different conditions on a countable family of non-self mappings.

2 Preliminaries

Now we present some basic definitions.

**Definition 2.1.** [18] Suppose \((\Sigma, \varrho)\) be a given metric space and \(Y : \Sigma \times \Sigma \times [0, 1] \rightarrow \Sigma\) is a function, then a triplet \((\Sigma, d, Y)\) is said to be hyperbolic space if \(Y\) satisfies the following properties:

(H1) \(\varrho(\mu, \Phi(\zeta, \nu, \Lambda)) \leq (1 - \Lambda)\varrho(\mu, \zeta) + \Lambda \varrho(\mu, \nu);\)

(H2) \(\varrho(\Phi(\zeta, \nu, \Lambda), \Phi(\zeta, \nu, \Lambda)) = |\Lambda - \xi|\varrho(\zeta, \nu);\)

(H3) \(\varrho(\zeta, \nu, \Lambda) = \varrho(\nu, \zeta, 1 - \Lambda);\)

(H4) \(\varrho(\Phi(\zeta, \mu, \Lambda), \Phi(\nu, \delta, \Lambda)) \leq (1 - \Lambda)\varrho(\zeta, \nu) + \Lambda \varrho(\mu, \delta),\)

\(\forall \zeta, \nu, \mu, \delta \in \Sigma\) and \(\Lambda, \zeta \in [0, 1].\)

The metric segment with the endpoints \(\zeta\) and \(\nu\) is given by the set

\[\text{seg}[\zeta, \nu] = \{Y(\zeta, \nu, \Lambda) : \Lambda \in [0, 1]\}.\]

**Remark 2.2.**

(1) The hyperbolic space \((\Sigma, \varrho, Y)\) is called as a convex metric space in the sense of Takahashi [38] if only the condition \((H1)\) is satisfied.

(2) Itoh [14] defined hyperbolic metric spaces using condition \((H4)\) as condition III and further used in [35] (with some restriction on \(\Lambda, \Lambda = 1/2\)).

(3) Goebel and Kirk [11] defined a hyperbolic space if the space \((\Sigma, \varrho, Y)\) satisfies conditions \((H1)-(H3)\).

(4) Using condition \((H3)\), we can ensure that the segment \(\text{seg}[\zeta, \nu]\) defined above is isometric to the real line segment \([0, \varrho(\zeta, \nu)]\).

Now we use some standard symbols and write \(Y(\zeta, \nu, \Lambda) = (1 - \Lambda)\zeta \oplus \Lambda \nu\). We will say that the subset \(E\) of \(\Sigma\) is convex if \((1 - \Lambda)\zeta \oplus \Lambda \nu \in E \forall \zeta, \nu \in E, \Lambda \in [0, 1]\). Hyperbolic metric spaces include all normed linear spaces and the Hilbert ball equipped with the hyperbolic metric [12].

Suppose \((\Sigma, \varrho, Y)\) be a given hyperbolic metric space, \(E\) a subset of \(\Sigma\). A non-self mapping \(T : E \rightarrow \Sigma\) is said to be Lipschitz if \(\forall \zeta, \nu \in E\), there exists a real constant \(L \geq 0\) such that \(\varrho(T(\zeta), T(\nu)) \leq L\varrho(\zeta, \nu)\).

If Lipschitz constant \(L = 1\), then the mapping \(T\) is said to be a non-self nonexpansive mapping. A point \(p \in E\) is said to be a fixed point of the mapping \(T\) if \(T(p) = p\). The set of all fixed points for mapping \(T\) is denoted by \(F(T)\).

**Definition 2.3.** [13,15]. Suppose \((\Sigma, \varrho, Y)\) be a given hyperbolic metric space, then for each \(b \in \Sigma, \epsilon \in (0, 2]\), and \(t > 0\) set

\[\delta(t, \epsilon) = \inf \left\{1 - \frac{1}{t} \varrho \left(\frac{\epsilon}{2} \oplus \frac{\epsilon}{2} \nu, b\right) : \varrho(\zeta, b) \leq t, \varrho(\nu, b) \leq t, \varrho(\zeta, \nu) \geq t\epsilon \right\}.\]
We say that the given space $\Sigma$ is uniformly convex if $\delta(t, \varepsilon) > 0$, $\forall \varepsilon \in (0, 2]$, and $t > 0$.

A mapping $\eta : (0, \infty) \times (0, 2) \to (0, 1]$ is said to be a modulus of uniform convexity if it proves a such $\delta = \eta(t, \varepsilon)$ for any given $t > 0$ and $\varepsilon \in (0, 2]$ [20].

Suppose $(\Sigma, \varrho)$ be a given hyperbolic metric space, and $\emptyset \neq \mathcal{E} \subset \Sigma$. Suppose $\{\zeta_n\}$ be a given bounded sequence in $\Sigma$, then, for any $\zeta \in \Sigma$, define

(i) the asymptotic radius of the sequence $\{\zeta_n\}$ at point $\zeta$ by $r(\{\zeta_n\}, \zeta) = \limsup_{n \to \infty} \varrho(\zeta_n, \zeta)$;

(ii) the asymptotic radius of the sequence $\{\zeta_n\}$ relative to the subset $\mathcal{E}$ by

$$r(\{\zeta_n\}, \mathcal{E}) = \inf \{r(\{\zeta_n\}, \zeta) : \zeta \in \mathcal{E}\};$$

(iii) the asymptotic center of the sequence $\{\zeta_n\}$ relative to the subset $\mathcal{E}$ as

$$A(\{\zeta_n\}, \mathcal{E}) = \{\zeta \in \mathcal{E} : r(\{\zeta_n\}, \zeta) = r(\{\zeta_n\}, \mathcal{E})\}.$$

The idea of $\Delta$-convergence in metric spaces was first stated by Lim [21] in 1976. Using Lim’s [21] idea in CAT(0) spaces, Kirk and Panyanak [16] demonstrated that numerous Banach spaces theorems involving weak convergence have an exact analogue in this context.

**Definition 2.4.** [16] Suppose $\{\zeta_n\}$ be a given bounded sequence in $\Sigma$, then the sequence $\{\zeta_n\}$ is said to be $\Delta$-converge at point $\zeta \in \Sigma$ if $\zeta$ is the unique asymptotic center of each subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$.

