Abstract: We improve the rational contractive condition and prove the common fixed point in the dislocated metric spaces. In addition, the new generalized rational contraction has existence of a common solution for a system of integral equations.

Keywords: fixed point, dislocated metric spaces, common fixed point

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1 Introduction and preliminaries

Fixed point theory plays an important role in the analysis. It extends the research about differential equations, control theory, economics, optimization problem, etc. Over the past few decades, a number of researchers have shown more interest in generalizing and extending the fixed point theorems. Al-Rawashdeh and Ahmad [1] proved common fixed point theorems by setting up new types of contractions. Some unique fixed point results have been proved by Arshad et al. [2] using operator T to satisfy certain rational contractions in partially ordered metric spaces. Common fixed point theorems for a discontinuous, non-compatible pair of self-maps were generalized by Tomar and Sharma [3].

Tomar et al. [4] proved common fixed point theorems without exploiting the notion of continuity or containment of range space of involved maps or completeness of space. The dislocated metric space was introduced by Hitzler and Seda [5], who established a fixed point theorem for incomplete dislocated metric spaces in order to generalize the Banach contraction principle. Beloul and Tomar [6] introduced the subsequential continuous hybrid pairs of maps and also combined this concept with compatibility to establish a coincidence and common fixed point theorem. Many researchers mostly concentrated on proving a common fixed points [7–9] using various mappings such as non-continuous self-maps, single-valued and multi-valued maps, and weakly compatible mappings. The aim of this article is to establish the existence and uniqueness of...
common fixed points [3–5] in dislocated metric spaces. Applications can be found in the common fixed point as well.

**Definition 1.1.** [5] Let the non-void set be $X$. A function $d_c : X \times X \to [0, \infty)$ is a dislocated metric space if $a, b, c \in X$

(i) If $d_c(a, b) = 0$ then $a = b$,

(ii) $d_c(a, b) = d_c(b, a)$,

(iii) $d_c(a, b) \leq d_c(a, c) + d_c(c, b)$. 

Then, $(X, d_c)$ is known as dislocated metric space.

**Example 1.1.** Let $X = R^+ \cup \{0\}$, then $d_c(a, b) = a + b$ defines a dislocated metric space on $X$.

**Definition 1.2.** [5] \{a_n\} is a sequence in a dislocated metric space if for any given $\varepsilon > 0$, there exists $n_0 \in N$ such that for each $n \geq n_0$, we obtain $d_c(a_n, a) < \varepsilon$.

**Definition 1.3.** [5] \{a_n\} is a Cauchy sequence in a dislocated metric space if for given $\varepsilon > 0$, there exists $n_0 \in N$ such that for each $m, n \geq n_0$, we obtain $d_c(a_m, a_n) < \varepsilon$.

**Definition 1.4.** Let $X = R$ be any set and $d_c : X \times X \to [0, \infty)$ is defined by $d_c(a, b) = |a| + |b|$ for each $a, b \in X$.

Note that $d_c$ is a dislocated metric space but not a metric space since $d_c(2, 2) = 4 > 0$.

**Definition 1.5.** We say that the metric space $(X, d_c)$ is a complete dislocated metric space if every Cauchy sequence of points in $X$ converges to a point in $X$.

**Definition 1.6.** Let $(X, d_c)$ be a dislocated metric space. If $T : X \to X$ is a contraction on $X$ then there exists $\lambda \in R$ with $0 \leq \lambda < 1$ such that $d(T(a), T(b)) \leq \lambda d(a, b)$ for each $a, b \in X$ with $a \neq b$. Then, $T$ has a unique fixed point in $X$.

### 2 Main results

The following rational contraction condition proved our common fixed point result.

