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Vibration Analysis of Composite Beams with Sinusoidal Periodically Varying Interfaces

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Abstract: As an increasing variety of composite materials with complex interfaces are emerging, we develop a theory to investigate composite beams and shed some light on new physical insights into composite beams with sinusoidal periodically varying interfaces. For the natural vibration of composite beams with continuous or periodically varying interfaces, the governing equation has been derived according to the generalised Hamiltonian principle. For composite beams having different boundary conditions, we transform the governing equations into integral equations and solve them by using the sinusoidal functions as test functions as well as the basis of the vibration modes. Due to the orthogonality of the sinusoidal functions, expansion coefficients in closed form can be found. Therefore, the proposed iterative schemes, with the help of the Rayleigh quotient and boundary functions, can quickly find the eigenvalues and free vibration modes. The obtained natural frequencies agree well with those obtained using the finite element method. In addition, the proposed method can be extended easily to laminated composite beams in more general cases or complex components and geometries in vibration engineering. The effects of different material properties of the upper and lower components and varying interface geometry function on the frequency of the composite beams are examined. According to our investigation, the natural frequency of a laminated beam with a continuous or periodically varying interface can be changed by altering the density or elastic modulus. We also show the responses of the frequencies of the components to the varying periodic interface.

Keywords: Bernoulli–Euler Composite Beam; Boundary Functions; Distinct Material Properties; Iterative Scheme; Natural Frequencies.

1 Introduction

Composite laminates were successfully applied in many engineering applications due to their excellent characteristics like light weight, fatigue resistance and high strength and stiffness [1]. For the same reason those composite beams act as light-weight-bearing structures, of which the most significant topic that has emerged from composite design is vibration analysis. Hajianmaleki and Qatu [2] presented a review that focused on the past 40 years’ investigation on the topic of vibration analysis of composite laminates. Other inspiring works include, but not limited to, Lenci and Rega [3] who studied the free vibration of two-layer beams using the asymptotic development method, while neglecting the shear deformations, interface normal compliance and axial and rotational inertia. Li and Qiao [4] studied on interesting laminated composite beams that were shear deformable and anisotropic for their geometrical nonlinearity. The composites rested on an elastic foundation, and the beam properties were fibre-reinforced and linearly elastic. Nazemnezhad [5] adopted the nonlocal Timoshenko model to investigate the free vibration of cantilever multi-layer graphene nano-ribbons by using molecular dynamics simulations. Szekrényes [6] analysed the free vibration of a delaminated composite beam, of which the single delamination could be described by six other equivalent single layers in a certain circumstance.

In recent decades, composite beams with variable cross-sections have received much attention, for they are needed to achieve superior structural performance to better fit the stress distribution. For example, some micro-electromechanical devices that are manufactured in the shapes of non-uniform composite beams [7] or nanotubes [8] are widely adopted. To avoid or achieve a certain controllable vibrational performance and architectural design in composite structures, composite beams with sinusoidal periodically varying interfaces are first investigated in this research. We try to control the vibration of
the beams by changing not only the physical properties but also the geometrical structure of the components. The composite beam with a continuous or periodic interface is shown in Figure 1. The interface geometry function can be written as \( f(x) = f_0 \sin \frac{2\pi Nx}{l} \), where \( f_0 \) is the amplitude, \( N \) is the number of wave, and \( l \) is the length of the beam. When the number of wave \( N \) is an integer, the interface is periodic; otherwise, it is continuous. The configuration of this beam is similar to a buckling stiff nanowire on a soft substrate in the work of Chen et al. [9]. A growing number of composite materials with periodically varying interfaces are emerging in both literature and practice. These include, but not limited to, sandwich panels with foam-filled sinusoidal plate cores [10, 11] used as energy absorption materials, double-ceramic-layer thermal barrier coatings [12], and bio-composite materials such as muscles and vertebrae of fish and snakes. It has been shown that the resonance effect of the fish/snake body can lead to optimal efficiency [13]. The periodically varying interfaces in the above-mentioned composite materials can be complex, depending on the application circumstances; here, we focus on a structure composed of two materials with a sinusoidal interface and expect that the basic principles and mathematical calculation methods that are developed in this work may also be applicable to composites with more complex interface structures.

