Stationary Solutions of Incompressible Stokes Equation for a Circular Pipe

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Stationary modes of the incompressible Stokes equation are derived using the method of potentials. Their relation to instationary modes is discussed.

In a recent paper [1] a method developed by Hansen, Stratton, Morse, Feshbach [2] (“HSMF-method”) for solving linear vector-wave equations in electrodynamics was applied to find analytical expressions for three-dimensional time-dependent solutions of the Stokes equation for a pipe with circular cross-section. In this paper we derive analytical expressions for stationary modes of the stationary Stokes equation

$$\frac{1}{\varrho_0} \frac{\partial p}{\partial \mathbf{r}} = -\nu_0 \operatorname{rot} \operatorname{rot} \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad (1)$$

with the help of “potentials” a , b for the same geometry and boundary condition as in [1].

Expressing $\mathbf{u}$ and p by potentials $a(\mathbf{r}, t)$, $b(\mathbf{r}, t)$ ([1], ref. 24)

$$\begin{aligned} \mathbf{u} &= \operatorname{rot} \mathbf{a} + \operatorname{rot} \operatorname{rot} \mathbf{b}, \quad p = p_0 + \varrho_0 \nu_0 \operatorname{div} (\mathbf{e}_z \Delta b), \\ \mathbf{a} &= a \mathbf{e}_z, \quad \mathbf{b} = b \mathbf{e}_z, \end{aligned} \quad (2)$$

we get for a , b the differential equations

$$\Delta a = 0, \quad \Delta \Delta b = 0. \quad (3)$$

Introducing cylindrical coordinates r , φ , z and separating $a(r, \varphi, z)$ and $b(r, \varphi, z)$ into $a(r, \varphi, z) = a_r(r) a_\varphi(\varphi) a_z(z)$, $b(r, \varphi, z) = b_r(r) b_\varphi(\varphi) b_z(z)$ we find for $k \neq 0$ the solutions

$$\begin{aligned} a(r, \varphi, z) &= a_m J_m(i k r) e^{i m \varphi} e^{i k z}, \\ b(r, \varphi, z) &= b_m J_m(i k r) e^{i m \varphi} e^{i k z} \\ &\quad + \frac{c_m r}{i k} J'_m(i k r) e^{i m \varphi} e^{i k z}, \end{aligned} \quad (4)$$

which give

$$\begin{aligned} u_r &= \left[a_m \frac{i m}{r} J_m + b_m k^2 \left(J_{m+1} - \frac{m}{i k r} J_m \right) \right. \\ &\quad \left. + c_m \left(\frac{m^2}{i k r} - i k r \right) J_m \right] e^{i m \varphi} e^{i k z}, \\ u_\varphi &= \left[a_m i k \left(J_{m+1} - \frac{m}{i k r} J_m \right) - b_m \frac{m k}{r} J_m \right. \\ &\quad \left. + c_m i m \left(\frac{m}{i k r} J_m - J_{m+1} \right) \right] e^{i m \varphi} e^{i k z}, \\ u_z &= [-b_m k^2 J_m \\ &\quad + c_m \{(2+m) J_m - i k r J_{m+1}\}] e^{i m \varphi} e^{i k z}. \end{aligned} \quad (5)$$

The boundary condition $\mathbf{u}(r=R, \varphi, z) = \mathbf{0}$ leads to the dispersion relation

$$\begin{aligned} D_S(k, m) &:= i k R m J_m^3(i k R) \\ &\quad + [k^2 R^2 - 2m(2+m)] J_m^2 J_{m+1} \\ &\quad + i k R (2+3m) J_m J_{m+1}^2 \\ &\quad + k^2 R^2 J_{m+1}^3 = 0, \\ m &= 0, 1, 2, \dots \end{aligned} \quad (6)$$

For $m=0$ we find

$$J_1 \left[J_0^2 + \frac{2i}{k R} J_0 J_1 + J_1^2 \right] = 0, \quad (7)$$

which means either

$$\begin{aligned} \text{a) } J_1(i k^{(j)} R) &= 0, \quad k^{(j)} = i k_i^{(j)}, \\ (k_i^{(j)}) &> 0 \text{ for } z > 0, \\ a_0 &\neq 0, \quad b_0 = c_0 = 0, \quad j = 1, 2, 3, \dots \end{aligned} \quad (7a)$$

or

$$\text{b) } J_0^2 + \frac{2i}{k^{(j)} R} J_0 J_1 + J_1^2 = 0 \Leftrightarrow J_0 J_2 = J_1^2, \quad (7b)$$

$$a_0 = 0, \quad b_0 \neq 0, \quad c_0 \neq 0.$$

See also [3], p. 547.