**Theorem 2.1.** If $(X, d_c)$ is a complete dislocated metric space and the mappings $S, T : X \to X$ satisfy

$$
d_c(Sa, Tb) \leq a_1d_c(a, b) + a_2 \frac{d_c(a, Sa)d_c(b, Tb)}{d_c(a, b)} + a_3 \frac{d_c(b, Sa)d_c(a, Tb)}{d_c(a, b)} + a_4 \frac{d_c(a, Sa)d_c(b, Tb)}{d_c(a, Tb) + d_c(a, b) + d_c(b, Sa)} + a_5 \frac{d_c(b, Tb)d_c(a, Tb)}{d_c(a, Sa) + d_c(b, Tb)}
$$

for each $a, b \in X$, where $a_1, a_2, a_3, a_4, a_5$ are positive reals with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then there exists a unique common fixed point in $X$.

**Proof.** Let $a_0 \in X$. Define $a_1 = Sa_0$ and $a_2 = Ta_1$ such that $d_c(a_1, a_2) = d_c(Sa_0, Ta_1)$. 

Then,
\[
d_c(a_1, a_2) \leq a_1 d_c(a_0, a_1) + a_2 \frac{d_c(a_0, a_1) d_c(a_1, T a_1)}{d_c(a_0, a_1)} + a_3 \frac{d_c(a_1, S a_0) d_c(a_0, T a_1)}{d_c(a_0, a_1)} + a_4 \frac{d_c(a_0, S a_0) d_c(a_1, T a_1)}{d_c(a_0, a_1)} + a_5 \frac{d_c(a_1, T a_1) d_c(a_0, a_1)}{d_c(a_0, a_1)} + a_6 \frac{d_c(a_0, S a_0) + d_c(a_1, T a_1)}{d_c(a_0, a_1)}
\]

\[
= a_1 d_c(a_0, a_1) + a_2 \frac{d_c(a_0, a_1) d_c(a_1, a_2)}{d_c(a_0, a_1)} + a_3 \frac{d_c(a_1, a_2) d_c(a_0, a_2)}{d_c(a_0, a_1)} + a_4 \frac{d_c(a_0, a_1) d_c(a_0, a_2)}{d_c(a_0, a_1)} + a_5 \frac{d_c(a_1, a_2) d_c(a_0, a_2)}{d_c(a_0, a_1)} + a_6 \frac{d_c(a_0, a_1) + d_c(a_0, a_1)}{d_c(a_0, a_1)} + a_5 \frac{d_c(a_1, a_2) + d_c(a_1, a_2)}{d_c(a_0, a_1) + d_c(a_1, a_2)}
\]

Since,
\[
d_c(a_2, a_1) \leq d_c(a_2, a_0) + d_c(a_0, a_1)
\]

and
\[
d_c(a_0, a_2) \leq d_c(a_0, a_1) + d_c(a_1, a_2).
\]

Therefore,
\[
d_c(a_1, a_2) \leq a_1 d_c(a_0, a_1) + a_2 d_c(a_1, a_2) + a_3 d_c(a_0, a_1) + a_5 d_c(a_1, a_2)
\]

\[
d_c(a_1, a_2) \leq a_1 \left( \frac{a_1 + a_4}{1 - (a_2 + a_5)} \right) d_c(a_0, a_1) = \nu d_c(a_0, a_1),
\]

where
\[
\nu = \frac{a_1 + a_4}{1 - (a_2 + a_5)}.
\]

By repeating the same process, we obtain
\[
d_c(a_2, a_1) \leq \nu^2 d_c(a_0, a_1).
\]

Consequently, we obtain that
\[
d_c(a_{2n+1}, a_{2n+2}) \leq \nu d_c(a_{2n}, a_{2n+1}) \leq \nu^2 d_c(a_{2n-1}, a_{2n}) \leq \nu^{2n+1} d_c(a_0, a_1).
\]

For any \( m > n \),
\[
d_c(a_n, a_m) \leq d_c(a_n, a_{n+1}) + d_c(a_{n+1}, a_{n+2}) + \ldots + d_c(a_m, a_m)
\]

\[
\leq (\nu^n + \nu^{n+1} + \ldots + \nu^{m-1}) d_c(a_0, a_1)
\]

\[
\leq \frac{\nu^n}{1 - \nu} d_c(a_0, a_1).
\]
Therefore,
\[ d_c(a_n, a_m) \leq \frac{\sqrt{n}}{1 - \sqrt{\epsilon}} d_c(a_0, a_1) \to 0 \quad \text{as} \ n, m \to \infty. \]

Thus, \( \{a_n\} \) be a Cauchy sequence. Since \( X \) is complete, there exists \( e \in X \) such that \( a_n \to e \). Therefore, \( e = Se \).