**Definition 2.5.** [5] Suppose $(\Sigma, \varrho)$ be a given hyperbolic metric space, and $\mathcal{E}$ a nonempty subset of $\Sigma$. Type function is a function $\tau : \mathcal{E} \to (0, \infty)$ if $\exists$, a bounded sequence $\{\zeta_n\} \in \Sigma$ in such a way that

$$r(\zeta) = \limsup_{n \to \infty} \varrho(\zeta_n, \zeta) \quad \forall \zeta \in \mathcal{K}.$$

It can be noted that each bounded sequence has a unique type function.

We are now revisiting the concept of $\Delta$-convergence in the context of hyperbolic metric spaces.

**Definition 2.6.** Suppose $\{\zeta_n\}$ be a given bounded sequence in $\Sigma$, then the sequence $\{\zeta_n\}$ is said to be $\Delta$-converge at point $\zeta \in \Sigma$ if $\zeta$ is unique asymptotic center of each subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$.

**Remark 2.7.** [21] **Relation between $\Delta$-convergence and strong convergence:** Let $(\Sigma, \varrho)$ be a metric space. A sequence $\{\zeta_n\}$ in $\Sigma$ is said to $\Delta$-converge to a point $\zeta \in \Sigma$, written as $\zeta_n \to^\Delta \zeta$ if for every subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$

$$\limsup_{i} \varrho(\zeta_{n_k}, \zeta) \leq \limsup_{i} \varrho(\zeta_{n_k}, \nu)$$

for every $\nu \in \Sigma$. It means that $\zeta$ is an asymptotic center of every subsequence of $\{\zeta_n\}$.

A sequence $\{\zeta_n\}$ is said to converge strongly to $\zeta \in \Sigma$, if

$$\lim_{i} \varrho(\zeta_{n_k}, \zeta) \leq \liminf_{i} \varrho(\zeta_{n_k}, \nu)$$

for every $\nu \in \Sigma$. This is equivalent to saying that all subsequences of $\{\zeta_n\}$ have a common asymptotic center (=$\zeta$) and asymptotic radius (=$\lim d(\zeta, \cdot)$). In general, $\zeta$ is not unique. The strong convergence implies $\Delta$-convergence.

**Lemma 2.8.** [30,39]. Let us assume that $\{\omega_n\}$, $\{\Omega_n\}$, and $\{x_n\}$ be any sequences of positive real numbers in such a way that

$$\omega_{n+1} \leq (1 + \Omega_n)\omega_n + x_n \quad \forall n \in \mathbb{N}.$$ 

If $\sum_{n=1}^{\infty} \Omega_n < \infty$ and $\sum_{n=1}^{\infty} x_n < \infty$, then the limit $\lim_{n \to \infty} \omega_n$ exists.
Lemma 2.9. [16] In a given complete hyperbolic space, each bounded sequence has a $\Delta$-convergent subsequence.

Lemma 2.10. [20] Suppose $(\Sigma, \varrho, \mathcal{Y})$ be a given uniformly convex complete hyperbolic space (UCCHS) having modulus of convexity $\eta$. Suppose $\eta$ increases along with $t$ (for fixed $\varepsilon$) and let $\{\beta_n\}$ is a sequence in $[a, b]$ for any $a, b \in (0, 1)$ and $\{\zeta_n, \{v_n\} \in \Sigma$ such that $\forall \zeta \in \Sigma \limsup_n \varrho(\zeta_n, \zeta) \leq t$, $\limsup_n \varrho(v_n, \zeta) \leq t$, and $\lim_n \varrho((1 - \beta_n)\zeta_n + \beta_n v_n, \zeta) = t$ for any $t \geq 0$. Then,

$$\lim_{n \to \infty} \varrho(\zeta_n, v_n) = 0.$$ 

Definition 2.11. [7] A nonempty subset $\mathcal{E}$ of a given hyperbolic space $\Sigma$ is said to be strictly convex if it is convex and satisfies the property

$$\delta \zeta \oplus (1 - \delta)\nu \in \text{interior}(\mathcal{E})$$

$\forall \delta \in (0, 1)$, and $\zeta, \nu \in \partial \mathcal{E}$. Otherwise, there are no segments that lie within the boundary $\partial \mathcal{E}$.

Definition 2.12. [28] A given Banach space $\Sigma$ is said to satisfy the Opial condition if, for any given sequence $\{\zeta_n\}$ that converges weakly and has the weak limit $\zeta \in \Sigma$, it satisfies

$$\liminf_{n \to \infty} \varrho(\zeta_n, \zeta) < \liminf_{n \to \infty} \varrho(\zeta_n, \nu)$$

$\forall \nu \in \Sigma$ with $\zeta \neq \nu$.

All the $\ell^p(1 \leq p < \infty)$ spaces, all Hilbert spaces, and all finite dimensional Banach spaces satisfy Opial condition. But $L_p$ ($0 < p < \infty$, $p \neq 2$) spaces do not satisfy Opial condition [7,29].

Lemma 2.13. (Demiclosedness principle) [6] Suppose $\Sigma$ be a given uniformly convex Banach space, $\mathcal{E}$ be a convex closed and nonempty subset of $\Sigma$ and $T : \mathcal{E} \to \Sigma$ be a mapping with a fixed point. Suppose $\{\zeta_n\}$ is a given sequence in $\Sigma$ in such a way that $\{\zeta_n\}$ converges weakly to $\zeta$, $\lim_{n \to \infty} \varrho(\zeta_n, T(\zeta_n)) = 0$. Then $T(\zeta) = \zeta$. That is, $I - T$ is demiclosed at zero.

Definition 2.14. Suppose $\mathcal{E} \neq \emptyset$ be a subset of a hyperbolic metric space $\Sigma$. Suppose $\{T_n\}$ be a family of mappings from $\mathcal{E}$ into $\Sigma$ with $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. The family of mappings $\{T_n\}$ is said to be uniformly $\Delta$-closed if, for any bounded sequence, $\{\zeta_n\} \in \mathcal{E}$ such that

$$\lim_{n \to \infty} \varrho(\zeta_n, T_n(\zeta_n)) = 0 \text{ implies } \Delta \text{-limit points of } \{\zeta_n\} \subset \bigcap_{n=1}^{\infty} F(T_n).$$

The following definitions are useful in dealing with countable family of mappings. Suppose $\{T_n\}$ and $\mathcal{T}$ are two families of non-self mappings from $\mathcal{E}$ into $\Sigma$ with $\mathcal{T} \neq F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n)$, where $F(T_n)$ denotes the set of all fixed points of mappings $T_n$ and $F(\mathcal{T})$ is the set of all common fixed points of all mappings in $\mathcal{T}$.

