Taylor-Couette Flow of an Oldroyd-B Fluid in an Annulus Due to a Time-Dependent Couple

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Taylor-Couette flow in an annulus due to a time-dependent torque suddenly applied to one of the cylinders is studied by means of finite Hankel transforms. The exact solutions, presented under series form in terms of usual Bessel functions, satisfy both the governing equations and all imposed initial and boundary conditions. They can easily be reduced to give similar solutions for Maxwell, second-grade, and Newtonian fluids performing the same motion. Finally, some characteristics of the motion, as well as the influence of the material parameters on the behaviour of the fluid, are emphasized by graphical illustrations.

Key words: Oldroyd-B Fluid; Velocity Field; Shear Stress; Time-Dependent Couple.

1. Introduction

The study of the motion of a fluid in the neighbourhood of a rotating or sliding body is of great interest for industry. The flow between rotating cylinders or through a rotating cylinder has many applications in the food industry, and is one of the most important and interesting problems of motion near rotating bodies. It has been intensively studied since G. I. Taylor (1923) reported the results of his famous investigations [1]. For Newtonian fluids, the velocity distribution for a fluid contained in an annular region between two cylinders, with a common axis, is given in [2]. The first exact solutions for motions of non-Newtonian fluids in cylindrical domains seem to be those of Ting [3] for second-grade fluids, Srivastava [4] for Maxwell fluids and Waters and King [5] for Oldroyd-B fluids. In the meantime many papers regarding such motions have been published but we mention here only a few of those regarding Oldroyd-B or more general fluids [6 – 15].

It is worth pointing out that almost all the above mentioned works deal with motion problems in which the velocity field is given on the boundary. To the best of our knowledge, the first exact solutions for motions of non-Newtonian fluids, due to a constant shear stress on the boundary, are those of Waters and King [16] over an infinite plate, and Bandelli and Rajagopal [17] between two co-axial circular cylinders. Similar solutions for the flow due to an infinite plate that applies a constant/time-dependent shear to a non-Newtonian fluid, have been also obtained by Bandelli et al. [18], Erdogan [19] and Fetecau and Kannan [20]. Although the computer techniques make the complete integration of the momentum equation feasible, the accuracy of the results can be established by comparison with an exact solution. Consequently, as in the case of the motion problems in which the velocity is given on the boundary, it is necessary to develop a large class of exact and approximate solutions for problems in which the boundary (or a part of the boundary) applies a shear stress to the fluid.

Our purpose here is to establish exact solutions for the velocity field and the shear stress corresponding to the motion of an Oldroyd-B fluid between two co-axial circular cylinders, one of them being fixed and the other one applying a time-dependent rotational shear stress to the fluid. More precisely, we extend the results of Bandelli and Rajagopal [17, Sect. 5] to rate type fluids, namely to Oldroyd-B fluids. The Oldroyd-B fluids store energy as linearized elastic solids and their dissipation is due to two dissipative mechanisms which arise from a mixture of two viscous fluids. They have been extensively used in many applications although an Oldroyd-B fluid cannot describe either shear thinning or shear thickening. However, they can describe stress-relaxation, creep, and the normal stress differences that develop during simple shear flows. This model is viewed as one of the most successful models for describing the response of a subclass of polymeric liquids. Furthermore, the general solutions to be obtained here for Oldroyd-B fluids can easily be reduced to give similar solutions for Maxwell, second-grade, and Newtonian fluids. Finally, the influence of the material constants on the fluid motion, as well as a comparison between the four models, is shown by graphical illustrations.

2. Governing Equations

The Cauchy stress $\mathbf{T}$ for an incompressible Oldroyd-B fluid is related to the fluid motion by the following constitutive equations:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},$$
$$\mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}) = \mu[\mathbf{A} + \lambda_{\mathrm{r}}(\dot{\mathbf{A}} - \mathbf{LA} - \mathbf{AL}^{\mathrm{T}})], \quad (1)$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ is the first Rivlin-Ericksen tensor, μ is the dynamic viscosity of the fluid, λ and λ_{r} are relaxation and retardation times, the superposed dot indicates the material time derivative and the superscript T denotes the transpose operation. The model characterized by (1) contains as special cases the upper convected Maxwell model for $\lambda = 0$ and the Newtonian fluid model for $\lambda = \lambda_{\mathrm{r}} = 0$. In some special flows, like those to be considered here, the governing equations corresponding to the Oldroyd-B fluids resemble those of second-grade fluids. Consequently, it is to be expected that the general solutions for Oldroyd-B fluids contain as special cases both the solutions corresponding to Maxwell and Newtonian fluids and those for second-grade fluids.

