On Hide's Magnetic Analogue of Ertel's Vorticity Theorem

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The relativistic formulation of Hide's "magnetic analogue" of Ertel's potential vorticity theorem is Dirac's "new classical theory of electrons".

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Dirac's "new classical theory of electrons" [1, 2] describes the relativistic hydrodynamics of conductive media with an "Ohm law"

$$u^\mu = \frac{e}{mc} (A^\mu - g^{\mu\nu} \partial_\nu S), \quad (\mu, \nu = 0, 1, 2, 3). \quad (1)$$

In (1), $u^\mu = \frac{dx^\mu}{dt}$ is the four-dimensional velocity, dt the differential of the proper time, A^μ the four-dimensional electro-magnetic vector potential, and S a gauge function. $g_{\mu\nu}$ means the metrical Minkowski tensor, and e and m are the charge and the rest mass of an electron, respectively.

According to (1), the norm of the velocity is

$$-c^2 = u_\mu u^\mu = \frac{e^2}{m^2 c^2} (A_\mu A^\mu + g^{\mu\nu} \partial_\mu S \partial_\nu S - 2 A^\mu \partial_\mu S). \quad (2)$$

The tensor of the four-dimensional rotation is

$$\omega_{\mu\nu} = \partial_\nu u_\mu - \partial_\mu u_\nu = \frac{e}{mc} (\partial_\nu A_\mu - \partial_\mu A_\nu) = \frac{e}{mc} F_{\mu\nu} = -\omega_{\nu\mu}, \quad (3)$$

where $F_{\mu\nu}$ is the anti-symmetric Maxwellian tensor of the electro-magnetic field.

The four-dimensional relativistic generalization of Helmholtz' theorem of the conservation of vorticity

means [3–5]

$$d\omega_{\alpha\beta} = u^\lambda \partial_\lambda \omega_{\alpha\beta} = \omega^\lambda_\alpha \partial_\beta u_\lambda - \omega^\lambda_\beta \partial_\alpha u_\lambda + \partial_\beta \frac{du_\alpha}{d\tau} - \partial_\alpha \frac{du_\beta}{d\tau}, \quad (4)$$

where $\frac{d\Phi}{d\tau} = u^\lambda \partial_\lambda \Phi$.

Because of the identity $\frac{d}{d\tau} dx^\alpha = u^\lambda \partial_\lambda dx^\alpha$ one can

write (4), with the help of the Cartan exterior differential [6]

$$[dx^\alpha \wedge dx^\beta] = -[dx^\beta \wedge dx^\alpha],$$

like a relativistic formulation of Ertel's vorticity theorem [7]; see also [8]

$$\frac{d}{d\tau} (\omega_{\mu\nu} [dx^\mu \wedge dx^\nu]) = 2 \partial_\nu \frac{du_\mu}{d\tau} [dx^\mu \wedge dx^\nu]. \quad (5)$$

By the Cartan-Stokes integral theorem, (5) yields the circulation theorem [6]

$$\frac{d}{d\tau} \oint_{C_2} \omega_{\mu\nu} [dx^\mu \wedge dx^\nu] = 2 \oint_{C_1} \frac{du_\mu}{d\tau} dx^\mu \quad (6)$$

(with $C_1 = \partial C_2$).

According to Poincaré's rules of exterior differentiation, (5, 6) with Dirac's law (1) yield an analogous equation for the electromagnetic field strength $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, namely the non-holonomic expression

$$\frac{d}{d\tau} (F_{\mu\nu} [A^\mu \wedge A^\nu]) = 2 \partial_\nu \frac{dA_\mu}{d\tau} [A^\mu \wedge A^\nu], \quad (7)$$

together with a circulation theorem of the electromagnetic field

$$\frac{d}{d\tau} \oint_{C_2} F_{\mu\nu} [dx^\mu \wedge dx^\nu] = 2 \oint_{C_1} \frac{dA_\mu}{d\tau} dx^\mu \quad (8)$$

according to the holonomic equations

$$\frac{d}{d\tau} (F_{\mu\nu} [dx^\mu \wedge dx^\nu]) = 2 \partial_\nu \frac{dA_\mu}{d\tau} [dx^\mu \wedge dx^\nu]. \quad (9)$$

These equations are a relativistic generalization of Hide's magnetic analogue [9, 10] of Ertel's vorticity theorem. The relativistic point of view generalizes the "magnetic analogue" of Hide to an "electro-magnetic" analogue.

Reprint requests to W. Schröder.

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