Many previous works were focused on analytical and computational methods for dealing with the non-linear vibration equations. An important review of the techniques involved in the development of advanced beam models has been presented by Carrera et al. [14]. In this review, six methods were briefly discussed: shear correction factors, which are introduced to enhance the classical beam theories; generalised beam theory; the so-called warping functions; the Saint-Venant model and the proper generalised decomposition; variational asymptotic methods; and the Carrera unified formulation. In addition, by using the spectral finite element method, Li et al. [15] performed free vibration analysis on a generally laminated beam when the boundary conditions were arbitrary. A pre-correction multiscale strategy was developed by Zhang et al. [16] to analyse the performances of heterogeneous multi-phase materials when the material properties were temperature-dependent. In this strategy, the microscopic heterogeneous properties were incorporated into the calculation of the macroscopic response by constructing the numerical basis functions based on the multiscale finite element formulation. Chiorean [17] proposed a novel computational method, based on the incremental-iterative arc-length technique, for analysing the ultimate strength of steel-concrete cross-sections that were subject to biaxial bending and axial forces. Brighenti and Bottoli [18] used a finite element method (FEM) to study the composite cross-section beam while considering the deformable connections between the constituents. The stiffness matrix was derived by the direct stiffness method, which came from theoretical solutions that were found in past research. Sayyad and Ghugal [19] reviewed the recent research on the free vibration of multi-layered composites by different numerical methods. Yang et al. [20] investigated the linear time-varying parameter system of deploying beams by the assumed-mode truncation method. Carrera et al. [21] proposed a higher-order multi-line approach for studying slender bodies. The method showed particular advantages for the investigation of reinforced structures with cross-sections composed of different physical components that were connected with each other by welded joints or mechanical fasteners.

Although the method we used in this work is based on the Rayleigh quotient, several unique and novel features in the algorithm are proposed: a variable transformation is realised to speed up the convergence of the iterative scheme; varying boundary functions are adopted, according to the boundary conditions; and the test functions and trial solution are both of sine-function type so that the orthogonality trick can be utilised. The iterative algorithm presented here is clear, reasonable and easy to follow and is even applicable to more complex cross-section composites, as long as the interface geometry function is continuous.

![Figure 1: The composite beam considered in this work is composed of two materials: one material for the upper beam and another for the lower beam. The blue solid line represents the interface of the composite, while the dashed line represents the neutral plane.](image)

### 2 Modelling and Mathematical Formulation

Consider a composite beam of rectangular cross-section that is composed of two sub-beams (Fig. 1). The Euler-Bernoulli hypothesis is applied here, under the assumptions that normals to the neutral surface remain normal
during the deformation and the deflections are small. Therefore, shear deformation is neglected.

Assume that the interface geometry function is
$$\sin, \quad f(x) = f_0 \sin \frac{2\pi Nx}{l},$$
where $f_0$ is the amplitude and $N$ is the number of wave. The cross-sectional area, elastic modulus, height, density and moment of inertia, of the upper and lower beams, are $A_i(x), E_i, h_i, \rho_i, I_i(x)$ and $A_i(x), E_i, h_i, \rho_i, I_i(x)$, respectively. The width of the beam is $b$ and the length is $l$, $y=0$ is located at the interface. Taking the plane of $y=0$ as the reference plane for the moment of inertia of the composites, the governing equation of the transverse vibration for the composite beam with an $N$-wave interface can be derived similarly to the one in [22]:

$$A(x) \frac{\partial^2 \omega(x, t)}{\partial t^2} + \frac{\partial^2 \omega(x, t)}{\partial x^2} \left( I(x) \frac{\partial^2 \omega(x, t)}{\partial x^2} \right) = 0, \quad t > 0, \quad 0 < x < l \quad (1)$$

where

$$A(x) := \rho_1 A_1(x) + \rho_2 A_2(x) \quad (2)$$

$$I(x) := E_i I_1(x) + E_2 I_2(x) \quad (3)$$

The cross-sectional areas and the moments of inertia of the upper and lower beams are

$$A_1(x) = \left[ h_1 - f(x) \right] b = \left[ h_1 - f_0 \sin \frac{2\pi N x}{l} \right] b$$

$$A_2(x) = \left[ h_2 + f(x) \right] b = \left[ h_2 + f_0 \sin \frac{2\pi N x}{l} \right] b$$

$$I_1(x) = \frac{1}{3} b \left[ h_1^3 - f_1^3(x) \right] = \frac{1}{3} b \left[ h_1^3 - f_0^3 \sin^3 \frac{2\pi N x}{l} \right]$$

$$I_2(x) = \frac{1}{3} b \left[ f_2^3 + h_2^3 \right] = \frac{1}{3} b \left[ f_0^3 \sin^3 \frac{2\pi N x}{l} + h_2^3 \right] \quad (4)$$

Assume the following harmonic vibration:

$$w(x, t) = y(x) \sin(\omega t) \quad (5)$$

We can derive

$$y^{(n)}(x) = \frac{1}{I(x)} \left[ \omega^2 A(x) y(x) - 2I'(x) y''(x) - I''(x) y'''(x) \right] \quad (6)$$

In this research, we investigate the free vibration of a beam with two simply supported ends only. The following boundary conditions are imposed:

$$y(0) = y(l) = 0 \quad (7)$$

$$y''(0) = y''(l) = 0 \quad (8)$$

Equation (6) under the above boundary conditions is a fourth-order Sturm–Liouville problem. As an immediately applicable method that is easy to implement, we hope the following algorithm will shed some light on other similar fourth-order Sturm–Liouville problems.