For the problem under consideration we assume a velocity field $\mathbf{v}$ and an extra-stress tensor $\mathbf{S}$ of the form

$$\mathbf{v} = \mathbf{v}(r,t) = w(r,t)\mathbf{e}_\theta, \quad \mathbf{S} = \mathbf{S}(r,t), \quad (2)$$

where $\mathbf{e}_\theta$ is the unit vector in the θ-direction of the cylindrical coordinates system r, θ, and z. For such flows the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{v}(r,0) = \mathbf{0}, \quad \mathbf{S}(r,0) = \mathbf{0}, \quad (3)$$

and $(1)_2$ and (2) imply $S_{rr} = S_{rz} = S_{zz} = S_{z\theta} = 0$ and we obtain the meaningful equation [6, 7]

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau(r,t) = \mu\left(1+\lambda_{\mathrm{r}}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)w(r,t), \quad (4)$$

where $\tau = S_{r\theta}$ is the non-trivial shear stress.

Neglecting body forces and in the absence of a pressure gradient in the axial direction, the balance of the linear momentum leads to the relevant equation [6]

$$\rho\frac{\partial w(r,t)}{\partial t} = \left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\tau(r,t), \quad (5)$$

where ρ is the constant density of the fluid. Eliminating $\tau(r,t)$ between (4) and (5), we obtain the governing equation for velocity

$$\lambda\frac{\partial^2 w(r,t)}{\partial t^2} + \frac{\partial w(r,t)}{\partial t} = \left(\nu + \alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)w(r,t), \quad (6)$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid and $\alpha = \nu\lambda_{\mathrm{r}}$.

The coupled partial differential equations (4) and (6), with suitable initial and boundary conditions, can be solved in principle by several methods, their efficiency depending on the domain definition. The integral transform technique represents a systematic, efficient, and powerful tool. The Laplace transform can be used to eliminate the time variable while the finite Hankel transform can be employed to eliminate the spatial variable. However, in order to avoid the lengthy and burdensome calculations of residues and contour integrals, we shall use the finite Hankel transform.

3. Taylor-Coutte Flow Due to a Time-Dependent Couple

Consider an incompressible Oldroyd-B fluid at rest in an annular region between two infinitely long coaxial circular cylinders. At time $t = 0^+$ let the inner cylinder of radius R_1 be set in rotation about its axis by a time-dependent torque per unit length $2\pi R_1\tau(R_1,t)$, where

$$\tau(R_1,t) = f(1-\mathrm{e}^{-\frac{t}{\lambda}}); \quad f = \text{constant}, \quad (7)$$

and let the outer cylinder of radius R_2 be kept stationary. Owing to the shear the fluid between cylinders is gradually moved. Its velocity is of the form $(2)_1$, the governing equations are given by (4) and (6) while the appropriate initial and boundary conditions are [4, 8–11, 17, 20]

$$w(r,0) = \frac{\partial w(r,0)}{\partial t} = 0, \;\; \tau(r,0) = 0; \;\; r \in (R_1,R_2], \quad (8)$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau(r,t)\Big|_{r=R_1} = \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)w(r,t)\Big|_{r=R_1} = f; \tag{9}$$
$$w(R_2,t)=0; \quad t>0.$$

Of course, $\tau(R_1,t)$ given by (7) is just the solution of the differential equation $(9)_1$.

