Emergence and Interaction of the Lump-Type Solution with the (3+1)-D Jimbo-Miwa Equation

Abstract: A kinky breather-soliton solution and kinky periodic-soliton solution are obtained using Hirota’s bilinear method and homoclinic test approach for the (3+1)-dimensional Jimbo-Miwa equation. Based on these two exact solutions, some lump-type solutions are emerged by limit behaviour. Meanwhile, two kinds of new dynamical phenomena, kinky breather degeneracy and kinky periodic degeneracy, are discussed and presented. Finally, the interaction between a stripe soliton and a lump-type soliton is discussed by the standardisation of the lump-type solution; the fusion and fission phenomena of soliton solutions are investigated and simulated by three-dimensional plots.

Keywords: Fusion and Fission; Interaction; Jimbo-Miwa Equation; Kinky Degeneracy; Lump-Type Solution.

1 Introduction

Since the concept of solitary wave was put forward in 1965 by Kruskal and Zabusky [1], a lot of nonlinear evolution equations (NLEEs) have been discovered in the field of applied science and natural science. NLEEs exhibit diversity and richness of exact solutions such as rational solution, periodic solution, rogue wave solution, trigonometric function solution and fractal dromion solution. In recent years, the investigation and research of lump solution for NLEEs has become more and more important and attractive. Lump solution is also called the vortex and anti-vortex solution, as a specific type of soliton solution, it was introduced in 1976 by Zaharov [2] and later by Craik and Adam [3]. In contrast to other forms of soliton solutions, lump solution is a kind of rational function solution, localised in all directions in the space. Very recently, Dai et al. [4] constructed homoclinic breather limit method based on the limit behaviour to obtain rogue wave solution, which is actually called the lump-type solution by some scholars. Ma et al. [5] proposed a new direct method based on quadratic function to obtain the lump and lump-type solution. Meanwhile, lump and lump-type solutions were presented for many nonlinear systems [6–15].

Moreover, the interaction about lump-type soliton has attracted much attention in recent years and has already got some useful results. It is known that the collision between solitons for some NLEEs is often completely non-elastic or completely elastic; such collisions are often accompanied by soliton fusion and fission phenomena. For example, in some cases, a single solitary wave may fission multiple solitons [16, 17]. However, for some NLEEs, multiple solitons will fuse a single solitary wave [18–20]. In recent years, in many nonlinear subject areas such as non-linear optics, thermal dynamics, fluid dynamics, solid state physics, a similar phenomenon has been noticed. Thus, it becomes very meaningful and important to discuss the interaction between solitary wave; it can also provide a large amount of physical information and more insight into the physical aspects of the problem, which leads to wider applications.

Now, we consider the (3+1)-dimensional Jimbo-Miwa (JM) equation \[ u_{txt} + 3u_x u_{xx} + 3u_y u_{yy} + 2u_{xt} - 3u_{yy} = 0, \] which comes from the second member of a KP-hierarchy called JM equation firstly introduced by Jimbo and Miwa [21]. It is known that the JM (1) is not Painlevé integrable. Recently, many methods are applied to study the JM equation, for instance, Ma and Yang obtained abundant lump-type solutions by Maple symbolic computations with direct method based on quadratic function [14, 15]. Besides, a large number of explicit solutions were investigated [22–26]. To the best of our knowledge, there are many good properties about (3+1)-D JM that have not been reported and studied. In this article, we will study the analytical solutions and the interaction between solitons by using Hirota’s bilinear method and extended homoclinic test approach (EHTA) [27]. Some dynamical phenomena, kinky soliton solution and lump-type solution of (3+1)-DJM equation will be studied in Section 2; in Section 3, we
will investigate the interaction between a stripe soliton and a lump-type solution; some new wave phenomena will be investigated and simulated by three-dimensional plots, and finally the conclusion is in Section 4.

2 The Emergence of Lump-Type Solution

In this section, by choosing different forms of test function, exact kinky breather-soliton solution and kinky periodic-soliton solution for the (3+1)-D JM equation are obtained using Hirota’s bilinear method and EHTA. Based on these exact solutions, some lump-type solutions are found by limit behaviour. Meanwhile, some new dynamical characteristics including kinky breather degeneracy and kinky periodic degeneracy are investigated.