3 Mathematical Transformation and Calculation

To derive the Rayleigh quotient [23, 24], (6) is rewritten as

$$\omega^2 A(x) y(x) = (I(x) y''(x))'' \quad (6')$$

By multiplying (6’) by $y(x)$ and integrating it from 0 to $l$, the left-side of the equation becomes $\omega^2 \int_0^l A(x) y^2(x) dx$. By using the boundary conditions (7, 8), the right-side of (6’) becomes

$$\int_0^l (I(x) y''(x))^2 y(x) dx = -\int_0^l (I(x) y''(x))' y'(x) dx + (I(x) y''(x)) y(x) \bigg|_0^l = -\int_0^l (I(x) y''(x))' y'(x) dx \quad (7)$$

$$= \int_0^l I(x) [y'''(x)]^2 dx - I(x) y''(x) y'(x) \bigg|_0^l = \int_0^l I(x) [y'''(x)]^2 dx \quad (8)$$

Thus,

$$\lambda = \omega^2 = \frac{\int_0^l I(x) [y'''(x)]^2 dx}{\int_0^l A(x) y^2(x) dx} \quad (9)$$

To obtain the vibration modes, we employ the following strategy of variable transformation. In the Sturm–Liouville problem, $y=0$ is a trivial solution. By using (6), the iterations converge to the trivial solution. To avoid this, we apply the following variable transformation:

$$u(x) = y(x) + B(x) \quad (10)$$

where

$$B(x) = l^3 x - 2lx^3 + x^4 \quad (11)$$

$B(x)$ is defined as a boundary function of a simply supported beam because it automatically satisfies boundary conditions (7) and (8). In this work, we mainly investigate
the first-order natural transverse frequencies. For the
higher-order frequencies of composite beams with differ-
ent boundary conditions, the technique presented below
is also applicable, but with the varying boundary func-
tion \(B(x)\); different boundary functions can be found in
Appendix A.

By substituting (10) into the governing equation and
using the boundary conditions, a new ordinary differen-
tial equation is obtained:

\[
\frac{d^4}{dx^4} u(x) - \lambda u(x) = F(x, u, u', u'', \lambda), \quad 0 < x < l
\]

(12)

\[
u(0) = u(l) = 0
\]

(13)

\[
u''(0) = u''(l) = 0
\]

(14)

Let \(F(x)\) denote \(F(x, u, u', u'', \lambda)\):

\[
F(x) := \lambda \left[ \frac{A(x)}{I(x)} - 1 \right] u(x) - \frac{\lambda A(x)}{I(x)} B(x)
\]

\[
- \frac{I''(x)}{I(x)} [u'''(x) - B''(x)]
\]

\[
- \frac{2I'(x)}{I(x)} [u''''(x) - B'''(x)] + B''''(x)
\]

(15)

The trick of adding the term \(-\lambda u(x)\) to the left-hand
side of (12) is essential for the fast convergence of the itera-
tive scheme.

Define the test functions as

\[
v(x, j) := \sin \left( \frac{j \pi x}{l} \right), \quad j \in \mathbb{N}
\]

(16)

Multiplying (12) by the test functions and integrating
it from 0 to \(l\) yields

\[
\int_0^l u^{(iv)}(x)v(x, j)dx - \lambda \int_0^l u(x)v(x, j)dx = \int_0^l \nu(x, j)F(x)dx
\]

(17)

By using (13) and (14), the first integral term in the left-
hand side can be expressed as

\[
\int_0^l u^{(iv)}(x)v(x, j)dx = \int_0^l u(x)v^{(iv)}(x, j)dx
\]

(18)

From (16), it follows that

\[
v^{(iv)}(x, j) = \left( \frac{j \pi}{l} \right)^4 v(x, j)
\]

(19)

Inserting (18, 19) into (17), we can derive the following

\[
\textbf{Theorem 1:} \quad \text{For the fourth-order Sturm–Liouville eigenvalue value problem (12)–(14), the eigenfunction } u(x), \text{ eigenvalue } \lambda \text{ and given function } F(x) \text{ satisfy the following equation:}
\]

\[
\left[ \left( \frac{j \pi}{l} \right)^4 - \lambda \right] \int_0^l u(x)v(x, j)dx = \int_0^l v(x, j)F(x)dx, \quad j \in \mathbb{N}
\]