(i) The family of mappings $\{T_n\}$ satisfies Aoyama Kimura Takahashi Toyoda (AKTT)-condition (I) if, for every bounded subset $\mathcal{B}$ of $\mathcal{E}$, $\sum_{n=1}^{\infty} \sup \{\varrho(T_n, \zeta), T_n(\zeta) : \zeta \in \mathcal{B}\} < \infty$ [3].

(ii) The family of mappings $\{T_n\}$ satisfies AKTT-condition (II) if for every bounded subset $\mathcal{B}$ of $\mathcal{E}$ and every increasing sequence $\{n_j\}$ of natural numbers $\mathcal{N}$, a mapping $T : \mathcal{E} \to \Sigma$ along with $I - T$ is demiclosed at 0 and a subsequence $\{n_j\}$ of $\{n_k\}$ in such a way that

$$\limsup_{j \to \infty} \{\varrho(T_n(\zeta), T(\zeta)) : \zeta \in \mathcal{B}\} = 0, \quad \bigcap_{n=1}^{\infty} F(T_n) = F(T) [4].$$

(iii) The family of mappings $\{T_n\}$ satisfies Nakajo Shimoji Takahashi (NST)-condition (I) along with $\mathcal{T}$ if, for every given bounded sequence $\{\zeta_n\}$ in $\mathcal{E}$,

$$\lim_{n \to \infty} \varrho(\zeta_n, T_n(\zeta_n)) = 0 \Rightarrow \lim_{n \to \infty} \varrho(\zeta_n, T_n(\zeta_n)) = 0$$

for each $T \in \mathcal{T}$ [25].
(iv) The family of mappings \( \{T_n\} \) is said to satisfy NST\(^{-}\)-condition with \( \mathcal{T} \) if, for every bounded sequence \( \{\zeta_n\} \) in \( \mathcal{E} \),

\[
\lim_{n \to \infty} g(\zeta_n, T_n(\zeta_n)) = 0 \quad \text{and} \quad \lim_{n \to \infty} g(\zeta_n, \zeta_{n+1}) = 0,
\]

which implies \( \lim_{n \to \infty} g(\zeta_n, T(\zeta_n)) = 0 \ \forall T \in \mathcal{T} \) [26].

Motivated by the above conditions, we consider a new type of following condition:

**Definition 2.15.** (1) The family of mappings \( \{T_n\} \) satisfies NST-condition (III) if for every bounded sequence \( \{\zeta_n\} \) in \( \mathcal{E} \),

\[
\lim_{n \to \infty} g(\zeta_n, T_n(\zeta_n)) = 0 \implies \text{\( \Lambda \)-limit points of} \ \{\zeta_n\} \subset \cap_{n=1}^{\infty} F(T_n).
\]

In the setting of Banach space, NST-condition (III) is equivalent to NST-condition [24].

**Example 2.16.** Let \( \Sigma = \mathbb{R} \) with the standard norm and \( \mathcal{E} = [0, \infty) \). Define a family of mappings \( T_n : \mathcal{E} \to \mathcal{E} \) by \( T_n(\zeta) = \left( 1 - \frac{1}{n} \right) \zeta \) for all \( \zeta \in \mathcal{E}, \ n \in \mathbb{N} \). Here, \( \cap_{n=1}^{\infty} F(T_n) = \{0\} \neq \emptyset \). Let the sequence \( \zeta_n = \frac{1}{n} \), and we have

\[
\lim_{n \to \infty} || \zeta_n - T_n(\zeta_n) || = \left\| \zeta_n - \left( 1 - \frac{1}{n} \right) \zeta_n \right\| = \frac{1}{n} || \zeta_n ||
\]

and \( \lim_{n \to \infty} || \zeta_n - T_n(\zeta_n) || = 0 \). Since \( \zeta_n = \frac{1}{n} \) also goes to 0 as \( n \to \infty \) and \( 0 \in \cap_{n=1}^{\infty} F(T_n) \).

Hence, the sequence of mappings \( \{T_n\} \) satisfies the NST-condition III for the sequence \( \zeta_n = \frac{1}{n} \).

**Remark 2.17.**

1. In the setting of Banach space, the NST-condition (III) is same as the NST-condition.
2. The NST-condition (I) with \( \mathcal{T} \) implies to NST\(^{-}\)-condition with \( \mathcal{T} \). However, the opposite is not true [26].
3. If \( \{T_n\} \) satisfies NST-condition (II), then \( \{T_n\} \) satisfies the NST\(^{-}\)-condition with \( \{T_n\} \).
4. It can be seen that if the family of mappings \( \{T_n\} \) is weakly closed, then \( \{T_n\} \) satisfies NST-condition.

In general, the NST-condition and NST\(^{-}\)-condition with \( \mathcal{T} \) are not related. But, under certain conditions, NST\(^{-}\)-condition with \( \mathcal{T} \) implies NST-condition. For instance, if \( \{T_n\} \) satisfies NST\(^{-}\)-condition with \( \{T_n\} \) and each mapping of \( \{T_n\} \) is demiclosed at zero, then \( \{T_n\} \) satisfies NST-condition.

**Lemma 2.18.** [3] Suppose \( \Sigma \) be a given Banach space, \( \mathcal{E} \) a closed, nonempty subset of \( \Sigma \). Suppose the family of mappings \( \{T_n\} : \mathcal{E} \to \Sigma \) satisfies AKTT-condition (I). Then, \( \forall \zeta \in \mathcal{E}, \ \{T_n(\zeta)\} \) converges strongly to some point of \( \Sigma \). Furthermore, suppose \( T : \mathcal{E} \to \Sigma \) be a mapping defined for \( \zeta \in \mathcal{E} \)

\[
T(\zeta) = \lim_{n \to \infty} T_n(\zeta).
\]

Then, \( \lim_{n \to \infty} \sup \{q(T(\zeta), T_n(\zeta)) : \zeta \in \mathcal{B} \} = 0 \) for any bounded subset \( \mathcal{B} \) of \( \mathcal{E} \). In particular, if \( I-T \) is demiclosed at 0, \( \cap_{n=1}^{\infty} F(T_n) = F(T) \neq \emptyset \), then the given family of mappings \( \{T_n\} \) satisfies AKTT-condition (II).

