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Exact Solutions to Several Nonlinear Cases of Generalized Grad–Shafranov Equation for Ideal Magnetohydrodynamic Flows in Axisymmetric Domain

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Abstract: In this paper, the steady-state equations of ideal magnetohydrodynamic incompressible flows in axisymmetric domains are investigated. These flows are governed by a second-order elliptic partial differential equation as a type of generalized Grad–Shafranov equation. The problem of finding exact equilibria to the full governing equations in the presence of incompressible mass flows is considered. Two different types of constraints on position variables are presented to construct exact solution classes for several nonlinear cases of the governing equations. Some of the obtained results are checked for their applications to magnetic confinement plasma. Besides, they cover many previous configurations and include new considerations about the nonlinearity of magnetic flux stream variables.

Keywords: Exact Equilibria; Grad–Shafranov Equation; Incompressible Flows; Magnetohydrodynamics.

1 Introduction

Magnetohydrodynamic (MHD) flows play an important role in astrophysics, geophysics, plasma confinement systems, and fusion research, and their applications have been widely studied [1–6]. MHD equilibria with incompressible flows in cylindrical and axisymmetric domains have been accomplished by several authors [7–26]. It was shown that the toroidal MHD plasma is governed by the well-known Grad–Shafranov (GS) equation for the poloidal magnetic flux function [7–11]. That equation contains five surface quantities in addition to an algebraic relation for the pressure.

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The construction of three-dimensional MHD equilibria is a difficult task, so many attempts have been made over the past years to investigate the different sides of the problem by means of mathematical simplifications such as symmetries. Lie symmetry analysis, conservation laws, and exact solutions on various problems in mathematical physics such as shallow-water waves, surface and internal waves, and magnetized dusty plasmas were carried out in [27–34]. Astrophysical jets were designed by an exact axisymmetric equilibrium of plasma in a gravitational field of an attracting center [35]. Intrinsic symmetries of the ideal MHD equilibrium equations were presented for the divergence-free plasma flows [36–38]. Magnetic dipole equilibria with flow effects was investigated by Catto and Krasheninnikov [39]. Axisymmetric stellar wind equilibria with open and closed magnetic field regions were examined by Keppens and Goedbloed [40].

In plasma confinement systems such as tokamaks, the poloidal and toroidal flows are of great interest because they are known to affect both neoclassical [41] and turbulent [42] transport and MHD stability [43]. For these configurations with stationary and ideally conducting plasmas, Maxwell’s equations together with the force balance equation reduce to a GS equation. Analytical solutions to the GS equation have been carried out in [16, 35, 37, 39].

There has been a number of applications of the GS, GS-type, and 2D MHD equilibrium theory to space plasma environment. Among those applications is the GS reconstruction that was appeared by Sonnerup and Guo [44]. It is a unique and advanced data analysis technique for the reconstruction of 2D coherent field and flow structures from data collected as the structures move past an observing platform. It was developed by Hau and Sonnerup [45]. The technical aspects of the GS reconstruction method were reviewed by Sonnerup et al. [46]. The method has been extended to the case where substantial field-aligned flow is present and to the case of streamline reconstruction in flow perpendicular to a unidirectional magnetic field [46]. The applications of the GS reconstruction technique to space plasma structures in the Earth’s magnetosphere and in the interplanetary space was reviewed by Hu [47].

In the present work, we consider the problem of finding nonlinear ideal MHD equilibria of axisymmetric
incompressible flows. The paper is organized as follows: In section 2, we review the derivation of a generalized GS equation for the steady-state MHD equations with incompressible flows. In Section 3, we explain how to obtain the equilibrium physical variables of the whole MHD system investigated in Section 2. In Section 4, we obtain nonlinear exact equilibria to several nonlinear cases for the equilibrium equations of MHD flows considered in Section 2. In Section 5, we summarize the results.

2 The Problem Formulation of the Equilibrium Equations

The ideal MHD incompressible flows are governed by the following set of equations, written in the steady-state and quite simple units:

The mass conservation equation:
\[ \rho \nabla \cdot \mathbf{v} = -\nabla \rho = 0, \] (1)

The momentum equation:
\[ \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \mathbf{j} \times \mathbf{B}, \] (2)

Faraday’s law:
\[ \nabla \times \mathbf{E} = 0, \] (3)

Ampère’s law:
\[ \nabla \times \mathbf{B} = \mathbf{j}, \] (4)

The divergence-free Gauss law:
\[ \nabla \cdot \mathbf{B} = 0, \] (5)