(20)

A new iterative algorithm will be developed according
to Theorem 1. Now, we can rewrite the Rayleigh quotient
using the variable transformation:

\[
\lambda := \omega^2 = \frac{\int_0^l I(x)[u''''(x) - B''''(x)]^2 dx}{\int_0^l A(x)[u(x) - B(x)]^2 dx}
\]

(21)

Notice that the sine functions automatically satisfy
boundary conditions (13) and (14). Hence, the trial solution
of the simply supported beam can be expressed as

\[
u(x) = \sum_{j=1}^m c_j \sin \left( \frac{j \pi x}{l} \right)
\]

(22)

Different trial solutions for varying boundary condi-
tions can be found in Appendix B. When the test functions
and the trial solution are both of sine-function type, the
orthogonality trick is adopted:

\[
\int_0^l \sin \left( \frac{j \pi x}{l} \right) \sin \left( \frac{k \pi x}{l} \right) dx = \frac{l}{2} \delta_{jk}
\]

(23)

where \(\delta_{jk}\) is the Kronecker delta symbol. Equation (22) is
substituted into (20) and the orthogonality is exploited.
Letting \(j = 1, \ldots, m\), the expansion coefficients \(c_j\)
\(j = 1, \ldots, m\) can be expressed as

\[
c_j = \frac{2}{l} \left[ \left( \frac{j \pi}{l} \right)^4 - \lambda \right] \int_0^l v(x, j)F(x)dx, \quad j = 1, \ldots, m
\]

(24)

In Appendix C, the iterative algorithm proposed in
this research is summarised step by step.

\section*{4 Results and Discussion}

\subsection*{4.1 Validation and Comparison}

Consider the free vibration of simply supported com-
posite beams with the interface \(f(x) = 0.01 \sin(2\pi N x/l)\), where
the number of wave \(N = 5\). The material constants are
The beam width is \( b = 0.1 \) (m). The upper height is \( h_1 = 10^{-10} \) (m), the iterative scheme converges after seven iterations, as shown in Figure 2a. In the last step, the relative norm is zero, so it cannot be plotted in the figure. We plot the convergence of another iterative scheme in Figure 2b, for which \( N = 2 \) and \( m = 5 \); other parameters remain the same. It converges after six steps in Figure 2b.

Furthermore, in Table 1, the results of the FEM using ABAQUS [25] are compared with the first-order transverse frequencies of composites with periodically varying interfaces that were obtained in this research. The relative differences between these results are also revealed. In the FEM simulations, we used a plane stress model instead of a 3D model for simplicity. An example FEM model with a 10-wave interface is shown in Figure 3 with a magnified view of the mesh density. CPS4 elements (4-node bilinear plane stress quadrilateral) are employed to mesh the system. The interface is assumed to be perfectly bonded. Test simulations of 3D models meshed by 3D elements (C3D8) are also conducted, which lead to essentially the same results as the plane stress model (Fig. 3).

Additionally, as both are numerical methods, we have compared the computation times: the computation time for FEM depends on the mesh density of the system. Generally, for a 2 (m)-long composite beam with a typical mesh size of \( 2 \times 10^{-3} \) (m), the computation time is approximately 15 s on a laptop with four 2.5 GHz CPUs and 8 GB of RAM. The computation time for the test function method depends on the number of terms of the trial solution \( m \) and the required precision \( \varepsilon \). Typically, with the parameters \( m = 30 \) and \( \varepsilon = 10^{-10} \), as adopted in this research, the calculation time for the test function method is approximately 1.6 s on the same laptop.

![Figure 3: An example FEM model with a 10-wave interface with a magnified view of the mesh density.](image)

Furthermore, for FEM, if we change the periodically varying interface of the composite beam, it will take us 7–8 m to remodel the system. However, simply changing values of parameters in the program of the test function method can conveniently lead to the required results and convergence.

**Table 1:** Comparison between the results obtained by the FEM and those obtained by the proposed method when \( f(x) = 0.01 \sin(2 \pi x / l) \) and \( b = 0.1 \) (m).