**Definition 2.19.** [17] The mapping \( T : \mathcal{E} \to \Sigma \) is called an inward mapping if \( \forall \zeta \in \mathcal{E} \), we have

\[
T(\zeta) \in I_c(\zeta) = \{ \zeta + c(\nu - \zeta) : c \geq 1, \ \nu \in \mathcal{E} \}.
\]

**Definition 2.20.** [17] The mapping \( T : \mathcal{E} \to \Sigma \) is called a weakly inward if \( \forall \zeta \in \mathcal{E} \), we have

\[
T(\zeta) \in \overline{I_c(\zeta)}, \quad \text{where} \ \overline{I_c(\zeta)} \ \text{is the closure of} \ I_c(\zeta).
\]

For more details and properties of weakly inward mappings, one may refer to [34].
\textbf{Definition 2.21.} [23] Suppose $\Sigma$ be a given hyperbolic space and $\mathcal{E}$ a convex, closed, and nonempty subset of $\Sigma$ and $T : \mathcal{E} \to \Sigma$ a mapping, define a function $h_{\mathcal{E}, T} : \mathcal{E} \to \mathbb{R}$ as
\[ h_{\mathcal{E}, T}(\zeta) = \inf \{ \Gamma \geq 0 : \Gamma \zeta \oplus (1 - \Gamma)T(\zeta) \in \mathcal{E} \}. \quad (2.3) \]

\textbf{Lemma 2.22.} [8,23] Suppose $\mathcal{E}$ be a convex, closed, and nonempty subset of a given complete hyperbolic space $\Sigma$. Suppose $T : \mathcal{E} \to \Sigma$ be a mapping and $h_{\mathcal{E}, T} : \mathcal{E} \to \mathbb{R}$ a function defined in (2.3). Then, the following properties hold:

1. $\forall \zeta \in \mathcal{E}$ and $\forall \Lambda \in [h_{\mathcal{E}, T}(\zeta), 1]$, $\Lambda \zeta \oplus (1 - \Lambda)T(\zeta) \in \mathcal{E}$;
2. $\forall \zeta \in \mathcal{E}$ and $\forall \beta \in [0, h_{\mathcal{E}, T}(\zeta)]$, $\beta \zeta \oplus (1 - \beta)T(\zeta) \notin \mathcal{E}$;
3. $\forall \zeta \in \mathcal{E}$, $T(\zeta) \in \mathcal{E}$ if and only if $h_{\mathcal{E}, T}(\zeta) = 0$;
4. If $T(\zeta) \notin \mathcal{E}$, then $h_{\mathcal{E}, T}(\zeta) \oplus (1 - h_{\mathcal{E}, T}(\zeta))T(\zeta) \in \partial \mathcal{E}$.

\textbf{Lemma 2.23.} Suppose $\mathcal{E}$ be a convex, closed, and nonempty subset of a given complete hyperbolic space $\Sigma$, $T : \mathcal{E} \to \Sigma$ a mapping that is weakly inward. Then, $h_{\mathcal{E}, T}(\zeta) < 1$ for all $\zeta \in \mathcal{E}$.

\textbf{Proof.} Since $T$ is a weakly inward mapping, so $T(\zeta) \in \overline{I_0(\zeta)}$. Then, there exists a sequence $\{v_n\} \subset I_0(\zeta)$ in such a way that $v_n \to T(\zeta)$ as $n \to \infty$. Now the sequence $\{v_n\}$ will be defined as $v_n = \zeta \oplus c_n(v_n - \zeta)$, where $v_n \in \mathcal{E}$ and $c_n \geq 1$ in such a way that $c_n \to c$ as $n \to \infty$. Now
\[ v_n = \frac{1}{c_n}v_n \oplus \left(1 - \frac{1}{c_n}\right)\zeta \to \frac{1}{c}T(\zeta) \oplus \left(1 - \frac{1}{c}\right)\zeta. \]

Since the subset $\mathcal{E}$ is closed
\[ \frac{1}{c}T(\zeta) \oplus \left(1 - \frac{1}{c}\right)\zeta \in \mathcal{E}. \]

Hence,
\[ h_{\mathcal{E}, T}(\zeta) = \inf \{ \Gamma \geq 0 : \Gamma \zeta \oplus (1 - \Gamma)T(\zeta) \in \mathcal{E} \} \leq \left(1 - \frac{1}{c}\right) < 1. \]

\textbf{Definition 2.24.} [33] The mapping $T : \mathcal{E} \to \Sigma$ is said to be semicompact if for each bounded sequence $\{\zeta_n\} \subset \mathcal{E}$ such that $\zeta_n - T(\zeta_n) \to v$ for some $v \in \Sigma$, there exists a convergent subsequence.

\textbf{Definition 2.25.} [27] The mapping $T : \mathcal{E} \to \Sigma$ is said to be closed if $\{\zeta_n\} \subset \mathcal{E}$ satisfying $\zeta_n \to \zeta$ and $T(\zeta_n) \to v$, then $\zeta \in \mathcal{E}$ and $T(\zeta) = v$.

\textbf{Definition 2.26.} [36] The mapping $T : \mathcal{E} \to \Sigma$ with $F(T) \neq \emptyset$ satisfies condition (I) if $\exists$ a nondecreasing function $f : [0, \infty) \to [0, \infty)$ along with $f(r) > 0$, $f(0) = 0 \forall r \in (0, \infty)$ in such a way that $f(g(\zeta, T(\zeta))) \leq g(\zeta, T(\zeta)) \forall \zeta \in \mathcal{E}$, where $g(\zeta, T(\zeta)) = \inf \{g(\zeta, v) : v \in F(T)\}$.

\textbf{Definition 2.27.} [27] Any finite family of mappings $\mathcal{T} : \mathcal{E} \to \Sigma$ along with $\emptyset \neq F(\mathcal{T})$ satisfies condition (II) if $\exists$ a nondecreasing function $f : [0, \infty) \to [0, \infty)$ along with $f(r) > 0$, $f(0) = 0 \forall r \in (0, \infty)$, in such a way that
\[ \max \{g(\zeta, T(\zeta)) : T \in \mathcal{T}\} \geq f(g(\zeta, F(\mathcal{T}))) \quad \forall \zeta \in \mathcal{E}. \]

\section{3 Weak convergence results}

Suppose $\mathcal{E}$ be a convex, closed, and nonempty subset of a given complete hyperbolic metric space $\Sigma$, $\{T_n\}$ be a family of $L_n$-Lipschitz weakly inward mappings from $\mathcal{E}$ into $\Sigma$. Assume that $\zeta_n \in \mathcal{E}$, we can generate a sequence $\{\zeta_n\}$ in $\mathcal{E}$ as follows:
\[ \begin{align*} 
\Theta_1 &= \max \left\{ \frac{1}{2}, h_{E,T}(\zeta) \right\} \\
\zeta_{m+1} &= \Theta_n \zeta_m \oplus (1 - \Theta_n)T_n(\zeta_m) \\
\Theta_{m+1} &= \max \left\{ \Theta_n, h_{E,T_n}(\zeta_m) \right\}. 
\end{align*} \] (3.1)

From Lemma 2.23, for any weakly inward mapping \( T : E \to \Sigma \), we have \( h_{E,T}(\zeta) < 1 \forall \zeta \in E \). Now it can be seen that for each \( n \in \mathbb{N} \), \( \Theta_{n+1} \in [h_{E,T_n}(\zeta_{m+1}), 1] \). Thus, from Lemma 2.22 (21), we obtain
\[ \zeta_{m+1} = \Theta_n \zeta_m \oplus (1 - \Theta_n)T_n(\zeta_m) \in E. \]

Hence, the algorithm (3.1) is well defined.