By the dependent variable transformation

\[ u(x, y, z, t) = 2\ln(f) \],

where \( f(x, y, z, t) \) is an unknown real function. Substituting (2) into (1), we have obtained the following bilinear equation

\[ (D^2_x D_y + 2D_x D_y - 3D_x D_y) f = 0, \]

where the bilinear operator \( D \) [28] is defined by

\[ D_x^m D_y^n (f \cdot g) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^n f(x', y', t') \bigg|_{x'=x-y, y'=y, t'=t}. \]

2.1 Kinky Breather Degeneracy and Lump-Type Solution

To seek kinky breather-soliton solution of the (3+1)-D JM, we choose the test function in the form

\[ f(x, y, z, t) = e^{-\xi} + b_0 \cos(p \eta) + b_1 e^{\eta}, \]

where \( \xi = x + \alpha y + \beta z + \omega t \) and \( \alpha, \beta, \omega, \alpha_1, \beta_1, \omega_1, p, p_1, b_0, b_1 \) are some unknown real numbers. Substituting (5) into (3) with Maple, we can obtain the following relations:

\[ \beta = \frac{\alpha (-p_1^2 + 2p_1^3 + 3p^2)}{3}, \beta_1 = \frac{p^2 \alpha (3p_1^2 - p_1^3 - 2\omega)}{3p_1^2}, \]

\[ \alpha_1 = -\frac{p^2 \alpha}{p_1^2}, b_1 = \frac{b_0^2}{4}, \]

for some arbitrary real constants \( \alpha, \omega, \omega_1, p, p_1 \) and \( b_0 \). An exact analytical solution is obtained by inserting (5) and (6) into (2) as follows

\[ u(x, y, z, t) = \frac{2 \left[ \sqrt{b_1} \sinh \left( \frac{p_1^3}{2} \ln b_1 \right) - b_1 \sin(p \eta) \right]}{2 \sqrt{b_1} \cosh \left( \frac{p_1^3}{2} \ln b_1 \right) + b_1 \cos(p \eta)} \]

where \( \xi = x + \alpha y + \frac{\alpha (2\omega_1 - p_1^2 + 3p^3)}{3} + \omega t, \eta = x - \frac{p^2 \alpha}{p_1^2} y + \frac{p^2 \alpha (3p_1^2 - p_1^3 - 2\omega)}{3p_1^2} + \omega_1 t. \)

In order to obtain the rational solution from the exact solution (7), then taking \( b_0 = -2, p = kp \) and letting \( p \to 0 \) in (7), we get a rational solution as follows:

\[ u(x, y, z, t) = \frac{4(3k^2 + 1)x + 2\omega_1(-\omega)z + 3(\omega + \omega_1)k^2 t + \left(x + \alpha y + \frac{2\alpha}{3}\omega_1 z + \omega_1 t \right)^2 + \left(x - \frac{\alpha}{k^2} y - \frac{2\alpha \omega}{3k^2} z + \omega_1 t \right)^2}{\left(x + \alpha y + \frac{2\alpha}{3}\omega_1 z + \omega_1 t \right)^2 + \left(x - \frac{\alpha}{k^2} y - \frac{2\alpha \omega}{3k^2} z + \omega_1 t \right)^2}. \]

The analytical solution (7) is actually a kinky breather-soliton solution which takes on breather characteristics as trajectory along the straight line \( x = -(\alpha y + \beta z + \omega t) \) and shows kinky characteristics as trajectory along the straight line \( x = -(\alpha y + \beta z + \omega t) \) [29]. Figure 1a clearly shows the feature of kink and breather in spatiotemporal variable \( y - t \). However, this solution (8) represents a kind of exact solitary wave solution in the form of the rational solution, this kind of soliton solution is actually called the lump-type solution. That is, a lump-type solution has emerged from the kinky breather-soliton solution by limit behaviour. Figure 1b shows the lump-type solution has an upward peak and a downward deep hole, and the downward deep hole is hidden below the plane wave. This lump-type solution with bright-dark structure has the characteristics of pulse solution, and the feature of kinky breather has disappeared or degenerated. Where, the curve drawn at the bottom of the figure is the contour line in Figure 1a and b.