<table>
<thead>
<tr>
<th>( h_1 ) = 0.04 m, ( h_2 ) = 0.06 m, ( l = 2 ) m</th>
<th>Analysis (rad/s)</th>
<th>Abaqus (rad/s)</th>
<th>Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_1 = \rho_2 = 7.8 \times 10^3 ) kg/m(^3)</td>
<td>381.98</td>
<td>370.22</td>
<td>3.08</td>
</tr>
<tr>
<td>( E_1 = E_2 = 200 \times 10^9 ) Pa</td>
<td>299.38</td>
<td>303.27</td>
<td>1.30</td>
</tr>
<tr>
<td>( \rho_1 = \rho_2 = 7.8 \times 10^3 ) kg/m(^3)</td>
<td>248.00</td>
<td>257.71</td>
<td>3.92</td>
</tr>
<tr>
<td>( E_1 = E_2 = 200 \times 10^9 ) Pa</td>
<td>450.80</td>
<td>436.91</td>
<td>3.08</td>
</tr>
<tr>
<td>( \rho_1 = 2.0 \times 10^3 ) kg/m(^3)</td>
<td>508.58</td>
<td>492.81</td>
<td>3.10</td>
</tr>
<tr>
<td>( E_1 = E_2 = 200 \times 10^9 ) Pa</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 2:** For the fourth-order simply supported composite beam solved by iterative algorithm, (a) \( N = 5 \) and (b) \( N = 2 \).
4.2 Amplitude and Number of Wave

In this research, we have assumed that the interface geometry of the beam components is a periodic function, i.e. \( f(x) = f_0 \sin(2\pi Nx/l) \). In this section, we will discuss the effects of the amplitude \( f_0 \) and the number of wave \( N \) of function \( f(x) \) on the first-order natural frequencies. Figure 4 indicates that the number of wave of function \( f(x) \) is small; specifically, it is no more than three. When the number of wave approaches 2, it appears as though the first-order frequency becomes steady. In contrast, the super-asymmetry in shape caused by the small number of wave \( N \) induces an interesting periodic change in frequency. When \( f_0 \) is relatively small compared with the height of the beam (0.01 compared with 0.1), the frequency remains almost the same until the number of wave \( N \) reaches 200. When \( N \) reaches 450, the frequency increases significantly. Compared with the case when the amplitude \( f_0 \) is 0.03 (30% of the height of the beam), the frequency will not be steady when the number of wave \( N \) reaches 10. According to Figure 5, the curve tends to be linear as soon as the number of wave \( N \) reaches 12. It is safe to conclude that as the amplitude of the interface geometry function increases, the frequency tends to increase due to the smaller number of wave \( N \) of the interface geometry function. Thus, when we try to alter the frequency of the composite beam,

\begin{equation}
\omega = \frac{f}{l} = \frac{E}{m} \left( \frac{n}{l} \right)^2
\end{equation}

According to Figure 4, as the number of wave \( N \) increases, the frequency tends to be steady. However, in the extreme situation that the material characteristics of the upper and lower beams are different, e.g. \( E_1 = 2 \times 10^{11} \) (N/m\(^2\)), \( E_2 = 0.02 \times 10^{11} \) (N/m\(^2\)), the frequency rises when the number of wave \( N \) increases significantly. When the amplitude \( f_0 \) is almost the same until the number of wave \( N \) reaches 450, the frequency increases significantly.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Effects of the amplitude \( f_0 \) and the number of wave \( N \) of interface geometry function \( f(x) = f_0 \sin(2\pi Nx/l) \) on the first-order frequency with \( \rho_s = 7.8 \times 10^2 \) (kg/m\(^3\)), \( \rho_s = 4 \times 10^4 \) (kg/m\(^3\)), \( E_s = 2 \times 10^{11} \) (N/m\(^2\)), \( E_s = 0.5 \times 10^{11} \) (N/m\(^2\)), \( l = 1 \) (m), \( b = 0.1 \) (m), \( h_1 = 0.06 \) (m), \( h_2 = 0.04 \) (m), and \( m = 10 \).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Effects of the amplitude \( f_0 \) and the number of wave \( N \) of interface geometry function \( f(x) = f_0 \sin(2\pi Nx/l) \) on the first-order frequency with \( \rho_s = 8.0 \times 10^1 \) (kg/m\(^3\)), \( \rho_s = 8.0 \times 10^4 \) (kg/m\(^3\)), \( E_s = 2 \times 10^{11} \) (N/m\(^2\)), \( E_s = 0.02 \times 10^{11} \) (N/m\(^2\)), \( l = 2 \) (m), \( b = 0.1 \) (m), \( h_1 = h_2 = 0.05 \) (m), and \( m = 10 \).}
\end{figure}
increasing the number of wave $N$ may not be an efficient approach when the amplitude $f_0$ is small.