3.1. Calculation of the Velocity Field

Multiplying (6) by $rB(rr_n)$, where

$$B(rr_n)=J_1(rr_n)Y_2(R_1r_n)-J_2(R_1r_n)Y_1(rr_n),$$

r_n is a positive root of the transcendental equation $B(R_2r)=0$ and $J_p(\cdot)$, $Y_p(\cdot)$ with $p=1,2$ are standard Bessel functions, and using the identity

$$\int_{R_1}^{R_2} rB(rr_n)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)w(r,t)\mathrm{d}r = \frac{2}{\pi r_n}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)w(r,t)\big|_{r=R_1}-r_n^2w_{nH}(t), \tag{10}$$

as well as the boundary condition $(9)_2$, we find that

$$\lambda\ddot{w}_{nH}(t)+(1+\alpha r_n^2)\dot{w}_{nH}(t)+\nu r_n^2w_{nH}(t)=\frac{2f}{\rho\pi r_n}, \quad t>0. \tag{11}$$

In the above relations, the function $w_{nH}(\cdot)$ defined by

$$w_{nH}(t)=\int_{R_1}^{R_2} rw(r,t)B(rr_n)\mathrm{d}r, \quad n=1,2,3,\ldots \tag{12}$$

is the finite Hankel transform of $w(r,.)$. In view of $(8)_{1,2}$, it must satisfy

$$w_{nH}(0)=\dot{w}_{nH}(0)=0. \tag{13}$$

The solution of the ordinary differential equation (11), with the initial conditions (13), has the simple form

$$w_{nH}(t)=\frac{2f}{\mu\pi r_n^3}\left[1-\frac{q_{2n}\mathrm{e}^{q_{1n}t}-q_{1n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}\right], \tag{14}$$

where

$$q_{1n},q_{2n}=\frac{-(1+\alpha r_n^2)\pm\sqrt{(1+\alpha r_n^2)^2-4\nu\lambda r_n^2}}{2\lambda}.$$

Now, applying the inverse Hankel transform formula [21]

$$w(r,t)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2J_1^2(R_2r_n)B(rr_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}w_{nH}(t) \tag{15}$$

to (14) and using the identity

$$\int_{R_1}^{R_2}(r^2-R_2^2)B(rr_n)\mathrm{d}r=\frac{4}{\pi r_n^3}\left(\frac{R_2}{R_1}\right)^2, \tag{16}$$

we find for the velocity field $w(r,t)$ the simple expression

$$w(r,t)=\frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right) -\frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(rr_n)}{r_n[J_2^2(R_1r_n)-J_1^2(R_2r_n)]}\frac{q_{2n}\mathrm{e}^{q_{1n}t}-q_{1n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}. \tag{17}$$

3.2. Calculation of the Shear Stress

Solving (4) with respect to $\tau(r,t)$ and bearing in mind the initial condition $(8)_3$, we find that

$$\tau(r,t)=\frac{\mu}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}}\int_0^t\mathrm{e}^{\frac{\tau}{\lambda}}\left(1+\lambda_r\frac{\partial}{\partial\tau}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)w(r,\tau)\mathrm{d}\tau. \tag{18}$$

Substituting (17) into (18) and using the identities

$$q_{1n}q_{3n}=-\nu r_n^2\frac{1+\lambda_r q_{1n}}{\lambda}, \quad q_{2n}q_{4n}=-\nu r_n^2\frac{1+\lambda_r q_{2n}}{\lambda}, \quad q_{3n}q_{4n}=\nu r_n^2\frac{\lambda-\lambda_r}{\lambda^2},$$

where $q_{3n}=q_{1n}+1/\lambda$ and $q_{4n}=q_{2n}+1/\lambda$, we get after lengthy but straightforward computations the simple form of the shear stress

$$\tau(r,t)=f\left(1-\mathrm{e}^{-\frac{t}{\lambda}}\right)\left(\frac{R_1}{r}\right)^2 +\frac{\pi f}{\lambda}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)\widetilde{B}(rr_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}\frac{\mathrm{e}^{q_{2n}t}-\mathrm{e}^{q_{1n}t}}{q_{2n}-q_{1n}}, \tag{19}$$

where

$$\widetilde{B}(rr_n)=J_2(rr_n)Y_2(R_1r_n)-J_2(R_1r_n)Y_2(rr_n).$$

4. Limiting Cases

4.1. Maxwell Fluid

Taking the limit as $\lambda_{\rm r} \to 0$ in (17) and (19), the solutions

$$w_{\rm M}(r,t) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right) - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(rr_n)}{r_n[J_2^2(R_1r_n)-J_1^2(R_2r_n)]}\,\frac{q_{6n}e^{q_{5n}t}-q_{5n}e^{q_{6n}t}}{q_{6n}-q_{5n}}, \tag{20}$$

