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**Exact Solutions to Zakharov-Kuznetsov Equation with Variable Coefficients by Trial Equation Method**

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**Abstract:** By the trial equation method and the complete discrimination system for polynomial method, some exact solutions to Zakharov-Kuznetsov equation with variable coefficients are obtained. These solutions include solitary solutions, rational solutions, periodic solution and double periodic solutions.

**Keywords:** Complete Discrimination System for Polynomial Method; Exact Solution; Trial Equation Method; Zakharov-Kuznetsov Equation.

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1 Introduction

It is well known that sometimes the differential equations with variable coefficients are more suitable to describe some phenomenon in practice models than the differential equations with constant coefficients. To solve the exact solutions of such differential equations with variable coefficients is still an important problem in both theory and application. Many methods have been proposed to find the exact solutions for nonlinear differential equations. Among these, the complete discrimination system for polynomial method [1–10] and the trial equation method [11–16] proposed by Liu are two very powerful and useful tools. In Liu’s papers [1–6], a large number of applications have been given. In particular, in [12], Liu generalised his trial equation method to differential equations with variable coefficients. Liu’s methods have been extensively developed and applied to a lot of differential equations. For example, Liu [17], Du [18], and Gurefe et al. [19–24] generalised Liu’s method and proposed some new trial equation methods such as irrational trial equation method, extended trial equation method, modified trial equation method and multiple extended trial equation method to solve nonlinear differential equations arising from mathematical physics, and Yang [25] applied the complete discrimination system for polynomial method to give the envelope travelling wave solutions to (2+1)-dimensional Davey-Stewartson equation.

In this paper, by using the trial equation method and the complete discrimination system for polynomial method, we solve the three-dimensional Zakharov-Kuznetsov equation (ZK, for simplicity) with variable coefficients and give its exact solutions including solitary wave solutions, rational solutions, periodic solutions and elliptic function solutions.

The paper is organised as follows. In Section 2, we introduce the main steps of the trial equation method for three-dimensional differential equations with variable coefficients. In Section 3, we give the exact solutions for the ZK equation with variable coefficients. The final section is a short conclusion.

2 Trial Equation Method

According to Liu’s paper [10], we generalise the trial equation method to three-dimensional differential equation with variable coefficients as follows:

(i) **First step:** Consider the following three-dimensional real nonlinear differential equation with variable coefficients:

\[ Q(t, x, y, u, u_x, u_y, u_{xx}, ...) = 0. \]  

(1)

Take the travelling wave transformation

\[ u(x, y, t) = u(\xi), \quad \xi = k_1(t)x + k_2(t)y + \omega(t), \]  

(2)

where \( k_1(t), k_2(t) \) and \( \omega(t) \) are unknown functions to be determined. Substituting the transformation (2) into (1) yields an ordinary differential equation (ODE)

\[ N(t, x, k_1, k_2, \omega, k_1', k_2', \omega', u, u_1, u_2, ...) = 0. \]  

(3)
(ii) **Second step**: Take the following trial equation

\[ (u')^2 = F(u) = \sum_{i=0}^{m} a_i u^i, \] (4)

where \( m \) and \( a_i \) are constants to be determined for \( i = 0, \ldots, m \). Substituting the trial equation (4) and other derivative terms such as \( u'' \) or \( u''' \) and so forth into (3), we will get a polynomial \( H(u) \) of \( u \). By the balance principle, we will determine the value of \( m \). Furthermore, by letting the coefficients of \( H(u) \) be zero, we derive out a system of ODEs. We solve the ODEs system and determine the functions \( k_1(t), k_2(t), \omega(t) \) and the values of \( a_0, \ldots, a_m \).

(iii) **Last step**: Rewrite (4) by the integral form

\[ \int \frac{1}{\sqrt{F(u)}} du = \pm (\xi - \xi_0). \] (5)

By the complete discrimination system for \( m \)th order polynomial [1–10], we can give the classification of the roots of \( F(u) \), and furthermore solve the corresponding integral (5) and obtain the exact travelling wave solutions to (1).