### 4.3 Material Properties

In Figure 6, the first-order natural frequencies of composites with the interface geometry function $f(x) = 0.01 \sin(2\pi x/l)$ are depicted. The elastic modulus of the lower beam is fixed to $E_2 = 2 \times 10^{11}$ (N/m$^2$) in Figure 6a, and the upper beam has a smaller elastic modulus than the lower beam. The ratio ranges from 0.005 to 1. Decreasing the ratio causes the frequency to decrease. For comparison, the upper beam has a fixed elastic modulus of $E_1 = 2 \times 10^{11}$ (N/m$^2$) in Figure 6b, which is larger than that of the lower beam. The ratio ranges from 1 to 1000. According to Figure 6b, with a constant elastic modulus of the upper beams, the frequency decreases as the ratio rises. The curve in Figure 6a is considered to be linearity, while in Figure 6b, the nonlinear relationship between the ratio and the frequency is especially significant. According to the sharp curve, the frequency drops rapidly, especially in the range between 1 and 100. It seems that the most efficient way to change the frequency of a composite is to alter the elastic modulus ratio of the upper and lower beams from 1 to 100 while keeping the elastic modulus of the upper beam constant when no parameters except the elastic modulus can be fixed. Furthermore, Figure 6 not only reveals the effects of the material properties, e.g. the elastic modulus, on the natural frequency but also shows the efficiency of the numerical technique that is applied in this research, which can be used for calculations when the elastic modulus ratio of the components varies from $\frac{1}{200}$ to 1000.

In Figure 7, the first-order frequency is plotted as a function of the density ratio of the components of the two-layer beams. The frequency falls as the density of the upper beam is increased in Figure 7a. The red curve in Figure 7b shows that the frequency increases with decreasing density of the lower beam. In Figure 7a, the density ratio of the components varies from $\frac{1}{8}$ to 1, while in Figure 7b, $\rho_1/\rho_2$ changes from 1 to 8, i.e. $\rho_2/\rho_1$ changes from $\frac{1}{8}$ to 1. However, the two curves in these figures are neither correlative nor symmetric. The curve in Figure 7a is linear, while the one in Figure 7b is quadratic. It seems that even if we know the exact frequency when the ratio of $\rho_1/\rho_2$ is equal to a certain value, we cannot predict the frequency when we switch the densities of the upper and lower beams with each other. Comparing Figures 6 and 7, we can safely conclude that the larger the difference between the densities of the two components, the greater the frequency; and the smaller the difference between the

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**Figure 6:** Effect of the elastic modulus ratio of the upper and the lower beams on the first-order frequency with $f(x) = 0.01 \sin(2\pi x/l)$, $\rho_1 = \rho_2 = 7.8 \times 10^3$ (kg/m$^3$), $l = 2$ (m), $b = 0.1$ (m), $m = 10$, $h_1 = 0.04$ (m), and $h_2 = 0.06$ (m). (a) The upper beam has a smaller elastic modulus than the lower beam and (b) the elastic modulus of the upper beam is larger than that of the lower beam.
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Conclusions

Using the new iterative method proposed in this research, the equations describing composite beams with periodically varying interfaces are considered and potential design guidelines are given for the first time. Graphical depictions of the natural frequency of the transverse vibration of simply supported composite beams are presented. Finite element simulation, realised by the ABAQUS software (a product of Dassault Systèmes Simulia Corp., Providence, RI, USA), is adopted to validate the numerical techniques. The effects of different material properties of the upper and lower components and varying interface geometry function on the natural frequencies of composites are revealed. The most significant findings of this research are listed as follows:

(i) The super-asymmetry in shape caused by the small number of wave \( N(0.1\sim2) \) induces an interesting periodic change in frequency. As the amplitude of the interface geometry function \( f(x) \) increases, the change in the frequency with the varying number of wave \( N \) becomes more obvious.

(ii) When the amplitude of the interface geometry function increases, the frequency tends to increase, even with a small number of wave of the interface geometry function, when material properties of the upper and the lower components of beam are distinct.

(iii) The larger the difference between the densities of the components of the beams, the greater the frequency; and the smaller the difference between the elastic moduli of the components of the composite structure, the greater the frequency.

Above all, the natural frequencies of the composite beams can be manipulated in different ways, for example, by adjusting the elastic modulus or density. Furthermore, we have examined the responses of the frequencies of the components to varying periodic interfaces. By adopting composite structures with periodic interfaces, we can also avoid or achieve a certain vibrational performance. Moreover, the iterative technique proposed in this paper presents an easy way to calculate composite beams with varying periodic interfaces, and it can also be extended to study composite beams with a continuously differentiable interface function. Currently, our group is focusing on beams with zigzag interfaces and will explore beams with more complex interfaces in the near future.
Nomenclature

- $A$: cross-sectional area
- $b$: width of beam
- $E$: elastic modulus
- $f_o$: amplitude
- $h$: height of beam
- $I$: moment of inertia
- $l$: length of beam
- $N$: number of wave
- $w$: transverse vibration