2.2 Kinky Periodic Degeneracy and Lump-Type Solution

Here, to obtain the kinky periodic-soliton solution, we choose a different test function in the form

\[ f(x, y, z, t) = e^{-\xi} + b_0 \cos(\eta) + b_1 e^{\eta}, \]

where \( \xi = p(x + \beta z), \eta = p(y + \beta z + \omega t) \). Substituting (9) into (3) and equating all coefficients of different powers of
Substituting (10) with (9) into (2), we obtain a kinky periodic-soliton solution of (1) as follows

\[
2\beta_1 = \frac{b_1^2}{4}, \quad \beta = -\frac{2\omega_1 p_1^3}{3p^2}, \quad \beta_1 = \frac{p_1^2}{3}.
\]

Specially, if taking \( b_0 = \pm 2, p_1 = kp \) and then letting \( p \to 0 \) in (11), we get a new lump-type solution as follows:

\[
u(x, y, z, t) = \frac{4x - \frac{8}{3}k^2 \omega z}{(x - \frac{2}{3}k^2 \omega z)^2 + k^2 (y + \omega t)^2}.
\]

The exact solution (11) is a kinky periodic-soliton solution which has period \( \frac{2\pi}{p \omega_1} \) along \( t \) [29], at the same time shows kink characteristics with space variable \( x, z \) for the (3+1)-D JM equation (see Fig. 2a). Equation (12) is a new lump-type solution which is similar to the structural feature of (8). The asymptotic behaviour of the solution (12) can be found when \( u \to 0 \), either as \( x \to \pm \infty \) or \( y \to \pm \infty \).
y → ±∞ or z → ±∞ or t → ±∞, therefore, it no longer has the characteristics of the kinky wave. Meanwhile, notice \( k \) and \( \omega_1 \) are some free real numbers, the period of the kinky periodic-wave solution \( 2\pi/p_0(\omega_1) \) → ∞, when \( p \to 0 \). In other words, this shows that kinky periodic-wave solution is degenerated as a lump-type solution when the period tends to infinity in the kinky periodic-wave solution.

3 Interaction Between Stripe Soliton and Lump-Type Solution

3.1 Stripe Soliton Solution

Now, we consider the stripe soliton solution for the \((3+1)-D\) JM equation with the help of its bilinear (3). Here, we choose the test function in the form

\[
f(x, y, z, t) = 1 + \delta e^{ax + by + cz + dt},
\]

The following relationships are obtained:

\[
c_1 = \frac{a^2 b + 2bd_1}{3a_3},
\]

leads to

\[
u(x, y, z, t) = \frac{2\delta e^{ax + by + cz + dt}}{a_3},
\]

where \( a, b, d \), and \( \delta \) are some free real constants. The asymptotic behaviour of the stripe soliton solution (15) can be obtained when \( u \to 2a, \) as \( t \to +\infty; u \to 0, \) as \( t \to -\infty. \) Thus, the stripe soliton (15) takes on the kinky soliton characteristics. In fact, the solution of such a characteristic is called kink-type solitary wave solution [30].

3.2 Standardisation of Lump-Type Solutions

It is noted that the denominators in the lump-type solutions (8) and (12) are the homogeneous polynomial of \( x, y, z \) and \( t \). Inspired by this form of lump-type solution, we choose a new standardised test function which is different from the literature [5, 6, 8, 12–15] as follows,

\[
f(x, y, z, t) = 1 + a_x xy + b_x xz + c_x xt + a_y yz + b_y yt + c_z zt + a_x x^2 + b_y y^2 + c_z z^2 + d_1 t^2.
\]

Substituting (16) into (3), through the calculation of Maple 16, we get the following cases:

Case 1:

\[
a_0 = a_0, a_1 = \frac{c_1 d_1^2 + c_1}{6a_2},
\]

\[
b_1 = -\frac{a_1 (3a_1^2 + c_1)}{a_2},
\]

\[
c_0 = a_0, c_1 = \frac{2a_1 d_1}{3a_2},
\]

\[
d_2 = d_1,
\]

where \( \zeta = \frac{9a_0 + 4a_1 d_1 - c_1^2}{4a_1 d_1 - c_1^2}. \) Then, a standardised lump-type solution is obtained by inserting (17) with (16) into (2) (see Fig. 3a)

\[
u(x, y, z, t) = \frac{2\delta e^{ax + by + cz + dt}}{a_3}.
\]

where \( \Delta = a_x x^2 + \frac{a_y^2}{4a_2} y^2 + \frac{a_z^2 d_1^2 + c_1}{9a_2^2} z^2 + d_1 t^2. \)