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Solutions $\mathbf{u}(r, \varphi, z)$ which belong to a) or b) are given by

$$\begin{aligned} \text{a)} \quad \mathbf{u}^{(0,j)} &= -\mathbf{e}_\varphi a_0 i k^{(j)} J_1(i k^{(j)} r) e^{i k^{(j)} z} \\ &= -\mathbf{e}_\varphi \tilde{a}_0^{(j)} J_1(k_i^{(j)} r) e^{-k_i^{(j)} z}, \\ \tilde{a}_0^{(j)} &= a_0 k_i^{(j)}, \end{aligned} \quad (8a)$$

$$\begin{aligned} \text{b)} \quad u_r^{(0,j)} &= \tilde{b}_0^{(j)} \left[J_1(i k^{(j)} r) \right. \\ &\quad \left. - \frac{r}{R} \frac{J_0(i k^{(j)} r)}{J_0(i k^{(j)} R)} J_1(i k^{(j)} R) \right] e^{i k^{(j)} z}, \\ u_\varphi^{(0,j)} &= 0, \end{aligned} \quad (8b)$$

$$u_z^{(0,j)} = \tilde{b}_0^{(j)} \left[\frac{\{2J_0(i k^{(j)} r) - i k^{(j)} r J_1(i k^{(j)} r)\} J_0(i k^{(j)} R)}{2J_0(i k^{(j)} R) - i k^{(j)} R J_1(i k^{(j)} R)} - J_0(i k^{(j)} R) \right] e^{i k^{(j)} z},$$

$$a_0 = 0, \quad c_0 = \frac{b_0 (k^{(j)})^2 J_1(i k^{(j)} R)}{i k^{(j)} R J_0(i k^{(j)} R)}.$$

Solutions for $m \neq 0$ can be found from (5) with

$$\begin{aligned} c_m^{(j)} &= \tilde{b}_m^{(j)} \frac{J_m(i k^{(j)} R)}{(2+m) J_m(i k^{(j)} R) - i k^{(j)} R J_{m+1}(i k^{(j)} R)}, \\ a_m^{(j)} &= \tilde{b}_m^{(j)} \frac{i R}{m J_m(i k^{(j)} R)} \frac{\left(J_{m+1} - \frac{m}{i k^{(j)} R} J_m \right) \left[(2+m) J_m - i k^{(j)} R J_{m+1} \right] + \left(\frac{m^2}{i k^{(j)} R} - i k^{(j)} R \right) J_m^2}{(2+m) J_m(i k^{(j)} R) - i k^{(j)} R J_{m+1}(i k^{(j)} R)}, \end{aligned} \quad (9)$$

where $k^{(j)}$ are solutions of (6) and $\tilde{b}_m^{(j)} = b_m (k^{(j)})^2$. The stationary pressure functions $p^{(m,j)}(r, \varphi, z)$ for $k^{(j)} \neq 0$, which follow from (2), (4) are given by

$$\begin{aligned} p^{(m,j)}(r, \varphi, z) &= p_0 - 2 Q_0 v_0 i k^{(j)} c_m^{(j)} \\ &\quad \cdot J_m(i k^{(j)} r) e^{i m \varphi} e^{i k^{(j)} z}. \end{aligned} \quad (10)$$

If $k^{(j)}$ is a solution of (6) the same holds for $-k^{(j)}$. Numerical calculations for $m=0$ ([1]) with small but finite values of σ are in agreement with (7a) to (8b). As can be seen from (6), there exist no solutions for $k^{(j)}$ with

$$\begin{aligned} J'_m(i k^{(j)} R) &= -J_{m+1}(i k^{(j)} R) \\ &\quad + \frac{m}{i k^{(j)} R} J_m(i k^{(j)} R) = 0, m > 0, \end{aligned}$$

as might be supposed from Figs. 1, 2 in [1].