Suppose, on the contrary, that \( e = Se,0 \). therefore, \( e = Se \).

As \( n \to \infty \), \( d_c(e, Se) = z > 0 \). which implies that,
\[ d_c(e, Se) \leq d_c(e, a_{2n+2}) + \frac{z}{d_c(e, a_{2n+1})} d_c(a_{2n+1}, a_{2n+2}) + \frac{2}{d_c(e, a_{2n+1})} d_c(e, a_{2n+2}) + \frac{3}{d_c(e, a_{2n+1})} d_c(e, a_{2n+1}) \]

As \( n \to \infty \), \( d_c(e, Se) = 0 \), which is a contradiction, hence \( Se = e \). Similarly, we show that \( e = Te \), and hence its uniqueness.

**Example 2.2.** Consider \( X = [0, 1] \) be a dislocated metric space \( d_c : X \times X \to X \) defined as follows:
\[ d_c(a, b) = \frac{a}{4} + \frac{b}{4}. \]

Let \( S : X \to X \) be defined as follows:
\[ Sa = \frac{4a}{3}, \ a \in X. \]

Let \( T : X \to X \) be defined as follows:
\[ Tb = \frac{2b}{3}, \ b \in X. \]

Now, \( d_c(Sa, Tb) = \frac{a}{3} + \frac{b}{6} \), let \( a = \frac{1}{3} \) and \( b = \frac{1}{2} \), thus the contractive condition of Theorem 2.1 is satisfied, and 0 is a unique common fixed point of \( S \) and \( T \).

**Corollary 2.3.** If \( (X, d_c) \) is a complete dislocated metric space and the mappings \( S, T : X \to X \) satisfy
\[ d_c(Sa, Tb) \leq \alpha d_c(a, b) + \beta (d_c(a, Sa) + d_c(b, Tb)) \]
for each \( a, b \in X \), where \( \alpha, \beta, \) and \( \gamma \) are non-negative reals with \( \alpha + \beta + \gamma < 1 \), then there exists a unique common fixed point in \( X \).
Proof. Put $a_4 = 0 = a_5$ in the Theorem of 2.1, we obtain the result.

**Theorem 2.4.** If $(X, d_{c})$ is a complete dislocated metric space and the mappings $S, T : X \to X$ satisfy
\[
    d_{c}(S(a), T(b)) \leq a_1 d_{c}(a, b) + a_2 \frac{d_{c}(b, T(b))d_{c}(a, S(a))}{1 + d_{c}(a, S(a))}
\]
for each $a, b \in X$, where $a_1$ and $a_2$ are non-negative reals with $a_1 + a_2 < 1$, then there exists a unique common fixed point in $X$.

**Proof.** Let $a_0 \in X$. Define $a_1 = S(a_0)$ and $a_2 = T(a_0)$ such that
\[
    d_{c}(a_1, a_2) = d_{c}(S(a_0), T(a_1)).
\]
Then,
\[
    d_{c}(a_1, a_2) \leq a_1 d_{c}(a_0, a_1) + a_2 \frac{d_{c}(a_0, S(a_0))d_{c}(a_1, T(a_1))}{1 + d_{c}(a_0, S(a_0))}
\]
\[
    = a_1 d_{c}(a_0, a_1) + a_2 \frac{d_{c}(a_0, a_1)d_{c}(a_1, a_2)}{1 + d_{c}(a_0, a_1)}
\]
\[
    \leq a_1 d_{c}(a_0, a_1) + a_2 d_{c}(a_1, a_2),
\]
which implies that,
\[
    d_{c}(a_1, a_2) \leq \frac{a_1}{1 - a_2} d_{c}(a_0, a_1) \leq \nu d_{c}(a_0, a_1),
\]
where $\nu = \frac{a_1}{1 - a_2}$.