Now, we present some important lemmas that can be utilized to prove the main convergence results.

**Lemma 3.1.** Suppose \( E \) be a convex, closed, and nonempty subset of a UCCHS \( \Sigma \). Suppose \( \{T_n\} \) be any given family of \( L_\infty \)-Lipschitz weakly inward mappings from \( E \) into \( \Sigma \), satisfying the conditions \( \emptyset \neq \cap \neq \bigcap_{n=1}^{\infty} F(T_n) \) and \( \sum_{n=1}^{\infty} (L_n - 1) < \infty \). Suppose \( \{\zeta_n\} \) be a sequence given by (3.1). Then, for each \( \emptyset \neq \cap \neq \bigcap_{n=1}^{\infty} F(T_n) \), \( \lim_{n \to \infty} q(\zeta_n, p) \) exists.

**Proof.** Suppose \( p \in \bigcap_{n=1}^{\infty} F(T_n) \). From (3.1) \( \forall n \in \mathbb{N} \), we have
\[ q(\zeta_{n+1}, p) = q(\Theta_n \zeta_m \oplus (1 - \Theta_n)T_n(\zeta_m), p) \]
\[ \leq \Theta_n q(\zeta_m, p) + (1 - \Theta_n)q(T_n(\zeta_m) - p) \]
\[ = \Theta_n q(\zeta_m, p) + (1 - \Theta_n)q(T_n(\zeta_m), T_n(p)) \]
\[ \leq \Theta_n q(\zeta_m, p) + (1 - \Theta_n)L_nq(\zeta_m, p) \]
\[ = [1 + (1 - \Theta_n)(L_n - 1)]q(\zeta_m, p) \]
\[ \leq [1 + (L_n - 1)]q(\zeta_n, p). \]

Since \( \sum_{n=1}^{\infty} (L_n - 1) < \infty \), by Lemma 2.8, we obtain the required conclusion. \( \square \)

**Lemma 3.2.** Suppose \( E, \Sigma, \{\zeta_n\} \), and \( \{T_n\} \) be same as in Lemma 3.1 with \( \Theta_n \subseteq [a, b] \subseteq (0, 1) \), \( \emptyset \neq \bigcap_{n=1}^{\infty} F(T_n) \), and \( \sum_{n=1}^{\infty} (L_n - 1) < \infty \). Then, \( \lim_{n \to \infty} q(T_n(\zeta_m), \zeta_n) = 0. \)

**Proof.** Suppose \( p \in \bigcap_{n=1}^{\infty} F(T_n) \), by Lemma 3.1 \( \lim_{n \to \infty} q(\zeta_n, p) \) exists. Then, there exists \( r \geq 0 \) such that
\[ \lim_{n \to \infty} q(\zeta_n, p) = r. \]

Since \( q(T_n(\zeta_m), p) \leq L_nq(\zeta_m, p) \leq r \). Moreover,
\[ \lim_{n \to \infty} q(\Theta_n \zeta_m \oplus T_n(\zeta_m), p) = \lim_{n \to \infty} q(\zeta_n, p) = r. \]

Thus, using Lemma 2.10, we have \( \lim_{n \to \infty} q(T_n(\zeta_m), \zeta_n) = 0 \). \( \square \)

**Lemma 3.3.** Suppose \( E, \Sigma, \{\zeta_n\} \), and \( \{T_n\} \) be same as in Lemma 3.1 with \( \emptyset \neq \bigcap_{n=1}^{\infty} F(T_n) \), \( \sum_{n=1}^{\infty} (L_n - 1) < \infty \), and \( \{\Theta_n\} \subseteq [a, b] \subseteq (0, 1) \). Assume that \( \Xi \) is a family of mappings from \( E \) into \( \Sigma \) such that \( \bigcap_{n=1}^{\infty} F(T_n) = F(\Xi) \). If sequence of mappings \( \{T_n\} \) satisfies the NST*-condition with \( \Xi \), mapping \( T \) is defined by (2.1), then
\[ \lim_{n \to \infty} q(\zeta_n, T(\zeta_n)) = 0 \text{ for all } T \in \Xi. \]

**Proof.** From Lemma 3.2, we have
\[ \lim_{n \to \infty} q(T_n(\zeta_n), \zeta_n) = 0. \]

It follows that
\[
\lim_{n \to \infty} \varrho(\zeta_n, T(\zeta_n) = 0.
\]

Since the sequence of mappings \( \{T_n\} \) satisfies the NST\(^*\)-condition with \( \mathfrak{T} \), we obtain
\[
\lim_{n \to \infty} \varrho(\zeta_n, T(\zeta_n)) = 0
\]
for all \( T \in \mathfrak{T} \).

**Theorem 3.4.** Suppose \( \mathcal{E} \) be a convex, closed, and nonempty subset of a given UCCHS \( \Sigma \), which has Opial condition, and \( \{T_n\} \) be same as in Lemma 3.1. If \( \{T_n\} \) satisfies NST-condition (III), then sequence \( \{\zeta_n\} \) given by (3.1) \( \Delta \)-converges to a point in \( \cap_{n=1}^{\infty} F(T_n) \), provided \( \Theta_n \subseteq [a, b] \subseteq (0, 1) \).

