Reconstruction of the Potential Function and its Derivatives for the Diffusion Operator

Hikmet Koyunbakan and Emrah Yilmaz

Firat University, Department of Mathematics, 23119, Elazıg, Turkey

Reprint requests to H. K. and E. Y.; E-mail: hkoyunbakan@gmail.com
and emrah231983@gmail.com

Z. Naturforsch. **63a,** 127 – 130 (2008); received October 24, 2007

We solve the inverse nodal problem for the diffusion operator. In particular, we obtain a reconstruction of the potential function and its derivatives using only nodal data. Results are a generalization of Law's and Yang's works.

Key words: Diffusion Operator; Inverse Nodal Problem; Reconstruction Formula.
MSC 2000: 34L40, 34A55, 34B99

1. Introduction

The inverse nodal problem was initiated by McLaughlin [1], who proved that the Sturm-Liouville problem is uniquely determined by any dense subset of the nodal points. Some numerical schemes were given by Hald and McLaughlin [2] for the reconstruction of the potential. Recently Law, Yang and other authors have reconstructed the potential function and its derivatives of the Sturm-Liouville problem from the nodal points [3 – 7].

In this paper, we are concerned with the inverse nodal problem for the diffusion operator on a finite interval. We reconstruct the potential function and all its derivatives by using Law's and Yang's method [7].

The diffusion operator is written as

$$Ly = -y'' + [q(x) + 2\lambda p(x)]y, \tag{1.1}$$

where the function $q(x) \in L^2[0,\pi]$, $p(x) \in L^2[0,\pi]$. Some spectral problems were extensively solved for the diffusion operator in [8 – 11].

Consider the problem

$$L[y] = \lambda^2 y, \tag{1.2}$$

$$y(0) = 1, \quad y'(0) = -h, \tag{1.3}$$

$$y'(\pi,\lambda) + Hy(\pi,\lambda) = 0, \tag{1.4}$$

where h and H are finite numbers.

Let λ_n be the n-th eigenvalue and $0 < x_1^n < \ldots < x_i^n < \pi$, $i = 1,2,\ldots,n-1$, the nodal points of the n-th eigenfunction. Also let $I_i^n = [x_i^n, x_{i+1}^n]$ be the i-th nodal domain of the n-th eigenfunction and let $l_i^n = |I_i^n| = x_{i+1}^n - x_i^n$ be the associated nodal length. Let $j_n(x)$ be the largest index j such that $0 \le x_j^{(n)} < x$.

Δ denotes the difference operator $\Delta a_i = a_{i+1} - a_i$. Inductively, for $k > 1$, $\Delta^k a_i = \Delta^{k-1} a_{i+1} - \Delta^{k-1} a_i$, and we introduce the difference quotient operator δ:

$$\delta a_i = \frac{a_{i+1} - a_i}{x_{i+1} - x_i} = \frac{\Delta a_i}{l_i} \text{ and } \delta^k a_i = \frac{\delta^{k-1} a_{i+1} - \delta^{k-1} a_i}{l_i}.$$

2. Main Results

Lemma 1. [12] Assume that $q \in L^2[0,\pi]$. Then, as $n \to \infty$ for the problem (1.2) – (1.4),

$$x_i^n = \frac{\left(i - \frac{1}{2}\right)\pi}{\lambda_n} - \frac{h}{2\lambda_n^2} + \frac{1}{2\lambda_n^2} \int_0^{x_i^n} (1 + \cos 2\lambda_n t)[q(t) + 2\lambda_n p(t)]\mathrm{d}t + O\left(\frac{1}{\lambda_n^4}\right), \tag{2.1}$$

$$l_i^n = \frac{\pi}{\lambda_n} + \frac{1}{2\lambda_n^2} \int_{x_i^n}^{x_{i+1}^n} (1 + \cos 2\lambda_n t)[q(t) + 2\lambda_n p(t)]\mathrm{d}t + O\left(\frac{1}{\lambda_n^4}\right). \tag{2.2}$$