$$\tau_{\rm M}(r,t) = f\left(1-e^{-\frac{t}{\lambda}}\right)\left(\frac{R_1}{r}\right)^2 + \frac{\pi f}{\lambda}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)\widetilde{B}(rr_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}\,\frac{e^{q_{6n}t}-e^{q_{5n}t}}{q_{6n}-q_{5n}} \tag{21}$$

corresponding to a Maxwell fluid performing the same motion are obtained. In the above relations

$$q_{5n}, q_{6n} = \frac{-1\pm\sqrt{1-4\nu\lambda r_n^2}}{2\lambda}.$$

4.2. Second-Grade Fluid

When $\lambda \to 0$ in (17) and (19), we obtain the solutions

$$w_{\rm SG}(r,t) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right) - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(rr_n)}{r_n[J_2^2(R_1r_n)-J_1^2(R_2r_n)]}\exp\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2}\right), \tag{22}$$

$$\tau_{\rm SG}(r,t) = f\left(\frac{R_1}{r}\right)^2 + \pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)\widetilde{B}(rr_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}\,\frac{1}{1+\alpha r_n^2}\exp\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2}\right) \tag{23}$$

corresponding to a second-grade fluid, where $B(rr_n)$ and $\widetilde{B}(rr_n)$ have been defined previously.

4.3. Newtonian Fluid

Finally when $\lambda \to 0$ in (20) and (21) or $\lambda_{\rm r} \to 0$ and then $\alpha \to 0$ into (22) and (23), the solutions

$$w_{\rm N}(r,t) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right) - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(rr_n)}{r_n[J_2^2(R_1r_n)-J_1^2(R_2r_n)]}e^{-\nu r_n^2 t}, \tag{24}$$

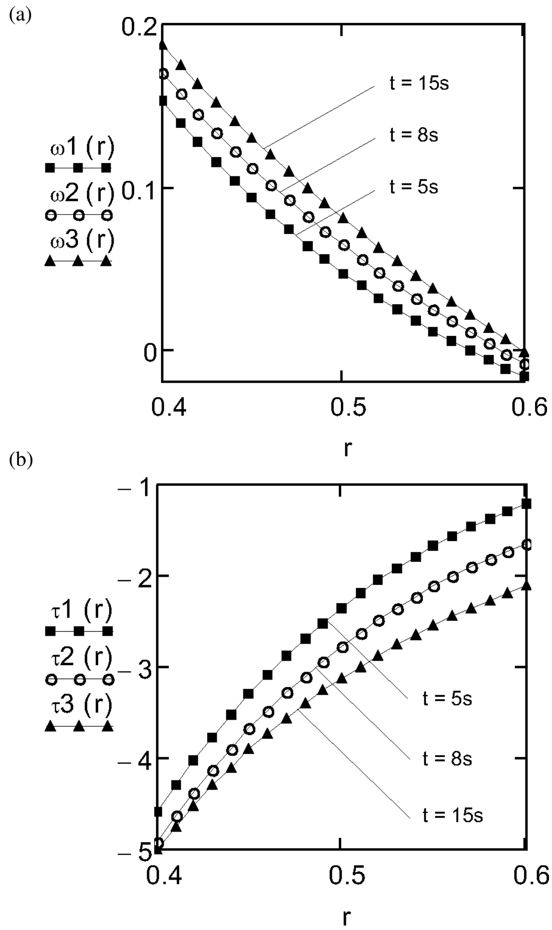

Fig. 1. Profiles of the velocity $w(r,t)$ given by (17) and shear stress $\tau(r,t)$ given by (19), for $\nu = 0.003$, $\mu = 2.916$, $R_1 = 0.4$, $R_2 = 0.6$, $f = -5$, $\lambda = 2$, $\lambda_{\rm r} = 1$, and different values of t.

$$\tau_{\rm N}(r,t) = f\left(\frac{R_1}{r}\right)^2 + \pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)\widetilde{B}(rr_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}e^{-\nu r_n^2 t} \tag{25}$$

for a Newtonian fluid are recovered.