**Proof.** Using Lemma 3.2, we obtain \( \lim_{n \to \infty} \varrho(\zeta_n, T_n(\zeta_n)) = 0 \). By the assumption that \( \{T_n\} \) satisfies the NST-condition (III), we have \( \Delta \)-limit points of \( \{\zeta_n\} \) in \( \cap_{n=1}^{\infty} F(T_n) \). In view of Lemma 3.1, sequence \( \{\zeta_n\} \) is bounded. Then, there is a subsequence \( \{\zeta_{n_k}\} \) of \( \{\zeta_n\} \) such that \( \{\zeta_{n_k}\} \) \( \Delta \)-converges to some \( \bar{p} \in \mathcal{E} \). Thus, \( \bar{p} \in \cap_{n=1}^{\infty} F(T_n) \). Now we prove that every subsequence of \( \{\zeta_n\} \) \( \Delta \)-converges to \( \bar{p} \). Arguing by contradiction, let \( \{\zeta_n\} \) be another subsequence of \( \{\zeta_n\} \) such that \( \{\zeta_{n_k}\} \) \( \Delta \)-converges to \( \bar{q} \) and \( \bar{p} \neq \bar{q} \). Since \( \lim_{n \to \infty} \varrho(\zeta_n, p) \) exists for each \( p \in \cap_{n=1}^{\infty} F(T_n) \), from the Opial condition, we have
\[
\lim_{n \to \infty} \varrho(\zeta_n, \bar{p}) = \lim_{k \to \infty} \varrho(\zeta_{n_k}, \bar{p}) < \lim_{k \to \infty} \varrho(\zeta_{n_k}, \bar{q}) = \lim_{j \to \infty} \varrho(\zeta_{n_j}, \bar{q}) = \lim_{j \to \infty} \varrho(\zeta_{n_j}, \bar{p}) = \lim_{n \to \infty} \varrho(\zeta_n, \bar{p}),
\]
a contradiction and thus \( \bar{p} = \bar{q} \). Hence, the sequence \( \{\zeta_n\} \) is \( \Delta \)-convergent to \( \bar{p} \in \cap_{n=1}^{\infty} F(T_n) \). This completes the proof.

**Theorem 3.5.** Suppose \( \mathcal{E}, \Sigma, \) and \( \{T_n\} \) be same as in Theorem 3.4. Suppose that \( \mathfrak{T} \) is a family of mappings from \( \mathcal{E} \) into \( \Sigma \) in such a way that \( I - T \) is demiclosed at 0 \( \forall T \in \mathfrak{T} \) and \( \cap_{n=1}^{\infty} F(T_n) = F(\mathfrak{T}) \). If the sequence of mappings \( \{T_n\} \) satisfies NST\(^*\)-condition with \( \mathfrak{T} \), then the sequence \( \{\zeta_n\} \) given by (3.1) \( \Delta \)-converges to a point in \( \cap_{n=1}^{\infty} F(T_n) \), provided \( \Theta_n \subseteq [a, b] \subseteq (0, 1) \).

**Proof.** Using Lemma 3.3, we have
\[
\lim_{n \to \infty} \varrho(\zeta_n, T(\zeta_n)) = 0 \quad \text{for all} \quad T \in \mathfrak{T}.
\]
Since \( \{\zeta_n\} \) is a bounded sequence, there exists a subsequence \( \{\zeta_{n_j}\} \) of \( \{\zeta_n\} \) in such a way that \( \zeta_{n_j} \) \( \Delta \)-converges to \( p \in \mathcal{E} \). From (3.3) and demiclosedness of family of mappings \( \mathfrak{T} \), it easily follows that \( p \in F(\mathfrak{T}) \). By using Opial condition as used in the proof of last Theorem 3.4, we can conclude that sequence \( \{\zeta_n\} \) \( \Delta \)-converges to \( p \in \cap_{n=1}^{\infty} F(T_n) \).

**Example 3.6.** Let \( H = \mathbb{R}^2 \) and \( \mathcal{E} = [0, 1] \times [0, 1] \). Define \( T_1, T_2, T_3 : \mathcal{E} \to H \) for all \( \zeta, \nu \in \mathcal{E} \) as
\[
T_1(\zeta, \nu) = (\zeta, 1 - \nu), \quad T_2(\zeta, \nu) = (1 - \zeta, \nu), \quad T_3(\zeta, \nu) = (1 - \zeta, 1 - \nu).
\]
Here \( T_1, T_2, \) and \( T_3 \) are Lipschitz mappings. Let \( \{\zeta_n\} \) be any bounded sequence in \( \mathcal{E} \) such that
\[
\lim_{n \to \infty} d(\zeta_n, T_n(\zeta_n)) = 0, \quad \lim_{n \to \infty} d(\zeta_n, T_n(\zeta_n)) = 0.
\]
We can easily say \( \lim_{n \to \infty} d(\zeta_n, T_n(\zeta_n)) = 0 \) for each \( i = 1, 2, \ldots, \infty \). Consequently,
\[
d(\zeta_n, T_n(\zeta_n)) \leq d(\zeta_n, \zeta_n) + d(\zeta_n, T_n(\zeta_n)) + d(T_n(\zeta_n), T_n(\zeta_n)) \leq 2d(\zeta_n, \zeta_n) + d(T_n(\zeta_n), T_n(\zeta_n)).
\]
Applying \( \lim n \to \infty \), we obtain \( \lim_{n \to \infty} d(\zeta_n, T_n(\zeta_n)) = 0 \) for each \( m = 1, 2, \ldots, \infty \), and hence, \( \{T_n\} \) for \( n = 1, 2, 3 \) satisfies NST\(^*\)-condition with \( \mathfrak{T} = \{T_1, T_2, T_3\} \), and
\[
F(T_1) \cap F(T_2) \cap F(T_3) = \left[ 0, 1 \times \left\{ \frac{1}{2} \right\} \right] \cap \left\{ \frac{1}{2} \times [0, 1] \right\} \cap \left[ \frac{1}{2}, \frac{1}{2} \right] = \left\{ \frac{1}{2}, \frac{1}{2} \right\} = F(\mathfrak{T}) \neq \emptyset.
\]
Let the sequence \( \{\zeta_n\} \) defined as follows:

\[
\zeta_n = \left\{ \left( \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \right) \right\}.
\] (3.4)

Here, all the conditions of the Theorem 3.5 are satisfied, and hence, the sequence \( \{\zeta_n\} \) generated by (3.4) converges to a point in \( \bigcap F(T_n) \) for \( n = 1, 2, 3 \).