3 **Exact Solutions to Zakharov-Kuznetsov Equation with Variable Coefficients**

The ZK equation arises from the plasma physics. In fact, in the plasma sheet boundary layer of the Earth’s magnetosphere and in the Van Allen radiation belts, there exists the plasmas with high-energy ion beams. Therefore, a topic interesting to study will be the propagation of ion-acoustic waves in a relativistic plasma with streaming ions. To consider the case of finite ion temperature is an important aim. In earlier studies, only the one-dimensional flow of the electrons and the ions had been invested. Kadomtsev and Petviashivili initially tried to model a soliton in a two-dimensional system [26], And then Zakharov and Kuznetsov modelled a soliton in a three-dimensional system [27]. For a non-relativistic magnetised plasma with \( T_e = 0 \), they obtained a three-dimensional differential equation now namely as the ZK equation. There are a large number of papers studying the ZK equation (see for example [21, 28–35]). Among these, in [32, 33], the symmetry and some exact solutions of the variable coefficients ZK equation had been studied.

In this paper, we use the previous trial equation method to solve the exact travelling wave solutions of the ZK equation with variable coefficients

\[ u_t + f(x, y, t)u_{xx} + g(x, y, t)u_{xxx} + h(x, y, t)u_{xy} = 0, \] (6)

where

\[ f(x, y, t) = f_1(t)x + f_2(t)y + f_3(t). \] (7)

This equation is widely used in various branches of physics, including plasma physics, fluid physics and quantum field theory [34–36].

Substituting travelling wave transformation (2) and trial equation (4) into (6), by the balance principle we get \( m = 3 \). The corresponding system of ODEs is given as follows:

\[ k_1'(t) x + k_2'(t) y + \omega'(t) + 2a_1 k_1^2(t) (g(x, y, t) k_1(t) + h(x, y, t) k_1(t)) = 0, \] (8)

\[ f(x, y, t) k_1(t) + 6 a_2 (g(x, y, t) k_2(t) + h(x, y, t) k_2(t)) = 0. \] (9)

From the form (7) of \( f(x, y, t) \), for the corresponding ODEs system, there exists a family of solutions if the following algebraic relation and ODEs hold,

\[ a_z = 3a_j, \] (10)

and

\[ k_z(t) = f_1(t) k_1(t), \] (11)

\[ k_2'(t) = f_2(t) k_1(t), \] (12)

\[ \omega'(t) = f_3(t) k_1(t). \] (13)

Solving the above ODEs system (11–13) gives

\[ k_1(t) = A_0 \exp \left( \int f_1(t) dt \right), \] (14)

\[ k_2(t) = A_0 \int f_2(t) \exp \left( \int f_1(t) dt \right) dt + B_0, \] (15)

\[ \omega(t) = A_0 \int f_3(t) \exp \left( \int f_1(t) dt \right) dt + C_0, \] (16)

where \( A_0, B_0 \) and \( C_0 \) are three arbitrary constants. In the following equations we provide all solutions to (5). For this purpose, let

\[ w = (a_j)^{\frac{1}{3}} u, \quad d_z = a_j (a_j)^{\frac{1}{3}}, \quad d_i = a_i (a_j)^{\frac{1}{3}}, \quad d_0 = a_0. \] (17)

Then (5) becomes
Denote
\[ F(w) = w^3 + d_2 w^2 + d_1 w + d_0. \]
Then
\[ \Delta = -27 \left( \frac{2d_1^3}{27} + d_0 - \frac{d_2 d_1^2}{3} \right) - 4 \left( d_1 - \frac{d_2^2}{3} \right)^3, \]
\[ D = d_1 - \frac{d_2^2}{3}, \]
make up a complete discrimination system for \( F(w) \). According to Liu [10], we have the following four cases:

**Case 1:** \( \Delta = 0, D < 0 \). Then we know that \( f \) has a real simple root and a real double root, that is, \( F(w) = (w-\alpha)^2(w-\beta) \), \( \alpha \neq \beta \). When \( w > \beta \), we get the following solitary wave solutions and periodic solution
\[
\begin{align*}
\alpha + \beta & \left\{ 1 - \tanh \left( \frac{\sqrt{\alpha - \beta}}{2} (a_j)^{\frac{1}{3}} (k_j(t)x + k_j(t)y + \omega(t) - \xi_0) \right) \right\} \\
= & \frac{\alpha + \beta}{(a_j)^{\frac{1}{3}}}, \quad \alpha > \beta; \\
\alpha + \beta & \left\{ 1 - \coth \left( \frac{\sqrt{\alpha - \beta}}{2} (a_j)^{\frac{1}{3}} (k_j(t)x + k_j(t)y + \omega(t) - \xi_0) \right) \right\} \\
= & \frac{\alpha + \beta}{(a_j)^{\frac{1}{3}}}, \quad \alpha > \beta; \\
\beta + (\alpha - \beta) & \left\{ 1 - \sec \left( \frac{\sqrt{\alpha - \beta}}{2} (a_j)^{\frac{1}{3}} (k_j(t)x + k_j(t)y + \omega(t) - \xi_0) \right) \right\} \\
= & \frac{\beta + (\alpha - \beta)}{(a_j)^{\frac{1}{3}}}, \quad \alpha < \beta.
\end{align*}
\]