Similarly, proceeding like that,
\[
    d_{c}(a_2, a_3) \leq \nu^2 d_{c}(a_0, a_1).
\]
Consequently, we obtain
\[
    d_{c}(a_n, a_{n+1}) \leq \nu d_{c}(a_{n-1}, a_n) \leq \nu^2 d_{c}(a_{n-2}, a_{n-1}) \leq \ldots \leq \nu^n d_{c}(a_0, a_1).
\]
Now, we wish to show that $\{a_n\}$ is a Cauchy sequence. For every $m > n$
\[
    d_{c}(a_n, a_m) \leq \sum_{k=n}^{m-1} d_{c}(a_k, a_{k+1}) \leq (\nu^n + \nu^{n+1} + \ldots + \nu^{m-1})d_{c}(a_0, a_1)
\]
\[
    \leq \frac{\nu^n}{1 - \nu} d_{c}(a_0, a_1)
\]
\[
    \to 0 \quad \text{as } m, n \to \infty.
\]
Thus, $\{a_n\}$ be a Cauchy sequence. Since $X$ is complete, there exists $e \in X$ such that $a_n \to e$. Suppose that $\theta = d_{c}(e, Te)$. Then,
\[
    d_{c}(e, Te) \leq d_{c}(e, a_{2n+2}) + d_{c}(a_{2n+2}, Te)
\]
\[
    = d_{c}(e, a_{2n+2}) + d_{c}(a_{2n+1}, Te)
\]
\[
    \leq d_{c}(e, a_{2n+2}) + a_1 d_{c}(a_{2n+1}, e) + a_2 \frac{d_{c}(e, T(e))d_{c}(a_{2n+1}, S(a_{2n+1}))}{1 + d_{c}(a_{2n+1}, S(a_{2n+1}))}
\]
\[
    = d_{c}(e, a_{2n+2}) + a_1 d_{c}(a_{2n+1}, e) + a_2 \frac{\theta + d_{c}(a_{2n+1}, a_{2n+2})}{1 + d_{c}(a_{2n+1}, a_{2n+2})}.
\]
As $n \to \infty$ and $a_n \to e$, we obtain
\[ (1 - a_2)\theta \leq 0. \]

Consequently,
\[ \theta = d_\alpha(e, Te) = 0. \]

Therefore, \( e = Te \). It follows similarly that \( e = Se \). Suppose \( g \in X \) be another common fixed point of \( S \) and \( T \). Then,
\[ d_\alpha(g, e) = d_\alpha(Sg, Te) \leq a_1d_\alpha(g, e) + \frac{a_2d_\alpha(e, T(e))d_\alpha(g, S(g))}{1 + d_\alpha(g, S(g))} \leq a_1d_\alpha(g, e), \]

which implies that
\[ (1 - a_1)d_\alpha(g, e) \leq 0. \]

Therefore,
\[ d_\alpha(g, e) = 0. \]

Hence \( g = e \). \( \square \)

### 3 Applications

We solve a system of integral equation as an application of Theorem 2.4. Consider the following system of integral equations:
\[ a(t) = \int_j^k k_j(t, r, a(r))dr + g(t) \]
\[ b(t) = \int_j^k k_j(t, r, a(r))dr + h(t), \]
where \( X = C[j, k], t \in [j, k] \subset R \), and \( a, g, h \in X \).

Suppose that \( k_1, k_2 : [j, k] \times [j, k] \times R \to R \) are continuous such that
\[ F_a(t) = \int_j^k k_j(t, r, a(r))dr \quad (1) \]
and
\[ G_a(t) = \int_j^k k_j(t, r, a(r))dr \quad (2) \]
for each \( a \in X \) and \( t \in [j, k] \). Then, the existence of a solution to (3.1) is equivalent to the existence of common fixed point of \( S \) and \( T \).