Ohm’s law for MHD:
\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \] (6)

where \( \rho, \mathbf{v}, P, \mathbf{B}, \) and \( \mathbf{j} \) stand as usual for the mass density, flow velocity field, gas pressure, magnetic field (induction), and current density. The system under consideration is a magnetically confined plasma of an axisymmetric incompressible flow. With the use of cylindrical coordinates \((r, \phi, z)\), the position of the surface of the conductor is specified by some boundary curve in the \(rz\)-plane. From the axisymmetric property, the equilibrium quantities do not depend on the azimuthal coordinate \(\phi\). Consequently, the divergence-free magnetic field \(\mathbf{B}\), current density \(\mathbf{j}\), and mass flow \(\rho \mathbf{v}\) can be expressed, with the aid of Ampère’s law (4), in terms of the stream functions \(\psi(r, z), I(r, z), G(r, z), \) and \(\Theta(r, z)\) as
\[ \mathbf{B} = \frac{1}{r} \left( \frac{\partial \psi}{\partial z} \mathbf{e}_r - \frac{\partial \psi}{\partial r} \mathbf{e}_\phi + \mathbf{j} \mathbf{e}_z \right), \] (7)

\[ \mathbf{j} = \frac{1}{r} \left( -\frac{\partial \psi}{\partial z} \mathbf{e}_r + \frac{\partial \psi}{\partial r} \mathbf{e}_\phi + \Delta \psi \mathbf{e}_z \right), \] (8)

and
\[ \rho \mathbf{v} = \frac{1}{r} \left( \frac{\partial G}{\partial z} \mathbf{e}_r - \frac{\partial G}{\partial r} \mathbf{e}_\phi + \Theta \mathbf{e}_z \right), \quad \rho = \rho(\psi). \] (9)

Here, \(\Delta\) is the Grad–Schlüter–Shafranov elliptic operator defined by
\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \]

The electric field is expressed by \(\mathbf{E} = -\nabla \Phi\).

In general, from the symmetry assumption, the functions \(\psi(r, z), I(r, z), G(r, z), \Theta(r, z), \rho, \) and \(\Phi\) are functions of \(r\) and \(z\). There is a functional dependence between these functions. First, from the incompressibility assumptions (1) and (9), we imply that \(\rho = \rho(\psi)\). Second, the Ohm’s law (6) is projected along the symmetry direction \(e_\phi\) and \(\mathbf{B}\), respectively, yielding
\[ \nabla \psi \cdot (\nabla \times \nabla G) \wedge (\nabla \times \nabla \psi) = 0 \] (10)

and
\[ \mathbf{B} \cdot \nabla \Phi = 0. \] (11)

Equations (10) and (11) imply
\[ G = G(\psi), \quad \Phi = \Phi(\psi); \] (12)

hence, the electric field is perpendicular to a magnetic surface. Third, two additional dependence on \(\psi\) are found from the component of Ohm’s law (6) perpendicular to a magnetic surface,
\[ \frac{1}{\rho r} (IG' - \Theta) = \Phi'(\psi), \quad \therefore \quad \frac{d}{d\psi}, \] (13)

and from the component of the momentum conservation (2) along \(e_\phi\),
\[ I \left( 1 - \frac{G^2}{\rho} \right) + G' \Phi' = X(\psi), \] (14)

where \(X(\psi)\) is a surface quantity.

Using (7)–(9) and (12)–(14), the components of (2) perpendicular to a magnetic surface and along \(\mathbf{B}\) yield, respectively, the generalized GS equation [8–12]:
\[ (1 - M^2) \Delta \psi - \frac{1}{2} (M^2)' |\nabla \psi|^2 + \frac{1}{2} \left( \frac{X^2}{1 - M^2} \right)' + r^2 \left( p - \frac{X G' \Phi'}{1 - M^2} \right)' \]
\[ + \frac{r^4}{2} \left( \frac{\rho(\Phi')'}{1 - M^2} \right)' = 0, \] (15)
with an expression for the pressure as

\[ P = p_0(\psi) - \rho \left( \frac{v^2}{2} + \frac{\Phi'^2}{\rho} \right), \]  

(16)

where the function \( p_0(\psi) \) is a surface quantity and the symbol \( M \) denotes the poloidal Alfvénic Mach number that is defined by

\[ M^2 = \frac{v_p^2}{v_{ap}^2} = \frac{G^2(\psi)}{\rho(\psi)} = M^2(\psi), \]

(17)

where \( v_p \) and \( v_{ap} \) are the poloidal flow velocity and Alfvén velocity, respectively.

For the above MHD flows, there are six surface quantities \( G(\psi), \Phi(\psi), X(\psi), \rho(\psi), p_0, \) and \( M(\psi); \) five of them are arbitrary. The solution \( \psi \) can be determined from (15), which can be solved analytically. Using the new dependent variable

\[ U = U(\psi), \quad U' = \sqrt{1 - M^2}, \quad M^2 < 1, \]

(18)

(15) is reduced to

\[ \Delta U + \frac{1}{2} \frac{d}{dU} \left( \frac{X^2}{1 - M^2} \right) + r^2 \frac{dP}{dU} + r^4 \frac{d}{dU} \left[ \rho \frac{d\Phi}{dU} \right]^2 = 0. \]

(19)

Several classes of exact solutions to (19) can be derived for some choices of \( M \), as we show in the next sections. Moreover, we show how to obtain the associated physical quantities to the full MHD system (1)–(6). The flux-function term \( P_s = \frac{XG'\Phi'}{1 - M^2} \) in (15) is replaced by \( P_s \) in (19).