### 4 Strong convergence results

**Theorem 4.1.** Suppose \( E, \Sigma, \) and \( \{T_n\} \) be same as in Lemma 3.1. If \( \{T_n\} \) satisfies NST\(^\ast\) -condition with \( \mathcal{S} \), and \( \mathcal{S} \) is a family of closed mappings from \( E \) into \( \Sigma \) with \( \cap_{n=1}^{\infty} F(T_n) = F(\mathcal{S}) \). If there exists a mapping \( \bar{T} \in \mathcal{S} \) such that \( \bar{T} \) is semicompact, then sequence \( \{\zeta_n\} \) given by (3.1) converges strongly to a point in \( \cap_{n=1}^{\infty} F(T_n) \), provided \( \Theta_n \subseteq [a, b] \subset (0, 1) \).

**Proof.** Using Lemma 3.3, we have

\[
\lim_{n \to \infty} g(\zeta_n, T(\zeta_n)) = 0 \quad \forall T \in \mathcal{S}.
\] (4.1)

In particular,

\[
\lim_{n \to \infty} g(\zeta_n, T(\zeta_n)) = 0.
\]

Since mapping \( \bar{T} \in \mathcal{S} \) is semicompact, we can find a subsequence \( \{\zeta_{n_j}\} \) of \( \{\zeta_n\} \) in such a way that \( \zeta_{n_j} \) converges strongly to \( p \in E \) as \( j \to \infty \). From (4.1),

\[
\lim_{j \to \infty} g(\zeta_{n_j}, T(\zeta_{n_j})) = 0 \quad \forall T \in \mathcal{S}.
\]

Using triangle inequality,

\[
g(T(\zeta_{n_j}), p) \leq g(T(\zeta_{n_j}), \zeta_{n_j}) + g(\zeta_{n_j}, p).
\]

Thus, \( g(T(\zeta_{n_j}), p) \to 0 \) as \( j \to \infty \). Since each mapping \( T \in \mathcal{S} \) is closed, it confirms that, \( T(p) = p \forall T \in \mathcal{S} \), hence \( p \in F(\mathcal{S}) \). Using Lemma 3.1 and \( \cap_{n=1}^{\infty} F(T_n) = F(\mathcal{S}) \), it can be easily seen that the sequence \( \{\zeta_{n_j}\} \) converges strongly to \( p \in \cap_{n=1}^{\infty} F(T_n) \).

**Theorem 4.2.** Suppose \( E, \Sigma, \) and \( \{T_n\} \) be same as in Lemma 3.1. If the family of mappings \( \{T_n\} \) satisfies NST\(^\ast\) -condition with \( \mathcal{S} \), and \( \mathcal{S} \) is a family of closed mappings from \( E \) into \( \Sigma \) with \( \cap_{n=1}^{\infty} F(T_n) = F(\mathcal{S}) \). If \( \mathcal{S} \) is finite and satisfies condition (II), then sequence \( \{\zeta_n\} \) given by (3.1) converges strongly to a point in \( \cap_{n=1}^{\infty} F(T_n) \), provided \( \Theta_n \subseteq [a, b] \subset (0, 1) \).

**Proof.** Lemma 3.1 confirms that, for each \( p \in \cap_{n=1}^{\infty} F(T_n) \), \( \lim_{n \to \infty} g(\zeta_n, p) \) exists. Let

\[
M = \sup \{g(\zeta_n, p) : n \in \mathbb{N}\}.
\] (4.2)

Now, we prove that the sequence \( \{\zeta_n\} \) is a Cauchy sequence. From (3.2) and (4.2), we obtain

\[
g(\zeta_{n+1}, p) \leq g(\zeta_n, p) + M(n - 1).
\]

Following this way, \( \forall n, m \in \mathbb{N} \), we can write

\[
g(\zeta_{n+m}, p) \leq g(\zeta_n, p) + M \sum_{i=m}^{n+m-1} (L_i - 1).
\] (4.3)

Using Lemma 3.3, we have

\[
\lim_{n \to \infty} g(\zeta_n, T(\zeta_n)) = 0 \quad \forall T \in \mathcal{S}.
\] (4.4)
Since \( \Sigma \) satisfies condition (II), \( \exists \) a function \( f \) in such a way that
\[
f(g(\zeta_n, F(\Sigma))) \leq \max\{g(\zeta_n, T(\zeta_n)) : T \in \Sigma\}.
\]
From (4.4), we obtain
\[
\lim_{n \to \infty} g(\zeta_n, F(\Sigma)) = 0. \tag{4.5}
\]
Thus, for any given \( \epsilon > 0 \), there exists a natural number \( n_0 \) such that
\[
g(\zeta_{n_0}, F(\Sigma)) < \frac{\epsilon}{4} \quad \text{and} \quad \sum_{i=n_0}^{\infty} \frac{1}{(L_i - 1)} < \frac{\epsilon}{4M}.
\]
Since \( F(\Sigma) \) is closed, from (4.5), \( \exists \) a point \( \zeta^* \in F(\Sigma) \) in such a way that \( g(\zeta_{n_0}, \zeta^*) < \frac{\epsilon}{4} \). From (4.3), we have for all \( n \geq n_0 \) and \( m \geq 1 \),
\[
g(\zeta_{n+m}, \zeta_n) \leq g(\zeta_{n+m}, p) + g(\zeta_n, p)
\leq 2g(\zeta_n, p) + M \sum_{i=n}^{n+m-1} (L_i - 1)
\leq 2g(\zeta_{n_0}, p) + 2M \sum_{i=n_0}^{n-1} (L_i - 1) + M \sum_{i=n}^{n+m-1} (L_i - 1)
\leq 2g(\zeta_{n_0}, p) + 2M \sum_{i=n_0}^{\infty} (L_i - 1) < \frac{2\epsilon}{4} + 2M \frac{\epsilon}{4M} = \epsilon.
\]
Thus, the sequence \( \{\zeta_n\} \) is a Cauchy sequence in \( E \). Since \( \Sigma \) is a complete space and \( E \) is a closed subset of \( \Sigma \), \( E \) is also complete. Therefore, the sequence \( \{\zeta_n\} \) converges to some \( \tilde{p} \in E \). Since the mapping \( T \) is closed, from (4.4), it can be easily followed that \( T(\tilde{p}) = \tilde{p} \forall \in \Sigma \) and \( \tilde{p} \in F(\Sigma) \). Since \( g_{\zeta_{n_0}}^\infty F(T_n) = F(\Sigma) \), sequence \( \{\zeta_n\} \) converges strongly to \( \tilde{p} \in \cap_{n=1}^{\infty} F(T_n) \).