Lemma 2. Suppose that $f \in L^2[0,\pi]$. Then for almost every $x \in [0,\pi]$, with $j = j_n(x)$,

$$\lim_{n\to\infty} \frac{\lambda_n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t = f(x).$$

Theorem 1. [12] Suppose that $q \in L^2[0,\pi]$, then

$$q(x) = \lim_{n\to\infty} \lambda_n \left(\frac{2\lambda_n^2 l_j^n}{\pi} - 2\lambda_n - 2p(x) \right).$$

Proposition 1. If q is a continuous function, then

(a) $\lim\limits_{n\to\infty} \sqrt{\lambda_n} l_i^{(n)} = \pi$ and $l_i^{(n)} = \dfrac{1}{n} + O\left(\frac{1}{n^2}\right)$;

(b) $l_{i+k}^{(n)} / l_{i+m}^{(n)} = 1 + O\left(\frac{1}{n}\right)$ for any fixed $k, m \in N$;

(c) $q_m \le \lambda_n - \frac{\pi^2}{(l_i^{(n)})^2} \le q_M$, where $q_m = \min\limits_{[0,\pi]} q(x)$ and $q_M = \max\limits_{[0,\pi]} q(x)$.

Lemma 3. [7] If $q \in C^N[0,\pi]$, then for $k = 1,\dots,N, \Delta^k l_j = O(n^{-(k+3)})$ as $n \to \infty$ and the order estimate is independent of j.

Lemma 4. [7] Let $\Phi_j = \sum\limits_{i=1}^{m} \phi_{j,i}$ with each $\phi_{j,i} = \prod\limits_{p=1}^{k_i} \varphi_{j,i,p}$, where each $\varphi_{j,i,p} \in U_j^{(n)}$. Suppose $\Phi_j = O(n^{-\nu})$ and q is sufficiently smooth. Then $\delta^k \Phi_j = O(n^{-\nu})$ for all $k \in N$.

Lemma 5. [7] Suppose $f \in C^N[0,\pi]$ and $\Phi_j = \int\limits_{x_j}^{x_{j+1}} f(x)\mathrm{d}x$. Then $\delta^k \Phi_j = O(n^{-1})$ for any $k = 0,1,\dots,N$.

Theorem 2. [7] Let $\Phi_m(x_j) = \psi_1(x_j)\psi_2(x_j)\dots\psi_m(x_j)$, where $\psi_i(x_j) = x_{j+k_i}$ and $k_i \in N \cup \{0\}$. If q is C^k on $[0,\pi]$, then

$$\delta^k \Phi_m(x_j) = \left\{ \begin{array}{cc} O(1), & 0 \le k \le m-1, \\ m! + O(n^{-1}), & k = m, \\ O(n^{-2}), & k \ge m+1. \end{array} \right\}$$

Theorem 3. If $q \in C^{N+1}[0,\pi]$, then $q^{(k)}(x) = \delta^k q(x_j) - 2\lambda_n p(x) + O(n^{-1})$ for $k = 0,1,\dots,N$, where $j = j_n(x)$. The order estimate is uniformly valid for compact subsets of $[0,\pi]$.

Remark: For $k = 1,\dots,N$,

$$V_k(x_j) = \begin{bmatrix} 1 & x_j & \cdots & x_j^k \\ 1 & x_{j+1} & \cdots & x_{j+1}^k \\ \vdots & \vdots & & \vdots \\ 1 & x_{j+k} & \cdots & x_{j+k}^k \end{bmatrix}$$

be a $(k+1)\times(k+1)$ Vandermonde matrix. It is well known that

$$\det V_k(x_j) = \prod_{m=1}^{k} \left[\prod_{i=0}^{m-1} \left(\sum_{p=i}^{m-1} l_{j+p} \right) \right].$$

To prove Theorem 3, we need the following lemma.