The analytical limit process $\sigma \rightarrow 0$ applied to (19) of [1], where σ is the time-separation constant, does not give the dispersion relation for the stationary

modes, since $\lim_{\sigma \rightarrow 0} D(k, m, \sigma) \equiv 0$. $D(k, m, \sigma)$ denotes the right hand side of equation (19) in [1] for incompressible flow. Instead, the following relation holds ($k \neq 0$):

$$D_S(k, m) = \text{const} \frac{\partial D(k, m, \sigma)}{\partial \sigma} \bigg|_{\sigma=0}. \quad (11)$$

As can be shown by simple but lengthy calculations a similar relation can be found between the stationary modes $\mathbf{u}^{(m,j)}(r, \varphi, z)$ and the instationary solutions $\mathbf{u}^{(m)}(r, \varphi, z, t, k, \sigma)$ (equation (17) in [1]) which fulfill the boundary condition $\mathbf{u}^{(m)}(r=R,$

$\varphi, z, t, k, \sigma) = \mathbf{0}$:

$$\begin{aligned} \mathbf{u}^{(m,j)}(r, \varphi, z) &= \frac{\partial [\mathbf{u}_{(r, \varphi, z, t, k, \sigma)}^{(m)} e^{\sigma t}]}{\partial \sigma} \bigg|_{\substack{\sigma=0 \\ k=k^{(j)}}} \\ &\quad \cdot A(m, k^{(j)}) \quad (a_0 = 0 \text{ for } m=0). \end{aligned} \quad (12)$$

$A(m, k^{(j)})$ are constants and $k^{(j)}$ is a solution of (6). The only exception of relation (12) is the stationary solution for $m=0$, $a_0 \neq 0$, $b_0 = c_0 = 0$. In this special case $\mathbf{u}^{(0,j)}(r, z)$ follows directly from

$$\begin{aligned} \mathbf{u}^{(0)}(r, z, t, k, \sigma) &= -\mathbf{e}_\varphi a_0 \sqrt{\frac{\sigma}{v_0} - k^2} J'_0 \left(\sqrt{\frac{\sigma}{v_0} - k^2} \right) e^{i k z} e^{-\sigma t} \end{aligned} \quad (13)$$

for $\sigma=0$ since $\mathbf{u}^{(0)}(r, z, t, k, \sigma=0) \neq \mathbf{0}$. For $\sigma=0$ ($a_0=0$ for $m=0$) one gets

$$\lim_{\sigma \rightarrow 0} \mathbf{u}^{(m)}(r, \varphi, z, t, k, \sigma) = \mathbf{0}.$$

The relation between the stationary pressure functions (10) and the corresponding instationary ones

(Eq. (16) of [1] for $c_c = \infty$) is found to be

$$p^{(m,j)}(r, \varphi, z) = p_0 + \frac{\partial [p^{(m)}_{(r, \varphi, z, t; k, \sigma)} e^{\sigma t}]}{\partial \sigma} \bigg|_{\substack{\sigma=0 \\ k=k^{(j)}}} \cdot B(m, k^{(j)}), \quad (14)$$

where $B(m, k^{(j)})$ are constants. Without explicit knowledge of the stationary functions $D_S(k, m)$, $\mathbf{u}^{(m,j)}(r, \varphi, z)$, $p^{(m,j)}(r, \varphi, z)$ it does not seem possible to state the relations (11), (12), (14). In order to get the relations (12) and (14), the process $\sigma \rightarrow 0$ with $b_m \sigma = b_m^* = \text{const}$ must be used, where the constants b_m enter the functions $\mathbf{u}^{(m)}$, $p^{(m)}$ according to equation (18) of [1] for $c_c = \infty$. If one interprets the functions $\mathbf{u}^{(m)}$ or $\mathbf{u}^{(m,j)}$ for $k \neq 0$ as flow patterns in the pipe, the corresponding Reynolds-numbers must be assumed to be very small, which means that the coefficients a_0 , b_m must be sufficiently small.

The case of vanishing separation constant k is of certain interest. If one solves (1) to (3) for $k = 0$ and $\mathbf{u}(r = R, \varphi, z) = \mathbf{0}$, only the trivial solution $\mathbf{u} \equiv \mathbf{0}$, $p = \text{const.}$ results: especially the Hagen-Poiseuille-solution for $m = 0$ does not appear. The reason is that the Hagen-Poiseuille-solution $\mathbf{u}_{\text{HP}}(r) = u_0(1 - r^2/R^2)\mathbf{e}_z$ cannot be represented by (3). For $k = 0, m = 0$, however, (1) is solved by