In what follows, we investigate the problem with \( M^2 > 1 \) and \( M^2 = 1 \). For MHD plasma with \( M^2 > 1 \), we rewrite (15) as

\[ (M^2 - 1)\Delta \psi + \frac{1}{2} (M^2)' |\nabla \psi|^2 + \frac{1}{2} \left( \frac{X^2}{M^2 - 1} \right)' \]

\[ -r^2 \rho' + \frac{r^4}{2} \left( \frac{\rho(\Phi')^2}{M^2 - 1} \right)' = 0. \]

(20)

Hence, changing the new dependent variable \( U \) in (18) to be \( U \),

\[ U = U(\psi), \quad U' = \sqrt{M^2 - 1}, \quad M^2 > 1, \]

(21)

(20) becomes

\[ \Delta U + \frac{1}{2} \frac{d}{dU} \left( \frac{X^2}{M^2 - 1} \right) - r^2 \frac{dP}{dU} + r^4 \frac{d}{dU} \left[ \rho \frac{d\Phi}{dU} \right]^2 = 0, \quad M^2 > 1. \]

(22)

We note that (15), (19), (20), and (22) have singularities when \( M^2 = 1 \). From the component of the momentum conservation (2) along the symmetry direction, we obtain, with the aid of (14), \( X(\psi) = rG'\Phi' \). Consequently, the only possible equilibria of this kind are of cylindrical shape with circular cross section. Hyperbolic flows in which the effective Mach number \( M^2 > 1 \) and elliptic flows for which \( M^2 < 1 \) were obtained [48, 49]. The near-Alfvénic flow solutions with \( M^2 = 1 \) were found to have more pronounced spiral structure than both the high Alfvénic Mach number solutions \((M^2 > 1)\) and the low Alfvénic Mach number solutions \((M^2 < 1)\).

### 3 The Physical Variables in Terms of the New Variable \( U \)

In this section we explain how to obtain the equilibrium physical variables \( B, j, E, \alpha, P, \) and \( \rho \) for the MHD system (1)–(6) given in Section 2. From (18) we have

\[ \Delta \psi = \frac{1}{\sqrt{1 - M^2}} \left( \Delta U + \frac{M}{1 - M^2} \frac{dM}{dU} |\nabla U|^2 \right). \]

(23)

To express the physical quantities in terms of the new variable \( U \), we introduce the following poloidal field:

\[ B = \frac{1}{r} \left( \frac{dG}{dU} \right) e_z - \frac{dU}{dr} e_z - \frac{dU}{d\phi} \Omega e_z. \]

(24)

From (9) and (24), the velocity field reads

\[ v = \frac{1}{r \rho} \left( \frac{dG}{dU} B + \Omega e_z \right). \]

(25)

Using (23), the equilibrium physical variables \( v, B, j, \) and \( P \) are obtained as

\[ B = \frac{B}{\sqrt{1 - M^2}} + r \nabla \phi, \]

(26)

\[ j = \frac{1}{r \sqrt{1 - M^2}} \left( \frac{dG}{dU} B + r \frac{d\Phi}{dU} \nabla \phi \right), \]

(27)

\[ E = -\frac{dG}{dU} \nabla U, \]

(29)

\[ P = P(U) - \frac{1 - M^2}{2} B^2 \frac{dG}{dU} + r^2 \rho \frac{d\Phi}{dU}^2 - \sqrt{1 - M^2} e_z \frac{d\Phi}{dU}. \]

(30)
With the aid of (18), we can obtain exact solutions to the original (15). Table 1 shows some solutions \(\psi(r,z)\) corresponding to solutions \(U(r,z)\) in accordance with choices for the Mach number. The physical quantities \(v, B, j,\) and \(P\) can be obtained using (26)–(30).

### Table 1: Some variable Mach numbers and corresponding solution \(\psi\) as a function of \(U\).

<table>
<thead>
<tr>
<th>(M(U))</th>
<th>(M^2 &lt; 1)</th>
<th>(\psi(U))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A + BU)</td>
<td>(-(1 + A) &lt; BU &lt; 1 - A)</td>
<td>(\frac{1}{B} \arcsin(A + BU) + \text{const})</td>
</tr>
<tr>
<td>(\sqrt{A + BU})</td>
<td>(-A &lt; BU &lt; 1 - A)</td>
<td>(\frac{2}{B} \sqrt{1 - A - BU} + \text{const})</td>
</tr>
<tr>
<td>((A + BU)^n, n \geq 3)</td>
<td>((-1)^n A &lt; BU &lt; 1 - A)</td>
<td>(\frac{n}{B} \sin^{-1}v + \text{const}, \quad \sin v = (A + BU)^{\frac{1}{2n}})</td>
</tr>
<tr>
<td>(\sqrt{1 - A'U^2})</td>
<td>(</td>
<td>AU</td>
</tr>
<tr>
<td>(\sqrt{1 - A'U'}, n \neq 2)</td>
<td>(0 &lt; U' &lt; 1/A^2)</td>
<td>(\frac{2}{A(2-n)} U^{1\frac{n}{2}} + \text{const})</td>
</tr>
<tr>
<td>(\sqrt{1 + C + BU - A'U^2})</td>
<td>(0 &lt; 1 + C + BU - A'U^2 &lt; 1)</td>
<td>(\frac{1}{A} \ln</td>
</tr>
<tr>
<td>(e^{AU})</td>
<td>(AU &lt; 0)</td>
<td>(\frac{1}{A} \arctanh(\sqrt{1 - e^{2AU}}) + \text{const})</td>
</tr>
<tr>
<td>(\sin U)</td>
<td>Any function (U)</td>
<td>(\ln</td>
</tr>
<tr>
<td>(\cos U)</td>
<td>Any function (U)</td>
<td>(\ln</td>
</tr>
<tr>
<td>(\tan U)</td>
<td>(-\pi/4 &lt; U &lt; \pi/4)</td>
<td>(\frac{1}{\sqrt{2}} \arcsin(\sqrt{2}\sin U) + \text{const})</td>
</tr>
<tr>
<td>(\tanh U)</td>
<td>Any function (U)</td>
<td>(\sinh U + \text{const})</td>
</tr>
<tr>
<td>(\sech U)</td>
<td>Any function (U)</td>
<td>(\ln</td>
</tr>
</tbody>
</table>