Of course when $\lambda \to 0$ in (7), we find that

$$\tau(R_1,t) = f. \tag{26}$$

Consequently, the solutions (22) and (23), as well as (24) and (25), correspond to a constant couple on the boundary. For large values of t these solutions, as well as the solutions corresponding to Maxwell and Oldroyd-B fluids, tend to the steady solutions

$$w_{\rm s}(r) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r-\frac{R_2^2}{r}\right), \quad \tau_{\rm s}(r) = f\left(\frac{R_1}{r}\right)^2, \tag{27}$$

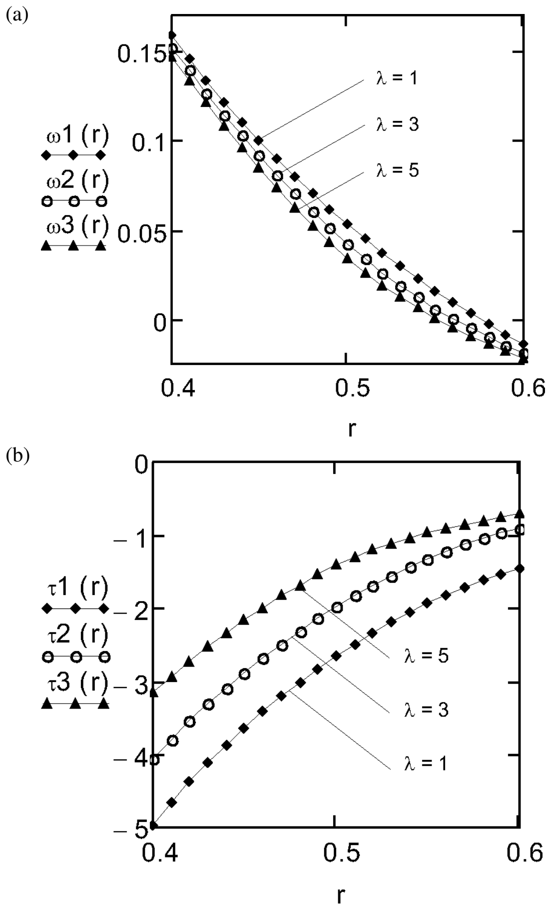

Fig. 2. Profiles of the velocity $w(r,t)$ given by (17) and shear stress $\tau(r,t)$ given by (19), for $\nu = 0.003$, $\mu = 2.916$, $R_1 = 0.4$, $R_2 = 0.6$, $f = -5$, $\lambda_r = 0.5$, $t = 5$ s and different values of λ.

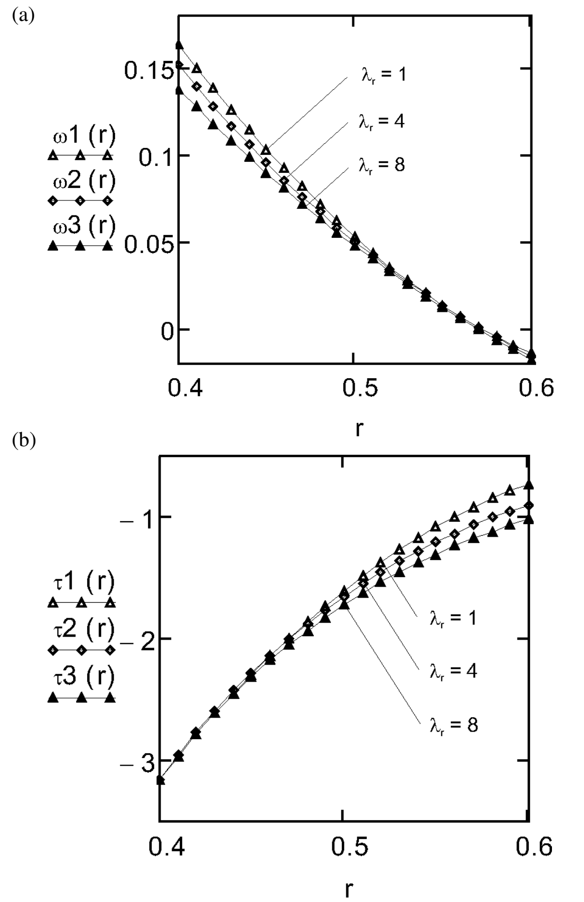

Fig. 3. Profiles of the velocity $w(r,t)$ given by (17) and shear stress $\tau(r,t)$ given by (19), for $\nu = 0.003$, $\mu = 2.916$, $R_1 = 0.4$, $R_2 = 0.6$, $f = -5$, $\lambda = 10$, $t = 10$ s and different values of λ_r.