**Theorem 3.1.** Let \( X = C([j, k], R) \), where \( j > i \geq 0 \) and \( d_\alpha : X \times X \to R \) be defined as follows:
\[ d_\alpha(a, b) = \max_{r \in [j, k]} ||a(t) - b(t)||_\omega \sqrt{1 + j^2 e^{a(t)}}. \]

There exist \( a_1, a_2 \geq 0 \) with \( a_1 + a_2 < 1 \) such that
\[ ||F_a(t) - G_a(t) + g(t) - h(t)||_\omega \sqrt{1 + j^2 e^{a(t)}} \leq a_1f(a, b)(t) + a_2k(a, b)(t), \]
where
\[ J(a, b)(t) = \|a(t) - b(t)\|_{\infty} \sqrt{1 + \frac{t^2}{e^{\cot^{-1}a}}} \]

and

\[ K(a, b)(t) = \frac{\|F_a(t) + g(t) - a(t)\|_{\infty} \|G_b(t) + h(t) - b(t)\|_{\infty} \sqrt{1 + \frac{t^2}{e^{\cot^{-1}a}}}}{1 + \|F_a(t) + g(t) - a(t)\|_{\infty} \sqrt{1 + \frac{t^2}{e^{\cot^{-1}a}}}}. \]

Then, the system of integral equations (1) and (2) have a unique common solution.

**Proof.** One can easily check that \((X, d_c)\) is a dislocated metric space.

Let us define the two functions \(S, T : X \times X \rightarrow X\) by \(S_a = F_a + g\) and \(T_a = G_a + h\).

Then,

\[ d(S(a), T(b)) = \max_{t \in [j, k]} \|F_a(t) - G_b(t) + g(t) - h(t)\|_{\infty} \sqrt{1 + \frac{t^2}{e^{\cot^{-1}a}}}, \]

\[ d(a, S(a)) = \max_{t \in [j, k]} \|F_a(t) + g(t) - a(t)\|_{\infty} \sqrt{1 + \frac{t^2}{e^{\cot^{-1}a}}}, \]

and

\[ d(b, T(b)) = \max_{t \in [j, k]} \|G_b(t) + h(t) - b(t)\|_{\infty} \sqrt{1 + \frac{t^2}{e^{\cot^{-1}a}}}. \]

Now,

\[ d_c(Sa, Tb) = \max_{t \in [j, k]} \|F_a(t) - G_b(t) + g(t) - h(t)\|_{\infty} \sqrt{1 + \frac{t^2}{e^{\cot^{-1}a}}} \]

\[ \leq \max_{t \in [j, k]} \left( \|a(t) - b(t)\|_{\infty} \sqrt{1 + \frac{t^2}{e^{\cot^{-1}a}}} + \|F_a(t) + g(t) - a(t)\|_{\infty} \|G_b(t) + h(t) - b(t)\|_{\infty} \sqrt{1 + \frac{t^2}{e^{\cot^{-1}a}}} \right) \]

\[ = \max_{t \in [j, k]} (a J(a, b)(t) + a K(a, b)(t)) \]

\[ \leq a \max_{t \in [j, k]} J(a, b)(t) + a \max_{t \in [j, k]} K(a, b)(t) \]

\[ = a d_c(a, b) + a K(a, b)(t) \]

Therefore, all the hypotheses of Theorem 2.4 are satisfied. Hence, the system of integral equation (3.1) have a unique common fixed point.

### 4 Conclusion

In this article, we made more generalizations about the rational contraction mapping and proved the common fixed point theorem in dislocated metric spaces. We also proved the existence of a common fixed point for a pair of mappings satisfying new generalizations of rational contraction mappings in that spaces. Examples are also verified. We also showed that the system of integral equations has a unique common fixed point. It is an open problem to prove common fixed point theorems in dislocated b-metric spaces and dislocated quasi b-metric spaces.

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