The second column shows the range of the function \(U\) that provided the last inequality in (18).

### 4 Nonlinear Exact Equilibria

In this section we consider some nonlinear cases of the equilibrium equations given in Section 2. We introduce two different transformations of new variables depending on \(r\) and \(z\) to obtain several classes of exact solutions to (19). Consequently, we obtain the associated physical quantities to the full MHD system (1)–(6). Using the representations

\[
\frac{1}{2} \frac{d}{dU} \left( \frac{X^2}{1 - M^2} \right) = f(U), \quad \frac{dP}{dU} = g(U),
\]

\[
\frac{1}{2} \frac{d}{dU} \left[ \left( \frac{dP}{dU} \right)^2 \right] = h(U),
\]

in (19), it is rewritten as

\[
\frac{\partial^2 U}{\partial r^2} - \frac{1}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = -f(U) - r^2 g(U) - r^3 h(U). \tag{32}
\]

**Case 1.** We seek exact solutions to (32) using the following new independent variable:

\[
\xi(r, z) = 9ar^2 + \eta(z), \tag{33}
\]

where \(a\) is a real constant and \(\eta\) is a function of \(z\). Using (33) into (32), we get

\[
-x \left( 36a(\xi - \eta) + \left( \frac{d\eta}{dz} \right)^2 \frac{d^2U}{dz^2} + \frac{d^2\eta}{dz^2} \frac{dU}{dz} \right) - \frac{1}{9a}(\xi - \eta)^2 h(U). \tag{34}
\]

Now, we seek solutions to (34) such that the function \(\eta(z)\) satisfies the following equation:

\[
\frac{d\eta}{dz} = 4a\eta + b, \tag{35}
\]

which has the general solution

\[
\eta(z) = \frac{1}{4a}(\zeta + e^{4az} - b), \tag{36}
\]
where \( b \) is a real constant and \( \xi \) is an integration constant.

Inserting (35) into (34) yields

\[
(36a(\xi - \eta) + 4(\eta + b)\frac{d^2U}{d\xi^2}) + 4a(\eta + b)\frac{dU}{d\xi} - f(U) - \frac{1}{9a}(\xi - \eta)g(U) - \frac{1}{9a}(\xi - \eta)^2 h(U).
\]

(37)

Equating the coefficients of like powers of \( \eta \) in both sides of (37), we get

\[
\eta^2: (36a + b^2)\frac{d^2U}{d\xi^2} + 4ab\frac{dU}{d\xi} + f(U) + \frac{1}{9a}(\xi - \eta)g(U)
\]

\[
+ \frac{1}{9a}(\xi - \eta)^2 h(U) = 0,
\]

(38)

\[
\eta^3: (-36a + 8ab)\frac{d^2U}{d\xi^2} + (4a)^3\frac{dU}{d\xi} - \frac{2}{9a}(\xi - \eta)g(U) - \frac{2}{9a}(\xi - \eta)^3 h(U) = 0,
\]

(39)

\[
\eta^4: \frac{d^4U}{d\xi^4} = -\frac{1}{(36a)^2} h(U).
\]

(40)

Substitution of (40) into (39) yields

\[
g(U) = \frac{1}{9a} \left( \frac{-2b}{4a} - 2\xi \right) h(U) + 144a^3 \frac{dU}{d\xi}.
\]

(41)

Substitution of (40) and (41) into (38) yields

\[
f(U) = \frac{1}{9a} \left( \frac{b^2 + 8ab\xi}{16a^2} + \xi^2 \right) h(U) - 4a(\eta + b)\frac{dU}{d\xi}.
\]

(42)

To seek solutions satisfying (38)–(40), we assume the solution as

\[
U(\xi) = \alpha + \beta(F(\xi))^\gamma,
\]

(43)

where \( \alpha \) and \( \beta \) are real constants to be determined later. The function \( F(\xi) \) expresses a trigonometric or hyperbolic function, and \( \gamma \) is a real constant. The constant \( \gamma \) can be determined according to the choice of the function \( F(\xi) \) [23].