Lemma 6.

$$\frac{\prod\limits_{m=1}^{k} l_{j+k-m}^m}{\det V_k(x_j)} = \frac{1}{\prod\limits_{m=1}^{k} m!} + O(\frac{1}{n}). \tag{2.3}$$

Next, we consider the following $(k+1)\times(k+1)$ matrix:

$$A = \begin{bmatrix} 1 & x_j & \cdots & x_j^{k-1} & q(x_j) \\ 1 & x_{j+1} & \cdots & x_{j+1}^{k-1} & q(x_{j+1}) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{j+k} & \cdots & x_{j+k}^{k-1} & q(x_{j+k}) \end{bmatrix}.$$

After some operations, we obtain

$$\det A = l_j l_{j+1} \dots l_{j+k-1} \cdot \det \begin{bmatrix} 1 & x_j & \cdots & x_j^{k-1} & q(x_j) \\ \delta(1) & \delta x_j & \cdots & \cdot & \delta q(x_j) \\ \vdots & \vdots & & \vdots & \vdots \\ \delta(1) & \delta(x_{j+k-1}) & \cdots & \cdot & \delta q(x_{j+k-1}) \end{bmatrix}$$

$$= (l_j)^k (l_{j+1})^{k-1} \dots l_{j+k-1} \cdot \det \begin{bmatrix} 1 & x_j & \cdots & x_j^{k-1} & q(x_j) \\ \delta(1) & \delta x_j & \cdots & \cdot & \delta q(x_j) \\ \vdots & \vdots & & \vdots & \vdots \\ \delta^k(1) & \delta^k(x_j) & \cdots & \cdot & \delta^k q(x_j) \end{bmatrix}.$$

Let B be the matrix at the right-hand side. By Lemma 4 and Theorem 2,

$$\det B = (\delta x_j)(\delta^2 x_j^2)\dots(\delta^{k-1} x_j^{k-1})(\delta^k q(x_j)) + O\left(\frac{1}{n^2}\right) = \prod_{m=0}^{k-1} (m!) \delta^k q(x_j) + O\left(\frac{1}{n}\right).$$

Proof of Theorem 3: For $k = 1,2,\ldots,N$, let

$$g(x) = \det\begin{bmatrix} 1 & x & \cdots & x^k & q(x) \\ 1 & x_j & \cdots & x_j^k & q(x_j) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{j+k} & \cdots & x_{j+k}^k & q(x_{j+k}) \end{bmatrix}.$$

By Rolle's theorem, $g(x_j) = g(x_{j+1}) = \ldots = g(x_{j+k}) = 0$ implies that there is some $\xi_{1,j+i} \in (x_{j+i}, x_{j+i+1})$ such that $g'(\xi_{1,j+i}) = 0$. When we repeat the process, we can find that $\xi_{k,j} \in (x_j, x_{j+k})$ such that $g^{(k)}(\xi_{k,j}) = 0$. In view of the definition of g, $q^{(k)}(\xi_{k,j})\det V_k(x_j) = k!\det A$. Hence

$$q^{(k)}(\xi_{k,j}) = (k!)(l_j)^k(l_{j+1})^{k-1}\ldots l_{j+k-1}\frac{\det B}{\det V_k(x_j)}.$$

By Lemma 6 $q^{(k)}(\xi_{k,j}) = \delta^k q(x_j) + O(\frac{1}{n})$, since $q \in C^{k+1}, q^{(k)}(x) = q^{(k)}(\xi_{k,j}) - 2\lambda_n p(x) + O(\frac{1}{n}) = \delta^k q(x_j) - 2\lambda_n p(x) + O(\frac{1}{n})$.

Theorem 4. Suppose that q in (1.1) is C^{N+1} on $[0,\pi]$ $(N \geq 1)$, and let $j = j_n(x)$ for each $x \in [0,\pi]$. Then, as $n \to \infty$,

$$q(x) = \lambda_n\left(\frac{2\lambda_n^2 l_j^n}{\pi} - 2\lambda_n - 2p(x)\right) + O\left(\frac{1}{n}\right),$$

and, for all $k = 1,2,\ldots,N$,

$$q^{(k)}(x) = \frac{2\lambda_n^{3/2}\delta^k l_j}{\pi} - 2\lambda_n\delta^k p(x_j) - 2\lambda_n p^{(k)}(x) + O(1).$$

