Complex Quaternion Representation of Proper Lorentz Transformations

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The representation of the proper Lorentz group by nonsingular complex quaternions is, of course, a quite obvious one, since it is well known that spatial rotations may be represented by quaternions, while velocity transformations may be identified with rotations by imaginary angles, which, therefore, may be represented by quaternions with imaginary vector part and real scalar part. Since any proper Lorentz transformation is the product of one velocity transformation with one spatial rotation, the representation of this group by nonsingular complex quaternions is established. We wish to cast this representation in a form useful for applications.

1. Any \( x \in Q_c \) with \( x^* x = S(x^* x) \) is of the form \( x = \kappa (t + i t) \) with \( \kappa \in C, t \in E_3, \kappa \in R \). Furthermore, \( x \rightarrow x' = q x q^* \),

\[
(1)
\]
with \( q q^* = 0 \), is a linear mapping of this subspace on itself, which leaves the scalar product \( \langle x, y \rangle = -S(x^* y) \) invariant. Moreover, if \( \text{Im} (q q^*) = 0 \), \( \arg \kappa \) is also left invariant and (1) represents a Lorentz transformation.

2. Any \( q \in Q_c \) is of the form

\[
q = \lambda q_1 q_2,
\]

with \( \lambda \in C, q_j \in R, -1 < \gamma_j \leq 1, +1 \leq \gamma_j < +\infty \), which may be interpreted as follows.

\[
q = q_1^1 - 1/\gamma_1^1/2 + 1/\gamma_1^1/2, \quad q_1 \in E_3, \quad (q_j, q_j) = 1, -1 < \gamma_j \leq 1 \text{ is a real normalized quaternion, therefore.}
\]

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2 The representation of the proper Lorentz group by normalized complex quaternions, where any transformation is given by the product of one velocity transformation with one rotation.

Thus by (1) and (2) with \( \lambda = 1 \), we have reestablished the representation of the proper Lorentz group by normalized complex quaternions, where any transformation is given by the product of one velocity transformation with one rotation.

3. Eq. (2) may be applied to any product \( q = q_2 q_1 \), with \( -1 < \gamma_3, \gamma_4 < +\infty \), in which case \( \lambda = 1 \). The result is trivial if \( -1 < \gamma_3 \leq 1, +1 \leq \gamma_4 < +\infty \) or vice versa, and almost trivial if \( -1 < \gamma_3, \gamma_4 \leq 1 \), in the latter case the product of two rotations is one rotation times the identity velocity transformation. If \( +1 \leq \gamma_3, \gamma_4 < +\infty \), the product of two velocity transformations is given by the product of one rotation by the rotation vector

\[
\delta_\varphi = \begin{bmatrix} v_2 & v_3 \\ v_3 & v_4 \end{bmatrix}
\]

\[
\cdot \text{arc cos} \left| 1 - \frac{(1 + \gamma_3 (1 + \gamma_4 (1 + \gamma_4 v_3 v_4 - v_3, v_4)))}{(v_3, v_4)} \right|^2 v_3^2 v_4^2
\]

with one velocity transformation by the velocity

\[
v_1 = v_3 + v_4 (\gamma_3, v_3, v_4) (1 + v_3, v_4) \]

which, of course, is just the theorem of velocity addition.

SO(3, R), \( u_j = -i l_j \), e. g. irreducible representations with weight \( l \), the most familiar ones are those with weight \( l = 1 \) (\( l_j = \sigma_j \), Pauli-matrices) and \( l = 1 \) (\( l = i \)). The following notation is used. C: field of complex numbers, \( R \): field of real numbers, \( Q_c \): field of complex quaternions, \( E_3 \): three dimensional euclidean space, \( V(q) \) and \( S(q) \) are the vector part and the scalar part, respectively, of a quaternion \( q = V(q) + S(q) \). \( q^* \) is obtained from \( q \) by complex conjugation of coefficients.

\[
q^* = V(q) + S(q), \quad q^{-1} = q^* q q^*, \quad \bar{q} = (q^*)^{-1} = (q^{-1})^*.
\]

This is easily verified by performing the multiplications and comparing real and imaginary coefficients.