**Case 2.** \( h(U) \) is a polynomial of fourth degree and \( F(\xi) = \sin \xi, \cos \xi, \sinh \xi, \cosh \xi. \)

Consider the polynomial

\[
h(U) = \sum_{n=0}^{m} a_n U^n.
\]

(44)

For the above forms of \( F(\xi) \), we get \( \gamma = -2/3 \) (see [23]).

Using (43) and (44) into (40), we obtain four classes of exact solutions to (40) as shown in Tables 2 and 3. For these solutions, the coefficients \( a_i \), \( i = 0, \ldots, 4 \), that appeared in (44) are determined in Table 3. Using (41) and the second column of Table 2, the function \( g(U) \) is determined for each solution \( U_i \), \( i = 1, \ldots, 4 \), in Table 2, respectively, as

\[
g(U_i(\xi)) = \frac{1}{9a} \left( \frac{-2b}{4a} - 2\xi \right) h(U_i(\xi)) - 96a^3 \beta \csc^{2/3} \xi \cot \xi,
\]

\[
- m\pi < \xi < m\pi,
\]

(45)

\[
g(U_i(\xi)) = \frac{1}{9a} \left( \frac{-2b}{4a} - 2\xi \right) h(U_i(\xi)) + 96a^3 \beta \sec^{2/3} \xi \tan \xi,
\]

\[
- \frac{m\pi}{2} < \xi < \frac{m\pi}{2},
\]

(46)

\[
g(U_i(\xi)) = \frac{1}{9a} \left( \frac{-2b}{4a} - 2\xi \right) h(U_i(\xi)) - 96a^3 \beta \cot \xi,
\]

\[
\xi \in (\infty, 0) \cup (0, \infty),
\]

(47)

\[
g(U_i(\xi)) = \frac{1}{9a} \left( \frac{-2b}{4a} - 2\xi \right) h(U_i(\xi)) - 96a^3 \beta \tan \xi,
\]

\[
- \infty < \xi < \infty,
\]

(48)

where \( m \) is an integer. Using (42) and the second column of Table 2, the function \( f(U) \) is determined for each solution \( U_i \), \( i = 1, \ldots, 4 \), in Table 2, respectively, as

\[
f(U_i(\xi)) = \frac{1}{9a} \left( \frac{b^2 + 8ab\xi}{16a^2} + \xi^2 \right) h(U_i(\xi))
\]

\[
+ \frac{8a^2 b}{3} (4a\xi + b) \csc^{2/3} \xi \cot \xi, m\pi < \xi < (m + 1)\pi,
\]

(49)

\[
f(U_i(\xi)) = \frac{1}{9a} \left( \frac{b^2 + 8ab\xi}{16a^2} + \xi^2 \right) h(U_i(\xi))
\]

\[
- \frac{8a^2 b}{3} (4a\xi + b) \sec^{2/3} \xi \tan \xi, - \frac{m\pi}{2} < \xi < \frac{m\pi}{2},
\]

(50)
In this case, $\gamma = \frac{2}{1-n}$; hence, we obtain four classes of exact solutions to (40) as shown in Table 4. For these solutions, the constants $A$, $B$, and $C$ are determined in Table 5.

Using (41) and the second column of Table 4, the function $g(U)$ is determined for each solution $U_i, i = 1, \ldots, 4$, of Table 4, respectively, as

$$g(U_i(\xi)) = \frac{1}{9a} \left( \frac{9-2b}{4a} - 2\xi \right) h(U_i(\xi)) + \frac{288a^3\beta}{1-n} \sin^{\frac{1+n}{2}}\xi \cos \xi,$$

$$g(U_i(\xi)) = \frac{1}{9a} \left( \frac{9-2b}{4a} - 2\xi \right) h(U_i(\xi)) - \frac{288a^3\beta}{1-n} \cos^{\frac{1+n}{2}}\xi \sin \xi,$$

$$g(U_i(\xi)) = \frac{1}{9a} \left( \frac{9-2b}{4a} - 2\xi \right) h(U_i(\xi)) + \frac{288a^3\beta}{1-n} \sin^{\frac{1+n}{2}}\xi \cos \xi,$$

$$g(U_i(\xi)) = \frac{1}{9a} \left( \frac{9-2b}{4a} - 2\xi \right) h(U_i(\xi)) + \frac{288a^3\beta}{1-n} \cos^{\frac{1+n}{2}}\xi \sin \xi.$$
Using (42) and the second column of Table 4, the function \( f(U) \) is determined for each solution \( U_i, i = 1, \ldots, 4 \), of Table 4, respectively, as

\[
f(U_i(\xi)) = \frac{1}{(9a)^2} \left( \frac{b^2 + 8ab\xi + \xi^2}{16a^2} \right) h(U_i(\xi))
- \frac{8a\beta}{1-n} (4a\xi + b) \sin^{\frac{1}{n}} \xi \cos \xi,
\]