**Theorem 4.3.** Suppose \( E, \Sigma, \) and \( \{T_n\} \) be same as in Lemma 3.1. If \( \{T_n\} \) satisfies AKTT-condition (I), mapping \( T \) is defined by (2.1) and \( g_{\zeta_{n_0}}^\infty F(T_n) = F(\Sigma) \), then sequence \( \{\zeta_n\} \) given by (3.1) converges strongly to a point in \( \cap_{n=1}^{\infty} F(T_n) \), provided \( \sum_{n=1}^{\infty} (1 - \Theta_n) < \infty \).

**Proof.** By Lemma 3.1, \( \forall p \in \cap_{n=1}^{\infty} F(T_n) \), \( \lim_{n \to \infty} g(\zeta_n, p) \) exists. Since
\[
g(\zeta_n, \zeta_{n+1}) = (1 - \Theta_n)g(\zeta_n, T_n(\zeta_n)),
\]
using the boundedness of the sequences \( \{\zeta_n\} \) and \( \{T_n(\zeta)\} \),
\[
\sum_{n=1}^{\infty} g(\zeta_n, \zeta_{n+1}) < \infty.
\]
That is, \( \{\zeta_n\} \) is a strongly Cauchy sequence. Therefore, \( \exists \zeta^* \in E \) such that
\[
\lim_{n \to \infty} \zeta_n = \zeta^*.
\]
Now, it suffices to show that \( \zeta^* \in \cap_{n=1}^{\infty} F(T_n) \).

Since the family \( \{T_n\} \) is weakly inward \( \forall n \in \mathbb{N} \), \( T_n(\zeta) \in T_n(\Sigma) \) \( \forall n \in \mathbb{N} \). Hence,
\[
T(\zeta) = \lim_{n \to \infty} T_n(\zeta) \in T_n(\Sigma),
\]
and \( T \) is weakly inward mapping. Using Lemma 2.23, we know that \( h_{E,T}(\zeta) < 1 \forall \zeta \in E \). Lemma 2.22 (Zl) implies that for all \( \delta \in (h_{E,T}(\zeta^*), 1) \),
\[
\delta \zeta^* \cap (1 - \delta)T(\zeta^*) \in E. \tag{4.6}
\]
On the other hand, \( \sum_{n=1}^{\infty} (1 - \Theta_n) < \infty \) ensures that \( \lim_{n \to \infty} \Theta_n = 1 \), where \( \Theta_n = \max \{ h_n(z_n) \} \). Thus, we can choose a subsequence \( \{ z_n^r \} \) with the property that \( \{ h_T(z_n^r) \} \) is nondecreasing and \( h_T(z_n^r) \to 1 \). In particular, Lemma 2.22 (22) ensures that for any fixed \( \delta < 1 \),

\[
\delta z_n^r \oplus (1 - \delta) T z_n^r (z_n^r) \notin E \quad \text{for sufficiently large } j. \tag{4.7}
\]

Take two positive real numbers \( \delta_1, \delta_2 \in (h_T(z^r), 1) \) with \( \delta_1 \neq \delta_2 \) and set \( \rho_1 = \delta_1 z^r \oplus (1 - \delta_1) T z^r \) and \( \rho_2 = \delta_2 z^r \oplus (1 - \delta_2) T z^r \). Now, for any \( \delta \in [\delta_1, \delta_2] \) by (4.6), we obtain

\[
\rho = \delta z^r \oplus (1 - \delta) T z^r \in E. \tag{4.8}
\]

Now, we prove that \( T_n(z_n^r) \to T(z^r) \) as \( n \to \infty \). For this, take \( C = B_d(z^r) \cap E, \forall r > 0 \), then \( C \) is a bounded subset of \( E \). Using triangle inequality, we have

\[
\rho(T_n(z_n^r), T(z^r)) \leq \rho(T_n(z_n^r), T(z^r)) + \rho(T(z^r), T(z^r)) \\
\leq L \rho(z_n^r, z^r) + \sup \{ \rho(T(z^r), T(z^r)) : \zeta \in C \}.
\]

Since \( z_n^r \to z^r \) as \( n \to \infty \) and from Lemma 2.18, we have \( T_n(z_n^r) \to T(z^r) \) as \( n \to \infty \). By (4.7), \( \delta z_n^r \oplus (1 - \delta) T z_n^r (z_n^r) \notin E \). Since \( z_n \to z^r \) and \( T(z_n^r) \to T(z^r) \) as \( n \to \infty \), we obtain

\[
\lim_{j \to \infty} \delta z_n^r \oplus (1 - \delta) T z_n^r (z_n^r) = \rho, \quad \text{and } \rho \in \partial E.
\]

Since \( \rho \) is any arbitrary point in segment \( [\rho_1, \rho_2] \), the entire segment \( [\rho_1, \rho_2] \subset \partial E \). The strict convexity of \( E \) implies that \( \rho_1 = \rho_2 \), i.e.

\[
\delta z^r \oplus (1 - \delta) T z^r = \delta z^r \oplus (1 - \delta) T z^r,
\]

hence \( T(z^r) = z^r \), so \( z^r \in F(T) \). Since \( \cap_{n=1}^{\infty} F(T_n) = F(T), z^r \in \cap_{n=1}^{\infty} F(T_n) \).

\[
\square
\]

**Corollary 4.4.** Suppose \( E \) and \( \Sigma \) be same as in Lemma 3.1. Suppose \( \{ T_n \} \) be a family of weakly inward nonexpansive mappings from \( E \) into \( \Sigma \) with \( \emptyset \neq \cap_{n=1}^{\infty} F(T_n) \). If sequence of mappings \( \{ T_n \} \) satisfies AKTT-condition (I) and mapping \( T \) defined by (2.1), \( \cap_{n=1}^{\infty} F(T_n) = F(T) \), then sequence \( \{ z_n \} \) given by (3.1) converges strongly to a point in \( \cap_{n=1}^{\infty} F(T_n) \), provided \( \sum_{n=1}^{\infty} (1 - \Theta_n) < \infty \).

## 5 Conclusion

In this article, Mann-Dotson’s algorithm for finding a common fixed point of a countable family of non-self mappings is studied in the setting of hyperbolic metric space. \( \Delta \)-convergence and strong convergence results for approximating a common fixed point of a countable family of non-self \( L_n \)-Lipschitz mappings are established under suitable conditions.

**Acknowledgements:** We are very thankful to the reviewers for their constructive comments and suggestions, which have been useful for the improvement of this article. The first author acknowledges with thanks the support from the University Research Grant from the University of Johannesburg, South Africa.

**Author contributions:** The authors contributed equally to this work.

**Conflict of interest:** The authors declare that they have no conflicts of interests.

## References