$$\begin{aligned} \mathbf{u}^{(0,0)} &= \text{rot} \cdot \text{rot} \mathbf{b}_r^{(0,0)}(r), \quad p^{(0,0)} = p_0 - c \varrho_0 v_0 z, \\ \mathbf{b}_r^{(0,0)} &= \mathbf{e}_z b_r^{(0,0)}(r), \quad \Delta_r \Delta_r b_r^{(0,0)} = c, \\ \Delta_r &= \frac{1}{r} \left(r \frac{d}{dr} \right), \end{aligned} \quad (15)$$

where c is an arbitrary constant. The solution of $\Delta_r \Delta_r b_r^{(0,0)}(r) = c$ consists of the general solution of the homogeneous equation and a special solution of the inhomogeneous equation. Taking into account the non-trivial, non-singular terms only, we have

$$\begin{aligned} b_r^{(0,0)}(r) &= b_0 r^2 + \frac{c}{64} r^4 + b_1 r^2 \ln r, \\ \Delta b_r^{(0,0)} &= \frac{c r^2}{4} + 4 b_0 \\ &\quad + 4 b_1 (\ln r + 1), \quad b_1 = 0, \end{aligned} \quad (16)$$

which gives

$$\mathbf{u}_{(z)}^{(0,0)} = \frac{c}{4R^2} \left(1 - \frac{r^2}{R^2} \right) \mathbf{e}_z, \quad p_{(z)}^{(0,0)} = p_0 - c \varrho_0 v_0 z. \quad (17)$$

As already Sexl [3] observed, the Hagen-Poiseuille solution can be found also from the time-dependent

solutions of the Stokes equation for $m = 0, k = 0$ by the process $\lim_{\substack{\sigma \rightarrow 0 \\ c \sigma = \text{const}}} \mathbf{u}_S^{(0)}(r, t, \sigma)$, where $\mathbf{u}_S^{(0)}$ denotes the

functions found by Sexl ([3], p. 575). In order to find all stationary solutions for $k = 0$ from the time-dependent solutions of the Stokes equation by using the HSMF-method we take the solutions found by Brosa [1]. Since he did not separate the Helmholtz-equation for $k = 0$ quite correctly, we summarize the three types of time-dependent solutions for $k = 0$:

a) $m = 0$, α_0 arbitrary, b_0 arbitrary:

$$\begin{aligned} J_1 \left(\sqrt{\frac{\sigma_1^{(j)}}{v_0}} R \right) &= 0, \quad j = 1, 2, 3, \dots, \quad u_r^{(0)} = 0, \\ u_\varphi^{(0)} &= a_0 \sqrt{\frac{\sigma_1^{(j)}}{v_0}} J_1 \left(\sqrt{\frac{\sigma_1^{(j)}}{v_0}} r \right) e^{-\sigma_1^{(j)} t}, \\ u_z^{(0)} &= b_0 \frac{\sigma_1^{(j)}}{v_0} \left[J_0 \left(\sqrt{\frac{\sigma_1^{(j)}}{v_0}} r \right) \right. \\ &\quad \left. - J_0 \left(\sqrt{\frac{\sigma_1^{(j)}}{v_0}} R \right) \right] e^{-\sigma_1^{(j)} t}. \end{aligned} \quad (18)$$

b) $m = 0$, b_0 arbitrary, σ arbitrary:

$$\begin{aligned} u_r^{(0)} &= 0, \quad u_\varphi^{(0)} = 0, \\ u_z^{(0)} &= b_0 \frac{\sigma}{v_0} \left[J_0 \left(\sqrt{\frac{\sigma}{v_0}} r \right) \right. \\ &\quad \left. - J_0 \left(\sqrt{\frac{\sigma}{v_0}} R \right) \right] e^{-\sigma t}. \end{aligned} \quad (19)$$

c) $m = 0, 1, 2, \dots$, b_m arbitrary:

$$\begin{aligned} J_m \left(\sqrt{\frac{\sigma_m^{(j)}}{v_0}} R \right) &= 0, \quad u_r^{(m)} = 0, \quad u_\varphi^{(m)} = 0, \\ u_z^{(m)} &= b_m \frac{\sigma_m^{(j)}}{v_0} J_m \left(\sqrt{\frac{\sigma_m^{(j)}}{v_0}} r \right) e^{im\varphi} e^{-\sigma_m^{(j)} t}. \end{aligned} \quad (20)$$