(58)

\[
f(U_i(\xi)) = \frac{1}{(9a)^2} \left( \frac{b^2 + 8ab\xi + \xi^2}{16a^2} \right) h(U_i(\xi))
+ \frac{8a\beta}{1-n} (4a\xi + b) \cos^{\frac{1}{n}} \xi \sin \xi,
\]

(59)

\[
f(U_i(\xi)) = \frac{1}{(9a)^2} \left( \frac{b^2 + 8ab\xi + \xi^2}{16a^2} \right) h(U_i(\xi))
- \frac{8a\beta}{1-n} (4a\xi + b) \cosh^{\frac{1}{n}} \xi \sinh \xi.
\]

(60)

\[
f(U_i(\xi)) = \frac{1}{(9a)^2} \left( \frac{b^2 + 8ab\xi + \xi^2}{16a^2} \right) h(U_i(\xi))
+ \frac{8a\beta}{1-n} (4a\xi + b) \cosh^{\frac{1}{n}} \xi \sinh \xi.
\]

(61)

The variable \( \xi \) in (54)–(61) should be \(-\infty < \xi < \infty\) for \( n = \frac{k-1}{k+1} \) where \( k \geq 0 \) is an integer.

**Case 4.** \( h(U) \) is a quadratic polynomial with \( F(\xi) = \tan \xi, \cot \xi, \tanh \xi, \cosh \xi \)

\[
h(U) = b_0 + b_1 U + b_2 U^2,
\]

(62)

where \( b_i, i = 0, 1, 2 \), are real constants.

In this case, \( \gamma = 2 \). Hence, we obtain four classes of exact solutions to (40) as shown in Tables 6 and 7. For these solutions, the constants \( b_i, i = 0, 1, 2 \), are determined in Table 7.

Using (41) and the second column of Table 6, the function \( g(U) \) is determined for each solution \( U_i, i = 1, \ldots, 4 \), of Table 6, respectively, as

\[
g(U_i(\xi)) = \frac{1}{9a} \left( \frac{9 - 2b}{4a} \right) h(U_i(\xi)) + 288a^2 \beta \tan \xi \csc^2 \xi,
\]

(63)

\[
g(U_i(\xi)) = \frac{1}{9a} \left( \frac{9 - 2b}{4a} \right) h(U_i(\xi)) - 288a^2 \beta \cot \xi \csc^2 \xi,
\]

(64)

\[
g(U_i(\xi)) = \frac{1}{9a} \left( \frac{9 - 2b}{4a} \right) h(U_i(\xi)) + 288a^3 \beta \tanh \xi \sech^2 \xi,
\]

(65)

\[
g(U_i(\xi)) = \frac{1}{9a} \left( \frac{9 - 2b}{4a} \right) h(U_i(\xi)) - 288a^3 \beta \coth \xi \csch^2 \xi,
\]

(66)

Using (42) and the second column of Table 6, the function \( f(U) \) is determined for each solution \( U_i, i = 1, \ldots, 4 \), in Table 6, respectively, as

\[
f(U_i(\xi)) = \frac{1}{(9a)^2} \left( \frac{b^2 + 8ab\xi + \xi^2}{16a^2} \right) h(U_i(\xi))
- 8a\beta (4a\xi + b) \tan \xi \sec^2 \xi,
\]

(67)

\[
f(U_i(\xi)) = \frac{1}{(9a)^2} \left( \frac{b^2 + 8ab\xi + \xi^2}{16a^2} \right) h(U_i(\xi))
+ 8a\beta (4a\xi + b) \cot \xi \csc^2 \xi,
\]

(68)

**Table 6:** Solutions for (40) where \( h(U) \) is a quadratic polynomial and \( F(\xi) = \tan \xi, \cot \xi, \tanh \xi, \cosh \xi \).

<table>
<thead>
<tr>
<th>( F(\xi) )</th>
<th>( U(\xi) )</th>
<th>( B )</th>
<th>( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tan \xi )</td>
<td>( U_i = \alpha + \beta \tan \xi )</td>
<td>( 2\beta \tan \xi \csc^2 \xi B )</td>
<td>( ig(U_i) ) d\xi</td>
</tr>
<tr>
<td>( \cot \xi )</td>
<td>( U_i = \alpha + \beta \cot \xi )</td>
<td>( -2\beta \cot \xi \csc^2 \xi B )</td>
<td>( ig(U_i) ) d\xi</td>
</tr>
<tr>
<td>( \tanh \xi )</td>
<td>( U_i = \alpha + \beta \tanh \xi )</td>
<td>( 2\beta \tanh \xi \sech^2 \xi B )</td>
<td>( ig(U_i) ) d\xi</td>
</tr>
<tr>
<td>( \cosh \xi )</td>
<td>( U_i = \alpha + \beta \coth \xi )</td>
<td>( -2\beta \coth \xi \csch^2 \xi B )</td>
<td>( ig(U_i) ) d\xi</td>
</tr>
</tbody>
</table>

The corresponding vector field \( B \) and the static pressure \( P \) are determined.
Case 5. Consider now the following new variable:
\[ \xi = ar^2 + \chi(z), \]  
(71)

with
\[ \left( \frac{dy}{dz} \right)^2 = a\chi^2 + k, \]  
(72)

where \( a \) and \( k \) are real constants that have the following general solutions:

If \( k > 0 \) and \( a > 0 \),
\[ \chi_1 = \sqrt{-\frac{k}{a}} \sinh(c_1 \pm \sqrt{a}z); \]  
(73)

if \( k > 0 \) and \( a < 0 \),
\[ \chi_2 = \sqrt{-\frac{k}{a}} \sin(c_1 \pm \sqrt{-a}z); \]  
(74)

if \( k < 0 \) and \( a > 0 \),
\[ \chi_3 = \sqrt{-\frac{k}{a}} \cosh(c_1 \pm \sqrt{a}z); \]  
(75)

where \( c_1, c_2 \) and \( c_3 \) are integration constants.