Proof: The uniform approximation for q is evident. Suppose that q is continuously differentiable on $[0,\pi]$. Apply the intermediate value theorem on Proposition 1c, then there is some $\xi_j^{(n)} \in (x_j^{(n)}, x_{j+1}^{(n)})$ such that

$$\frac{\lambda_n^2 l_j^{(n)}}{\pi} = \left(1 - \frac{q(\xi_j^{(n)})}{\lambda_n}\right)^{-1/2} = 1 + \frac{q(\xi_j^{(n)})}{2\lambda_n} + O\Big(\frac{1}{\lambda_n^2}\Big).$$

Hence

$$2\lambda_n\left(\frac{\lambda_n^2 l_j^n}{\pi} - 1\right) - q(\xi_j^{(n)}) = O\left(\frac{1}{n^2}\right),$$

$$2\lambda_n\left(\frac{\lambda_n^2 l_j^n}{\pi} - p(x)\right) + 2\lambda_n p(x) - q(\xi_j^{(n)}) = O\left(\frac{1}{n^2}\right).$$

Applying the mean value theorem, when n is sufficiently large, then

$$q(x) = q(\xi_j^{(n)}) - 2\lambda_n p(x) + O\left(\frac{1}{n}\right).$$

Then we employ a modified Prüfer substitution due to Ashbaugh-Benguria [13] to solve the boundary conditions h and H, $x = 0$ and $x = \pi$, respectively:

$$\begin{cases} y = r(x)\sin\sqrt{\lambda}\,\theta(x), \\ y' = \sqrt{\lambda}\,r(x)\cos\sqrt{\lambda}\,\theta(x), \end{cases}$$

so that

$$\theta' = \cos^2\sqrt{\lambda}\,\theta(x) - \frac{q(x)\sin^2\sqrt{\lambda}\,\theta(x)}{\lambda} - 2p(x)\sin^2\sqrt{\lambda}\,\theta(x). \tag{2.4}$$

Integrating (2.4) from x_j to x_{j+1},

$$\frac{\pi}{\sqrt{\lambda_n}} = \int_{x_j}^{x_{j+1}}\cos^2\sqrt{\lambda}\,\theta(x)\mathrm{d}x - \frac{1}{\lambda}\int_{x_j}^{x_{j+1}} q(x)\sin^2\sqrt{\lambda}\,\theta(x)\mathrm{d}x - 2\int_{x_j}^{x_{j+1}} p(x)\sin^2\sqrt{\lambda}\,\theta(x)\mathrm{d}x, \tag{2.5}$$

it results by Lemma 5 from (2.5) that

$$\frac{\pi}{\sqrt{\lambda_n}} = -\frac{q(x_j)}{2\lambda_n}l_j + c_j + O\left(\frac{1}{n^{N+4}}\right) - p(x_j)l_j + d_j + O\left(\frac{1}{n^{N+4}}\right) + O\left(\frac{1}{n}\right), \tag{2.6}$$

where

$$c_j = \frac{1}{2\lambda_n}\sum_{k=1}^{N}\frac{q^{(k)}(x_j)}{(k+1)!}l_j^{k+1} = O\left(\frac{1}{n^4}\right),$$

$$d_j = \sum_{k=1}^{N}\frac{p^{(k)}(x_j)}{(k+1)!}l_j^{k+1} = O\left(\frac{1}{n^2}\right).$$

Summarizing from (2.6),

$$q(x_j) = -\frac{2\pi\sqrt{\lambda_n}}{l_j} - 2\lambda_n p(x_j) + 2\lambda_n(c_j + d_j) + O(1).$$

Therefore

$$\delta q(x_j) = -2\pi\sqrt{\lambda_n}\frac{\Delta l_j}{l_j^2 l_{j+1}} - 2\lambda_n \delta p(x_j) + O(1),$$

and so, for $k = 1, 2, \ldots, N$,

$$\delta^k q(x_j) = -2\pi\sqrt{\lambda_n}\delta^{k-1}\left(\frac{\Delta l_j}{l_j^2 l_{j+1}}\right) - 2\lambda_n \delta^k p(x_j) + O(1). \quad (2.7)$$