Case 2:

\[
a_0 = 0, a_1 = a_1, a_2 = a_2, b_0 = 0, b_1 = 0, b_2 = \frac{3a_1 d_1}{2c_0},
\]

\[c_0 = c_0, c_1 = 0, c_2 = \frac{a_1 c_1}{4a_2}, d_2 = \frac{c_1}{4a_2}.
\]

Similarly, we obtain a new lump-type solution (see Fig. 3b)

\[
u(x, y, z, t) = \frac{2(c_0 t + 2a_1 x)}{1 + c_0 xt + a_y y^2 + a_x x^2 + \frac{3a_1 a_y y^2}{2c_0} + \frac{a_1 c_1 z^2 + c_1 d_1}{6a_2} + \frac{c_1^2 t^2}{4a_2}}.
\]

From Figure 3, we can clearly observe that the lump-type solution has an upward peak and a downward deep hole. This standardised lump-type solution of bright-dark structure is similar to the solutions (8) and (12).

3.3 The Fusion and Fission Phenomena of Interaction

Here, we construct the interaction between the stripe soliton solution (13) and the lump-type solution (16) for...
the (3+1)-D JM equation. To obtain the fusion and fission phenomena of interaction between the stripe soliton and the lump-type solution, we turn the above function \( f(x, y, z, t) \) into the following new test function

\[
\begin{align*}
  f(x, y, z, t) &= 1 + a_0 xy + b_0 xz + a_0 yz + b_0 yt + c_0 zt \\
  &+ a_0 x^2 + b_0 y^2 + c_0 z^2 + d_1 t^2 + \delta e^{x^2+y^2+z^2+d_1 t^2}.
\end{align*}
\] (21)

The function \( f(x, y, z, t) \) is composed of a rational function (17) and an exponential function (13). Substituting (21) into (3), through the tedious and long calculation, we can get the following cases:

**Case 1:**

\[
\begin{align*}
  a_0 &= a_0, a_1 = \frac{a_0^{2} \Delta}{6a_1}, a_2 = a_2, a_3 = a_3, b_0 = \frac{a_0^{2} (3a_1 + c_0)}{3a_1^2}, \\
  b_1 &= \frac{a_0 (3a_1^2 - c_0)}{2a_1^2}, b_2 = \frac{a_0^{2} \Delta}{4a_1^2}, b_1 = \frac{a_0}{2a_1}, c_0 = c_0, c_1 = \frac{2a_0 d_1}{3a_1}, \\
  c_2 &= \frac{a_0^{2} \Delta}{9a_1^2}, c_3 = \frac{a_0 c_0}{6a_1^2}, d_2 = d_2, d_3 = -\frac{a_0^2 - c_0}{2a_1}, \delta = \delta,
\end{align*}
\] (22)

where \( \Delta = \frac{9a_1^4 + 4a_1^2 d_1 - c_0^2}{4a_1^2 d_1 - c_0} \). So, we can obtain a mixed type algebraic-exponential solitary wave solutions via inserting (22) and (21) into (2), where

\[
\eta = a_0 x + \frac{a_0 c_0}{6a_1^2} z + \frac{a_1}{2a_1} t, \quad \Delta = a_1^2 x^2 + \frac{a_1^2 \Delta}{4a_1^2} y^2 + \frac{a_0^{2} \Delta}{9a_1^2} z^2 + d_1 t^2.
\]