Inserting (71) and (72) into (32) yields
\[
[4a(\xi - \chi) + a\chi^2 + k] \frac{d^2U}{dz^2} + a\chi \frac{dU}{dz} + f(U) + \frac{1}{a}(\xi - \chi)g(U) + \frac{1}{a^2}(\xi - \chi)^2h(U) = 0.
\]  
(76)

Equating the coefficients of like powers of \( \chi \) in both sides of (76), we get

\[ \chi^0: (4a\xi + k) \frac{d^2U}{dz^2} + f(U) + \frac{1}{a} \xi g(U) + \frac{1}{a^2} \xi^2 h(U) = 0, \]  
(77)

\[ \chi^1: -4a \frac{dU}{dz} + a \frac{dU}{dz} + \frac{1}{a} g(U) - \frac{2}{a^2} \xi h(U) = 0, \]  
(78)

\[ \chi^2: \frac{dU}{dz} = -\frac{1}{a^2} h(U). \]  
(79)

Substitution of (79) into (78) yields
\[ g(U) = \frac{1}{a}(4 - 2\xi) h(U) + a^2 \frac{dU}{dz}. \]  
(80)

Substitution of (80) and (79) into (77) yields
\[ f(U) = \frac{1}{a} \left[ \xi^2 + 4(a^2 - 1) \xi + ak \right] h(U) - \frac{a^3}{2} \frac{dU}{dz}. \]  
(81)

To seek solutions satisfying (77)–(79), first we solve (79) by putting the solution as \( U(\xi) = \alpha + \beta F(\xi) \), where \( \alpha \) and \( \beta \) are real constants to be determined later. The function \( F(\xi) \) expresses a trigonometric or hyperbolic function, and \( \gamma \) is a real constant. The constant \( \gamma \) can be determined according to the choice of the function \( F(\xi) \). We consider the function \( h(U) \) as a quadratic polynomial with
\[ h(U) = \sum_{n=0}^{2} d_n U^n. \]  
(82)

In this case, \( \gamma = -2 \). Using (82) into (79), we obtain eight classes of exact solutions to (79) (see Tabs. 8 and 9). For
The constants $d_j, j = 0, 1, 2$, appeared in (44) for each solution $U_i, i = 1, \ldots, 8$, in Table 8.

<table>
<thead>
<tr>
<th>$U_i$</th>
<th>$F(\xi)$</th>
<th>$d_j, i = 0,1,2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_1$</td>
<td>$\sin \xi$</td>
<td>$d_0 = -\frac{2a^3(3\alpha + 2\beta)}{\beta}, d_1 = \frac{4a^3(3\alpha + \beta)}{\beta}, d_2 = -6a^3$</td>
</tr>
<tr>
<td>$U_2$</td>
<td>$\cos \xi$</td>
<td>$d_0 = -\frac{2a^3(3\alpha + 2\beta)}{\beta}, d_1 = \frac{4a^3(3\alpha + \beta)}{\beta}, d_2 = -6a^3$</td>
</tr>
<tr>
<td>$U_3$</td>
<td>$\sinh \xi$</td>
<td>$d_0 = -\frac{2a^3(3\alpha - 2\beta)}{\beta}, d_1 = \frac{4a^3(3\alpha - \beta)}{\beta}, d_2 = -6a^3$</td>
</tr>
<tr>
<td>$U_4$</td>
<td>$\cosh \xi$</td>
<td>$d_0 = -\frac{2a^3(3\alpha + 2\beta)}{\beta}, d_1 = \frac{4a^3(3\alpha + \beta)}{\beta}, d_2 = 6a^3$</td>
</tr>
<tr>
<td>$U_5$</td>
<td>$\tan \xi$</td>
<td>$d_0 = -\frac{2a^3(3\alpha - 4\alpha \beta + \beta^2)}{\beta}, d_1 = \frac{4a^3(3\alpha - 2\beta)}{\beta}, d_2 = -6a^3$</td>
</tr>
<tr>
<td>$U_6$</td>
<td>$\cot \xi$</td>
<td>$d_0 = -\frac{2a^3(3\alpha - 4\alpha \beta + \beta^2)}{\beta}, d_1 = \frac{4a^3(3\alpha - 2\beta)}{\beta}, d_2 = -6a^3$</td>
</tr>
<tr>
<td>$U_7$</td>
<td>$\tanh \xi$</td>
<td>$d_0 = -\frac{2a^3(3\alpha + 4\alpha \beta + \beta^2)}{\beta}, d_1 = \frac{4a^3(3\alpha + 2\beta)}{\beta}, d_2 = -6a^3$</td>
</tr>
<tr>
<td>$U_8$</td>
<td>$\coth \xi$</td>
<td>$d_0 = -\frac{2a^3(3\alpha + 4\alpha \beta + \beta^2)}{\beta}, d_1 = \frac{4a^3(3\alpha + 2\beta)}{\beta}, d_2 = -6a^3$</td>
</tr>
</tbody>
</table>