If we look for stationary solutions with $k = 0$ as result of a process $\sigma \rightarrow 0$, $b_m \sigma^n = \text{const}$, only solutions of type b) can be used since in the other two cases $\sigma = 0$ cannot be approached smoothly. Making explicit the argument used in [1] we get for

$$\begin{aligned} u_z^{(0)}(r, t, \sigma) \\ = b_0 \frac{\sigma}{v_0} \left[J_0 \left(\sqrt{\frac{\sigma}{v_0}} r \right) - J_0 \left(\sqrt{\frac{\sigma}{v_0}} R \right) \right] e^{-\sigma t} \end{aligned}$$

the relations

$$u_z^{(0)}(r, t, \sigma = 0) = 0, \quad \left(\frac{\partial u_z^{(0)}(\sigma)}{\partial \sigma} \right)_{\sigma=0} = 0,$$

$$\left(\frac{\partial^2 u_z^{(0)}(\sigma)}{\partial \sigma^2} \right)_{\sigma=0} = \frac{b_0}{2 v_0^2} (R^2 - r^2).$$

Thus we find

$$\lim_{\substack{\sigma \rightarrow 0 \\ b_0 \sigma^2 = b_0^*}} \mathbf{u}^{(0)}(r, t, \sigma) = \frac{b_0^*}{4 v_0^2} (R^2 - r^2) \mathbf{e}_z \\ = u_0 \left(1 - \frac{r^2}{R^2} \right) \mathbf{e}_z =: \mathbf{u}_{\text{HP}}(r),$$

$$u_0 = b_0^* R^2 / 4 v_0^2.$$

In the same way the stationary pressure $p_{\text{HP}}(z)$ follows from

$$p_{\text{HP}}(z) = \lim_{\substack{\sigma \rightarrow 0 \\ b_0 \sigma^2 = b_0^*}} p^{(0)}(z, t, \sigma),$$

where $p^{(0)}(z, t, \sigma)$ is given by ([1], Eq. (27))

$$p^{(0)}(z, t, \sigma) = p_0 - \varrho_0 b_0 \frac{\sigma^2}{v_0} J_0 \left(\sqrt{\frac{\sigma}{v_0}} R \right) z e^{-\sigma t}.$$

With

$$p^{(0)}(z, t, \sigma = 0) = p_0, \quad \left(\frac{\partial p^{(0)}(\sigma)}{\partial \sigma} \right)_{\sigma=0} = 0,$$

$$\left(\frac{\partial^2 p^{(0)}(\sigma)}{\partial \sigma^2} \right)_{\sigma=0} = -\frac{2 \varrho_0 b_0}{v_0} z$$

we get finally

$$p_{\text{HP}}(z) = p_0 - \frac{\varrho_0}{v_0} b_0^* z = p_0 - 4 \varrho_0 v_0 \frac{u_0}{R^2} z.$$

Contrary to the stationary solutions $\mathbf{u}^{(m,j)}(r, \varphi, z)$, the coefficient b_0^* (or u_0) which appears in $\mathbf{u}_{\text{HP}}(r)$ may take values which give large Reynolds-numbers since $\mathbf{u}_{\text{HP}}(r)$ solves the stationary non-linear Navier-Stokes equation.

With regard to the infinite pipe the Hagen-Poiseuille-solution remains as the only stationary solution of the Stokes equation which fulfills the (incomplete) boundary condition $\mathbf{u}(r=R)=\mathbf{0}$, which possesses continuous partial derivatives everywhere and which is bounded for $-\infty < z < \infty$. The functions $\mathbf{u}^{(m,j)}(r, \varphi, z)$ with $k \neq 0$ represent solutions of the Stokes equation for the semi-infinite pipe with $\mathbf{u}^{(m,j)}(r, \varphi, z \rightarrow \infty) = \mathbf{0}$ and a prescribed boundary condition at $z=0$.

We want to mention that also $\Delta a = 0$ (2) is only sufficient to solve (1). If for the case $m=0, k=0$ one lets $\Delta a = c$, one finds an additional solution $\mathbf{u}(r) = -\frac{1}{2} c r \mathbf{e}_\varphi$, which gives no contribution for homogeneous boundary conditions $\mathbf{u}^{(0,0)}(r) = \mathbf{0}$, but for a boundary condition

$$u_r^{(0,0)}(r=R) = u_z^{(0,0)}(r=R) = 0,$$

$$u_\varphi^{(0,0)}(r=R) = u_{\varphi 0},$$

which characterizes a rotating pipe.

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