If we use the results of Theorem 3 and Theorem 5, we get

$$q^{(k)}(x) = \delta^k q(x_j) - 2\lambda_n p(x) + O(n^{-1}),$$

$$q^{(k)}(x) = \frac{2\lambda_n^{3/2}\delta^k l_j}{\pi} - 2\lambda_n \delta^k p(x_j) - 2\lambda_n p(x) + O(1).$$

Theorem 5. Assume that q is C^{N+1} on $[0,\pi]$. Then, for $k = 1, 2, \ldots, N$,

$$\delta^k q(x_j) = \frac{2\lambda_n^{3/2}\delta^k l_j}{\pi} - 2\lambda_n \delta^k p(x_j) + O(1).$$

The estimate is independent of j.

Proof: In view of derivations and the fact that $\delta l_j = \frac{\Delta l_j}{l_j} = \frac{O(n^{-4})}{O(n^{-1})} = O(n^{-3})$, it suffices to show that

$$\delta^{k-1}\left(\frac{\pi^2 \Delta l_j}{l_j^2 l_{j+1}}\right) = -\delta^{k-1}\left(\frac{\lambda_n \Delta l_j}{l_j}\right) + O\left(\frac{1}{n^3}\right). \quad (2.8)$$

But

$$\begin{aligned}\delta\left(\frac{\lambda_n \Delta l_j}{l_j} - \frac{\pi^2 \Delta l_j}{l_j^2 l_{j+1}}\right) &= \delta\left[\left(\lambda_n - \frac{\pi^2}{l_j l_{j+1}}\right)\frac{\Delta l_j}{l_j}\right] \\ &= \pi^2 \frac{(\Delta l_j + \Delta l_{j+1})}{l_j^2 l_{j+1} l_{j+2}} + \left(\lambda_n - \frac{\pi^2}{l_j l_{j+1}}\right)\delta l_j \\ &= O\left(\frac{1}{n^3}\right).\end{aligned}$$

Thus, (2.8) follows by Lemma 4. If we write (2.8) in (2.7), then

$$\delta^k q(x_j) = -2\pi\sqrt{\lambda_n}\left[-\delta^{k-1}\left(\frac{\lambda_n \Delta l_j}{\pi^2 l_j}\right) + O\left(\frac{1}{n^3}\right)\right] - 2\lambda_n \delta^k p(x_j) + O(1), \quad (2.9)$$

$$\delta^k q(x_j) = \frac{2\lambda_n^{3/2}\delta^k l_j}{\pi} - 2\lambda_n \delta^k p(x_j) + O(1).$$

[1] J.R. McLaughlin, J. Differential Equations **73**, 354 (1988).
[2] O. Hald and J. R. McLaughlin, Inverse Problems **5**, 307 (1989).
[3] C.K. Law, C. Liang Shen, and C.F. Yang, Inverse Problems **15**, 253 (1999).
[4] Y.H. Cheng and C.K. Law, Inverse Problems **22**, 891 (2006).
[5] Y. Ting Chen, Y.H. Cheng, C.K. Law, and J. Tsa, Proc. Am. Math. Soc. **130**, 2319 (2002).
[6] H. Koyunbakan, Appl. Math. Lett. **19**, 995, (2006).
[7] C.K. Law and C.-F. Yang, Inverse Problems **14**, 299 (1998).
[8] E. Bairamov, Ö. Çakar, and O. Çelebi, J. Math. Anal. Appl. **216**, 303 (1997).
[9] M.G. Gasymov and G.S. Guseinov, SSSR Dokl. **37**, 19 (1981).
[10] H. Koyunbakan and E.S. Panakhov, J. Math. Anal. Appl. **326**, 1024 (2007).
[11] M. Jaulent and C. Jean, Commun. Math. Phys. **28**, 177 (1972).
[12] H. Koyunbakan, Numerical Functional Analysis and Optimization (2008), in press.
[13] M.S. Ashbaugh and R.D. Benguira, J. Differential Equations **103**, 205 (1993).