**Case 2:** \( \Delta = 0, D > 0 \). Then \( F(w) = (w-\alpha)^3 \). We get a rational solution
\[ u_i = \alpha + \frac{2}{(a_j)^{\frac{1}{3}} (k_j(t)x + k_j(t)y + \omega(t) - \xi_0)^3}. \]

**Case 3:** \( \Delta > 0, D < 0 \). Then \( F(w) = (w-\alpha) (w-\beta) (w-\gamma) \). Without loss of generality, we suppose \( \alpha < \beta < \gamma \). When \( \alpha < w < \beta \), we have the Jacobian elliptic function solutions
\[ \begin{align*}
u_i &= \frac{(\beta - \alpha) \text{sn}^2 \left( \frac{\sqrt{\gamma - \alpha}}{2} (a_j)^{\frac{1}{3}} (k_j(t)x + k_j(t)y + \omega(t) - \xi_0), m \right) + \alpha}{(a_j)^{\frac{1}{3}}}, \\
&\quad \text{when } w > \gamma,
\end{align*} \]
where \( m^2 = \frac{\beta - \alpha}{\gamma - \alpha} \).

**Case 4:** \( \Delta < 0 \). Then \( F(w) \) has a real root and a pair of complex roots, that is, \( F(w) = (w-\alpha)(w^2 + pw + q) \) and \( p^2 - 4q < 0 \). We have the following Jacobian elliptic function solution
\[ u_i = \frac{\alpha - \sqrt{\alpha^2 + p\alpha + q}}{(a_j)^{\frac{1}{3}}}, \quad 2\sqrt{2} \frac{\alpha - \sqrt{\alpha^2 + p\alpha + q}}{(a_j)^{\frac{1}{3}}} \left\{ 1 + \text{cn} \left( (\alpha^2 + p\alpha + q)^{\frac{1}{3}} (a_j)^{\frac{1}{3}} (k_j(t)x + k_j(t)y + \omega(t) - \xi_0), m \right) \right\} \]
where \( m^2 = \frac{1}{2} \left( 1 - \frac{\alpha + p}{\sqrt{\alpha^2 + p\alpha + q}} \right) \).

According to the expressions (22–28), we have given the corresponding exact travelling wave solutions \( u_i \) \( (i = 1, \ldots, 7) \) to the ZK equation (6), where \( k_j(t), k_j(t) \), and \( \omega(t) \) are represented by (14–16). We can see that these solutions include solitary wave solutions, periodic solutions in terms of trigonometric functions, rational solutions and double periodic solutions in terms of Jacobian elliptic functions.

**Remark:** For a general function \( f(x, y, t) \) which is analytic at \( (0, 0, t) \), we can expand it as a Taylor series at the point
\[ f(x, y, t) = f(0, 0, t) + \frac{\partial f}{\partial x}(0, 0, t)x + \frac{\partial f}{\partial y}(0, 0, t)y + o(x^2 + y^2). \]

If we denote \( f_i(t) = \frac{\partial f}{\partial x}(0, 0, t), f_j(t) = \frac{\partial f}{\partial y}(0, 0, t), f_k(t) = f(0, 0, t) \), then we can use the above trial equation method to obtain approximate analytic solutions to ZK equation with general variable coefficients.
4 Conclusions

We use the trial equation method and complete discrimination system for polynomial method to solve the three-dimensional ZK equation with variable coefficients, and obtain its exact travelling wave solutions including solitary solutions, rational solutions, periodic solutions and double periodic solutions. From the result, we can see that the trial equation method is a powerful tool for solving some differential equations with variable coefficients.

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References