- $x, y$: Cartesian coordinates along the beam and normal to it, respectively

Greek symbols

- $\rho$: density
- $\omega$: frequency
- $\delta_{jk}$: the Kronecker delta symbol

Subscripts

- 1 and 2: refer to quantities for upper and lower beams, respectively

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Appendix A

In the work above, the first-order transverse frequency of simply supported composite beams is calculated by a new technique that we proposed. The results are validated and compared with other data obtained by simulation software. In this new algorithm, it is important to select an appropriate boundary function $B(x)$ according to the boundary conditions and whether higher-order frequencies are needed. In the appendix, we only present the varying boundary function $B(x)$ for interested readers. The process used to find the proper boundary function $B(x)$ has been demonstrated in another work [26].

Different boundary conditions of composite beams are as follows:

For a simply supported beam, they are

$$y(0) = y(l) = 0, \quad y'(0) = y'(l) = 0$$  \quad (A1)

For a two-end fixed beam, they are

$$y(0) = y'(0) = y''(l) = 0$$  \quad (A2)

For a cantilever beam, they are

$$y(0) = y'(0) = 0, \quad y''(l) = y'''(l) = 0$$  \quad (A3)

Each boundary function $B_i(x)$ is a polynomial of at least fourth order that satisfies the boundary conditions automatically when we calculate the first-order frequency.

For a simply supported beam, the boundary function $B_i(x)$ is

$$B_i(x) = x^5 - 2l x^3 + l' x$$  \quad (A4)

For a two-end fixed beam, the boundary function $B_i(x)$ is

$$B_i(x) = x^5 - 2l x^3 + l' x^2$$  \quad (A5)

For a cantilever beam, the boundary function $B_i(x)$ is

$$B_i(x) = x^5 - 4l x^3 + 6l'^2 x^2$$  \quad (A6)

The above boundary functions are unique for each type beam because the leading coefficients are fixed to one. Other boundary functions can also be found; we have provided the simplest ones.

For calculating the second-order frequency, the following boundary function is adopted:

$$B_x = y B_2(x) + S_2(x)$$  \quad (A7)

where

$$\gamma = \frac{\int_0^l A(x) B_2(x) S_2(x) dx}{\int_0^l A(x) B_1^2(x) dx}$$  \quad (A8)

For a simply supported beam, $S_2(x)$ is

$$S_2(x) = x^5 - \frac{7l}{3} x^4 + \frac{4l^2}{3} x^3$$  \quad (A9)

For the two-end fixed beam, $S_2(x)$ is

$$S_2(x) = x^5 - 2l x^4 + l' x^3 = x B_1(x)$$  \quad (A10)

For a cantilever beam, $S_2(x)$ is

$$S_2(x) = x^5 - \frac{10l}{3} x^4 + \frac{10l'^2}{3} x^3$$  \quad (A11)

For calculating the third-order frequency, the following boundary function is adopted:

$$B_2(x) = \gamma_1 B_1(x) + \gamma_2 B_2(x) + S_2(x)$$  \quad (A12)

where
\[ \gamma_1 = \frac{\int_0^1 A(x) B_1(x) S_1(x) dx}{\int_0^1 A(x) B_1^2(x) dx} \]  \hspace{1cm} (A13) \\
\[ \gamma_2 = \frac{\int_0^1 A(x) B_2(x) S_2(x) dx}{\int_0^1 A(x) B_2^2(x) dx} \]  \hspace{1cm} (A14)

For a simply supported beam, \( S_1(x) \) is
\[ S_1(x) = x^6 - \frac{9l}{4} x^5 + \frac{5l}{4} x^4 \]  \hspace{1cm} (A15)

For the two-end fixed beam, \( S_1(x) \) is
\[ S_1(x) = x^6 - 2lx^5 + l^2 x^4 = x S_1(x) \]  \hspace{1cm} (A16)

For a cantilever beam, \( S_1(x) \) is
\[ S_1(x) = x^6 - 3lx^5 + \frac{5l^2}{2} x^4 \]  \hspace{1cm} (A17)

Based on the expressions given above, we can obtain the rule for calculating the \( i \)-th order frequency \( (i \geq 2) \). The boundary function \( B_i(x) \) should be of the form
\[ B_i(x) = \gamma_1 B_1(x) + \gamma_2 B_2(x) + \ldots + \gamma_{i-1} B_{i-1}(x) + S_i(x) \]  \hspace{1cm} (A18)

where
\[ \gamma_1 = -\frac{\int_0^1 A(x) B_1(x) S_1(x) dx}{\int_0^1 A(x) B_1^2(x) dx} \]  \hspace{1cm} (A19)