which are the same for all types of fluids. This is not a surprise, since for large values of t the boundary condition (7) tends to that given by (26). In conclusion, after some time, the behaviour of a non-Newtonian fluid can be approximated by that of a Newtonian fluid. Depending on the constant f, the radii R_1 and R_2 of the cylinders and the material constants, this time will be ascertained graphically.

5. Numerical Results and Conclusions

Our aim was to provide exact solutions for the velocity field $w(r,t)$ and the shear stress $\tau(r,t)$ corresponding to the flow of an Oldroyd-B fluid between two infinite co-axial cylinders, the inner one being subject to a time-dependent torque. The solutions that have

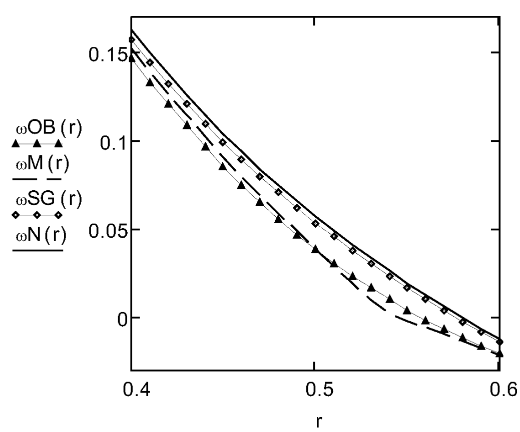

Fig. 4. Profiles of the velocity $w(r,t)$ for Oldroyd-B, Maxwell, second-grade, and Newtonian fluids, for $\nu = 0.003$, $\mu = 2.916$, $R_1 = 0.4$, $R_2 = 0.6$, $f = -5$, $\lambda = 4$, $\lambda_r = 1$, and $t = 5$ s.