Obviously, the exact solution (23) represents a kind of exact solitary wave solution in the form of the mixed type algebraic-exponential. The asymptotic behaviours of \( u(x, y, z, t) \) can be obtained with \( \frac{c_0 - a_1^2}{2a_1} > 0 \): \( u \rightarrow 2a_0 \), when \( t \rightarrow +\infty \); \( u \rightarrow 0 \), when \( t \rightarrow -\infty \). The asymptotic behaviours take on the kink-type solitary wave characteristics. In fact, the solution (23) is a kink-type solitary wave solution with \( \eta \), and is also a lump soliton with \( \eta \), and is also a lump soliton with \( \eta \), and is also a lump soliton with \( \eta \), and is also a lump soliton with \( \eta \). The interaction of fusion phenomena is clearly shown by means of the expression of the solution (23). From Figure 4, we can clearly observe this fusion phenomenon. Figure 4a shows the solution \( u \) consists of a stripe soliton and a lump-type soliton at time \( t = -8 \). With the development of time \( t \), the lump-type soliton is drowned or swallowed up by stripe soliton in the interaction between the stripe soliton and the lump-type soliton. When \( t = 8 \), the lump-type soliton is almost completely swallowed by stripe soliton. In fact, fusion occurs in the interaction between two solitons, the lump soliton is swallowed or drowned by stripe soliton. Our results reflect the completely non-inelastic interaction between the two solitary waves (see Fig. 4).
Case 2:

\[ a_0 = b_0 = b_1 = b_2 = c_1 = c_2 = 0, \quad a_1 = a_4, \quad a_2 = a_3, \quad a_3 = a_3, \quad c_0 = c_0. \]

\[ b_2 = \frac{3a_1 a_2}{2c_0}, \quad c_2 = \frac{a_1 c_0}{6a_2}, \quad d_2 = \frac{c_2^2}{4a_2}, \quad d_3 = \frac{a_1 (c_0 - a_3 a_3^2)}{2a_2}, \quad \delta = \delta. \]

(24)

Leads to

\[
\begin{align*}
& u(x, y, z, t) = \\
& \frac{2 a_1 a_2 x + a_1 a_3 z + a_1 a_3 x^2 + \frac{3a_1 a_2 y^2}{2c_0} + \frac{a_1 c_0 z^2}{6a_2} + \frac{c_2^2 t^2}{4a_2} + \delta e^{\frac{a_1 (c_0 - a_3 a_3^2) t}{2a_2}}}{2a_2}
\end{align*}
\]

(25)

where \( \delta, a_1, a_2, a_3, \) and \( c_0 \) are arbitrary constants; some dynamic characteristics of the solution (25) can be acquired with \( \frac{a_1 (c_0 - a_3 a_3^2)}{2a_2} > 0; \) \( u \to 2a_3, \) when \( t \to +\infty; \)

\( u \to 0, \) when \( t \to -\infty. \) From Figure 5, the fission phenomenon of the lump-type soliton is showed from the algebraic-exponential solution (25). Figure 5a shows the solution \( u \) is only one stripe soliton at time \( t = -20. \) With the development of time \( t, \) the lump-type soliton is completely separated from the mixed type algebraic-exponential solitary wave (see Fig. 5c). The phenomenon of the fission of lump-type soliton is simulated by three-dimensional plots (see Fig. 4).

From the above analysis and three-dimensional plots simulations, the reason for the fusion and fission of solitons is clearly displayed. Fusion and fission phenomena of solitons show that the mixed type algebraic-exponential solitary wave solutions are unstable. Comparing our results and previous work [13, 14, 23–26], we extend the result of the (3+1)-D JM equation. The fusion and fission phenomena of solitons are also reported for the first time for the (3+1)-D JM equation.

### 4 Conclusions

In conclusion, kinky breather-soliton solution, kinky periodic-soliton solution, stripe soliton solution,
lump-type solution and mixed type algebraic-exponential solution are obtained by using Hirota’s bilinear method, homoclinic test approach with limit behaviour. In addition, kinky breather degeneracy and kinky periodic degeneracy are discussed and studied in the process of the emergence of lump-type solutions. Finally, we also discuss the interaction between the stripe soliton and the lump-type soliton, the fusion and fission of mixed type algebraic-exponential solitary wave solutions are investigated. It is hoped that these results will provide some valuable information in the higher-dimensional nonlinear field and relevant fields in physics. One problem, which we will study, is whether similar results exist or not when applied to another real integrable or non-integrable system? Under what conditions will soliton of fusion and fission appear? These are very interesting topics for future research.

Acknowledgments: This work was supported by the Hunan Provincial Natural Science Foundation of China (No: 2016JJ6122), the National Natural Science Foundation of P.R. China (No: 11661037 and 11471109), and the Scientific Research Project of Education Department of Hunan Province (No: 17C1297).

References