Poisson Brackets and Nijenhuis Tensor *

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Many integrable models satisfy the zero Nijenhuis tensor condition. Although its application for discrete systems is then straightforward, there exist some complications to utilize the condition for continuous infinite dimensional models. A brief sketch of how we deal with the problem is explained with an application to a continuous Toda lattice.

1. Let M be a symplectic manifold with a Poisson bracket

$$[A, B]^{(0)} = f^{\mu\nu}\partial_\mu A\,\partial_\nu B \tag{1}$$

for two functions $A(x)$ and $B(x)$ of the coordinate x^μ. For the discrete system, the repeated indices on μ and ν imply of course automatical summations over some finite discrete values. For continuous cases, it must be replaced for example by

$$[A, B]^{(0)} = \iint \mathrm{d}x\,\mathrm{d}y\, f_{ab}(x, y)\,\frac{\delta A}{\delta\phi_a(x)}\,\frac{\delta B}{\delta\phi_b(y)} \tag{2}$$

for some field variables $\phi_b(x)$ with internal index specified by b. However, for simplicity, we will hereafter use the notation as in (1) for the discrete case even for the continuous one, unless it is explicitly stated otherwise.

It is well-known [1] that the inverse tensor $f_{\mu\nu}$ of the bi-vector field $f^{\mu\nu}$ obeys the condition

$$\partial_\lambda f_{\mu\nu} + \partial_\mu f_{\nu\lambda} + \partial_\nu f_{\lambda\mu} = 0\,. \tag{3}$$

By Darboux' theorem [2], there exists a local coordinate frame in M such that the $f_{\mu\nu}$'s are constants, i.e. $\partial_\lambda f_{\mu\nu} = 0$. In this note, we assume that we will be dealing with constant $f_{\mu\nu}$'s. Then, it is often more convenient to set

$$x_\mu = f_{\mu\nu}x^\nu\,, \tag{4a}$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = -f^{\mu\nu}\partial_\nu\,. \tag{4b}$$

Many integrable systems possesses another anti-symmetric tensor $F_{\mu\nu} = -F_{\nu\mu}$ satisfying

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0\,, \tag{5}$$

although we need not assume existence of its inverse $F^{\mu\nu}$. We can then construct infinite numbers of anti-symmetric tensors by

$$F^{(0)}_{\mu\nu} = f_{\mu\nu}\,,\quad F^{(1)}_{\mu\nu} = F_{\mu\nu}\,,\quad F^{(2)}_{\mu\nu} = F_{\mu\alpha}f^{\alpha\beta}F_{\beta\nu}\,, \tag{6a}$$

$$F^{(n+1)}_{\mu\nu} = F_{\mu\alpha}f^{\alpha\beta}F^{(n)}_{\beta\nu} \qquad (n = 0, 1, 2, 3, \ldots)\,. \tag{6b}$$

As a generalization of (1), we may then introduce new brackets by

$$[A, B]^{(n)} = -F^{(n)}_{\mu\nu}\partial^\mu A\,\partial^\nu B \tag{7}$$

for any non-negative integer n. For the special case of $n = 0$, it reproduces (1). As we shall see shortly, $[A, B]^{(n)}$ for any n will define Poisson brackets also under some conditions. Setting now

$$S^\nu_\mu = F_{\mu\lambda}f^{\lambda\nu}\,, \tag{8}$$

the Nijenhuis tensor [3] is given by

$$N^\lambda_{\mu\nu} = S^\alpha_\mu\partial_\alpha S^\lambda_\nu - S^\alpha_\nu\partial_\alpha S^\lambda_\mu - S^\lambda_\alpha(\partial_\mu S^\alpha_\nu - \partial_\nu S^\alpha_\mu)\,. \tag{9}$$

2. The following two Propositions have been proved elsewhere [4].

Proposition 1

We assume the validity of (5). Then, the following three conditions are equivalent, i.e. the validity of any one of them implies that of all others.

(i) $N^\lambda_{\mu\nu} = 0$,

(ii) $\partial_\lambda F^{(2)}_{\mu\nu} + \partial_\mu F^{(2)}_{\nu\lambda} + \partial_\nu F^{(2)}_{\lambda\mu} = 0$,

(iii) $[A, B]^{(1)} = -F_{\mu\nu}\partial^\mu A\,\partial^\nu B$ defines a Poisson bracket.