For a simply supported beam, \( S_i(x) \) is
\[ S_i(x) = x^{i+3} - \frac{2i + 3}{i + 1} i x^{i+2} + \frac{i + 2}{i + 1} i^2 x^{i+1} \]  \hspace{1cm} (A20)

For the two-end fixed beam, \( S_i(x) \) is
\[ S_i(x) = x^{i+3} - 2ix^{i+2} + l^2 x^{i+1} = x S_{i-1}(x) \]  \hspace{1cm} (A21)

For a cantilever beam, \( S_i(x) \) is
\[ S_i(x) = x^{i+3} - \frac{2(i+3)}{i+1} i x^{i+2} + \frac{(i+2)(i+3)}{i(i+1)} i^2 x^{i+1} \]  \hspace{1cm} (A22)

**Appendix B**

For a two-end fixed beam, the boundary conditions are
\[ y(0) = y(l) = 0 \]  \hspace{1cm} (B1)

The trial solution can be taken as
\[ u(x) = \sum_{j=1}^{m} c_j \sin \frac{j\pi x}{l} + c_{m+1} x \left( 1 - \frac{x}{l} \right) + c_{m+2} x^2 \left( 1 - \frac{x}{l} \right) \]  \hspace{1cm} (B2)

which automatically satisfies (B1). By comparing it with the case of the simply supported beam, we find that two additional unknown coefficients \( c_{m+1} \) and \( c_{m+2} \) appear in the trial solution. Thus, two more algebraic equations are necessary to determine \( c_j \) \((j = 1, \ldots, m + 2)\). \( u'(x) \) can be derived from (B3). By inserting it into (B2), we obtain the following two equations:
\[ \sum_{j=1}^{m} \frac{j\pi}{l} c_j + c_{m+1} = 0 \]  \hspace{1cm} (B4)

\[ -c_{m+1} - l c_{m+2} + \sum_{j=1}^{m} \frac{j\pi}{l} \cos(j\pi) c_j = 0 \]  \hspace{1cm} (B5)

For a cantilever beam, the boundary conditions are
\[ y(0) = y''(l) = 0 \]  \hspace{1cm} (B6)

\[ y'(0) = y'''(l) = 0 \]  \hspace{1cm} (B7)

The trial solution can be taken as
\[ u(x) = \sum_{j=1}^{m} c_j \sin \frac{j\pi x}{l} + c_{m+1} x + c_{m+2} \left( \frac{l x^2}{2} - \frac{x^3}{6} \right) \]  \hspace{1cm} (B8)

which automatically satisfies (B6). By the same token, we can obtain \( u'(x) \), \( u''(x) \), \( u'''(x) \) from (B8). By inserting the trial solution into (B7), we obtain the following two equations:
\[ \sum_{j=1}^{m} \frac{j\pi}{l} c_j + c_{m+1} = 0 \]  \hspace{1cm} (B9)

\[ c_{m+1} + \sum_{j=1}^{m} \left( \frac{j\pi}{l} \right)^3 \cos(j\pi) c_j = 0 \]  \hspace{1cm} (B10)

**Appendix C**

To obtain \( \lambda \) and \( y(x) \), we solve the eigenvalue problem by the following iterative algorithm:
(i) Given \( m \), \( \varepsilon \) and initial guesses of \( \lambda \) and \( c_j \);
(ii) For \( j = 1, \ldots, m \), calculate
\[ u(x) = \sum_{j=1}^{m} c_j \sin \left( \frac{j\pi x}{l} \right) \]

\[ u''(x) = -\sum_{j=1}^{m} c_j \left( \frac{j\pi}{l} \right)^2 \sin \left( \frac{j\pi x}{l} \right) \]

\[ u'''(x) = -\sum_{j=1}^{m} c_j \left( \frac{j\pi}{l} \right)^3 \cos \left( \frac{j\pi x}{l} \right) \]

Calculate \( F(x) \) by (15) and update \( c_j \) by (24).

(iii) Substitute \( u(x) \), \( u''(x) \) and the updated \( c_j \) of (ii) into (21) and calculate

\[ \lambda_{k+1} = \frac{\int_0^l I(x)[u''(x) - B''(x)]^2 \, dx}{\int_0^l A(x)[u(x) - B(x)]^2 \, dx} \]

If the convergence criterion

\[ \left| \lambda_{k+1} - \lambda_k \right| \leq \varepsilon \]

is satisfied, stop the iterations; otherwise, return to (ii). When \( u(x) \) is available, \( y(x) \) is computed by

\[ y(x) = u(x) - B(x) \]

References