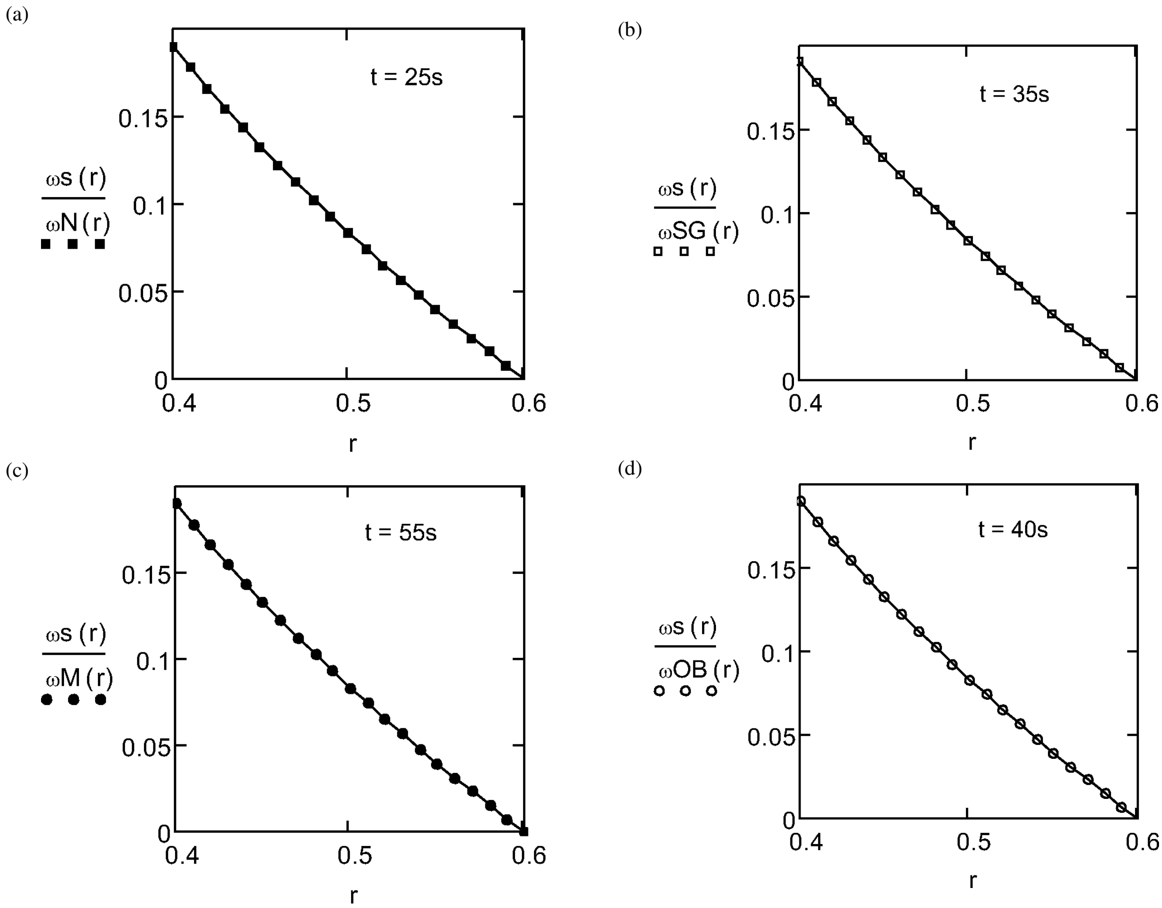

Fig. 5. Required time to reach the steady-state for Newtonian, second-grade, Maxwell, and Oldroyd-B fluids, for $\nu = 0.003$, $\mu = 2.916$, $R_1 = 0.4$, $R_2 = 0.6$, $f = -5$, $\lambda = 3$, and $\lambda_r = 1$.

been obtained, presented under series form in terms of Bessel functions $J_1(\cdot)$, $J_2(\cdot)$, $Y_1(\cdot)$, and $Y_2(\cdot)$, satisfy both the governing equations and all imposed initial and boundary conditions. They can easily be simplified to give similar solutions for Maxwell, second-grade, and Newtonian fluids. The solutions for second-grade and Newtonian fluids correspond to a constant torque f on the boundary. For large values of t, all solutions tend to the steady solutions $w_s(r)$ and $\tau_s(r)$, which are the same for all kinds of fluids, although the motion of the rate type fluids (Maxwell and Oldroyd-B) is due to a time dependent shear stress on the boundary. It is worth pointing out that, for large times, this time-dependent shear stress tends to the same constant value f.

In order to exhibit some relevant physical aspects of the obtained results, the diagrams of the velocity $w(r,t)$ and of the shear stress $\tau(r,t)$ are depicted against r for different values of t and of the material constants. From Figures 1a and 1b the influence of the rigid boundary on the fluid motion is clearly evident. The velocity of the fluid, as well as the shear stress in absolute value, is an increasing function of t and a decreasing one with respect to r. Figures 2 and 3 show the influence of the relaxation and retardation times λ and λ_r on the fluid motion. Their effect on the velocity field $w(r,t)$ and the shear stress $\tau(r,t)$ is qualitatively the same, excepting a small neighbourhood near the outer cylinder. On this neighbourhood the velocity of the fluid modifies its monotony with respect to λ_r.

Finally, for comparison, the diagrams of $w(r,t)$ corresponding to the four models (Oldroyd-B, Maxwell, second-grade, and Newtonian) are presented in Figure 4 for the same values of the common material constants and the time t. In all cases the velocity of the fluid is a decreasing function with respect to r and the

Newtonian fluid is the swiftest while the Oldroyd-B fluid is the smallest in the region near the moving cylinder. In practice, it is necessary to know the approximate time after which the fluid is moving according to the steady-state solutions. This time, as it results from Figures 5, is the smallest for the Newtonian fluids and the biggest for Maxwell fluids. The units of the parameters into Figures 1 – 5 are SI units, and the roots r_n have been approximated by $(2n-1)\pi/[2(R_2-R_1)]$.

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