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Moreover, any one of these conditions implies the validity first of

$$\partial_\lambda F^{(n)}_{\mu\nu} + \partial_\mu F^{(n)}_{\nu\lambda} + \partial_\nu F^{(n)}_{\lambda\mu} = 0 \tag{10}$$

for all $n = 0, 1, 2, 3, \ldots$. Secondly, $[A, B]^{(n)}$ given by (7) defines Poisson-brackets for any $n = 0, 1, 2, \ldots$ ∎

Proposition 2

We assume the same conditions of the Proposition 1. Then, if the trace $\mathrm{Tr}\, S^n = (S^n)^\mu_\mu$ exists, H_n's defined by

$$H_n = \frac{1}{n}\,\mathrm{Tr}(S^n), \quad (n = 1, 2, 3 \ldots) \tag{11}$$

satisfies the recursion relation

$$S^\lambda_\mu \partial_\lambda H_n = \partial_\mu H_{n+1} \tag{12}$$

as well as the involution property

$$[H_n, H_m]^{(p)} = 0 \tag{13}$$

for any non-negative integer p and for all $n, m = 1, 2, 3 \ldots$ ∎

Remark 1

The finite Toda lattice has been demonstrated elsewhere [5] to satisfy the conditions of Propositions 1 and 2. However, for continuous infinite dimensional cases such as the KdV system, $\mathrm{Tr}\, S^n$ gives in general a divergent result, so that we cannot apply the Proposition 2. For such a case, we must appeal to the following weaker Proposition without assuming existence of the trace.

Proposition 3

We assume again the conditions of the Proposition 1. Suppose now that we can find a pair functions H_1 and H_2 satisfying the relation

$$S^\lambda_\mu \partial_\lambda H_1 = \partial_\mu H_2, \tag{14}$$

then we can construct an infinite number of quantities H_n's ($n = 1, 2, 3, \ldots$) satisfying the recursion relation (12) with the involution property (13). ∎

Remark 2

We suppose now that the Hamiltonian H of the system is identified with H_ℓ for an arbitrary but fixed positive integer ℓ, so that $H = H_\ell$. Imposing the Hamiltonian equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} Q = [H, Q]^{(p)} = [H_\ell, Q]^{(p)} \tag{15}$$

for any dynamical variable Q, (13) and (15) imply that all K_n's ($n = 1, 2, \ldots$) are constants of motion and in mutual involution. Therefore, if we can find a sufficient number of algebraically independent terms among these K_n's, then the system is integrable by the Liouville's theorems at least for finite systems.

3. For continuous models we must modify the notations suitably. Instead of x_μ and ∂^μ, we must use $\phi_a(x)$ and $\frac{\delta}{\delta\phi_a(x)}$. Similarly, $f_{\mu\nu}$ and $F_{\mu\nu}$ must be replaced by $f_{ab}(x, y)$ and $F_{ab}(x, y)$. For example, consider the case of a continuous Toda lattice, where the internal index can assume two values $a = 1$ and 2 with

$$f_{ab}(x, y) = \varepsilon_{ab}\,\delta(x-y), \quad (a, b = 1, 2), \tag{16 a}$$

$$\varepsilon_{ab} = -\varepsilon_{ba}, \quad \varepsilon_{12} = -\varepsilon_{21} = 1. \tag{16 b}$$

Further, we set

$$F_{11}(x, y) = \{\xi(x)\exp\phi_2'(x) + \xi(y)\exp\phi_2'(y)\}\,\delta'(x-y), \tag{17 a}$$

$$F_{12}(x, y) = -F_{21}(y, x) = \phi_1(x)\,\delta(x-y), \tag{17 b}$$

$$F_{22}(x, y) = \varepsilon(x-y) = \begin{cases} 1, & x > y \\ -1, & x < y \end{cases}, \tag{17 c}$$

where $\xi(x)$ is an arbitrary function of x, which is, however, independent of $\phi_a(x)$, and the primes indicate the space derivatives. We can easily verify then that the conditions of the Propositions 1 and 3 are satisfied with

$$H_1 = \int \mathrm{d}x\,\phi_1(x), \tag{18 a}$$

$$H_2 = \int \mathrm{d}x\,\{\tfrac{1}{2}(\phi_1(x)^2 - \xi(x)\exp\phi_2'(x)\}, \tag{18 b}$$

$$H_3 = \int \mathrm{d}x\,\{\tfrac{1}{3}(\phi_1(x))^3 - 2\,\xi(x)\,\phi_1(x)\exp\phi_2'(x)\}. \tag{18 c}$$

The Hamiltonian equation of motion (15) for $p = 0$ and $\ell = 2$ gives then

$$\dot\phi_2 = \phi_1, \quad \dot\phi_1 = 2\frac{\partial}{\partial x}\exp(\phi_2') \tag{19}$$

when we set $\xi(x) = 1$. Especially $\psi = \phi_2'$ satisfies a second order differential equation

$$\frac{\partial^2}{\partial t^2}\psi = 2\frac{\partial^2}{\partial x^2}\exp\psi. \tag{20}$$

We can also apply the Proposition 3 for KdV and other cases, but the details will be given elsewhere.

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