The function $f(U_i)$ is determined from (43) for each solution $U_i, i = 1, \ldots, 8$, as

\[
g(U_i(\xi)) = \frac{1}{a^2}[(4 - 2\xi)h(U_i(\xi)) + 2a^2\beta\sec^2\xi \tan \xi],
\]

\[\tag{88}
m\pi < \xi < (m + 1)\pi,
\]

\[
g(U_i(\xi)) = \frac{1}{a^2}[(4 - 2\xi)h(U_i(\xi)) - 2a^2\beta\csc^2\xi \cot \xi],
\]

\[\tag{83}
m\pi < \xi < (m + 1)\pi,
\]

\[
g(U_i(\xi)) = \frac{1}{a^2}[(4 - 2\xi)h(U_i(\xi)) + 2a^2\beta\sec^2\xi \tan \xi],
\]

\[\tag{84}
-\frac{m\pi}{2} < \xi < \frac{m\pi}{2},
\]

\[
g(U_i(\xi)) = \frac{1}{a^2}[(4 - 2\xi)h(U_i(\xi)) - 2a^2\beta\csc^2\xi \cot \xi],
\]

\[\tag{85}
\xi \in (-\infty, 0) \cup (0, \infty),
\]

\[
g(U_i(\xi)) = \frac{1}{a^2}[(4 - 2\xi)h(U_i(\xi)) + 2a^2\beta\sec^2\xi \tan \xi],
\]

\[\tag{86}
-\infty < \xi < \infty,
\]

\[
g(U_i(\xi)) = \frac{1}{a^2}[(4 - 2\xi)h(U_i(\xi)) + 2a^2\beta \csc \xi \cot \xi],
\]

\[\tag{87}
m\pi < \xi < (m + 1)\pi,
\]

The conditions on $\xi$ in (91)–(98) are the same as shown in (83)–(90), respectively.
5 Application to Magnetic Confinement Plasma

In this section we discuss the applicability of some of the obtained solutions ($U_4$ and $U_8$ in Tab. 8) to magnetic confinement devices. Attempts have been made to heat the plasma very quickly and contain it in strong magnetic fields for a while, so that what is produced from energy is more than what is consumed before the plasma containment is destroyed. There are several models to contain the plasma, including the tokamak device. This device depends on maintaining the plasma ions in the orbits of the magnetic field and increase its density and save them away from the plasma-containing walls. In this system the energy collisions between the nuclei of atoms of hydrogen isotopes (deuterium and tritium) have to be high for the occurrence of the fusion. Also, it should satisfy the optimum conditions for the characteristics of the plasma, which is only when nuclear fusion occurs. Figures 1a–d and 2a–d describe magnetic fluxes for the solutions $U_4$ and $U_8$ in Table 8. In all figures, we have used the variable $\xi = ar^2 + \chi(z)$ in (71) with $\chi = \sqrt{-\frac{a}{c_2}} \text{sin}^{-1}(\frac{k}{a} \pm \sqrt{-a^2})$ as in (74).

Figure 1a–d describe the solution $U_4$ and the corresponding original magnetic flux $\psi(r, z)$ with the first choice of Mach number as $M = U$ in Table 1. The functions $\psi(r, z)$ and $U_4$ are plotted in Figure 1a and b, respectively, with the choices of parameters $a = -9$, $k = 9$, $c_2 = A = \alpha = 0$, and $\beta = B = 1$. In (c) and (d), $k = 9/4$ is used.

Figure 1: Contour plots of magnetic surfaces describe plasma in axisymmetric confinement systems. The magnetic surfaces are plotted for the solution $U_4$ in Table 8 and the corresponding magnetic flux $\psi(r, z)$ with the first choice of Mach number as $M = U$ in Table 1. (a) and (c) show $U_4(r, z)$ while (b) and (d) show $\psi(r, z) = \sin^{-1}(U_4)$. The values of parameters used are $a = -9$, $k = 9$, $c_2 = A = \alpha = 0$, and $\beta = B = 1$. In (c) and (d), $k = 9/4$ is used.
and parameters used in Figure 1a–d. The formations of O-shaped field lines in Figures 1a–d and 2a–d provide a strong toroidal magnetic field in the plasma. It is shown from these figures that the magnetic field lines formed in the shape of toroids around the interior plasma, and the poloidal cross-section boundary is a closed line in space. This means that the ions and electrons in the plasma have to travel tightly around the field lines, which prevents them from escaping from the vessel, and any impurities arising from fusion products or the wall interactions are ionized inside the main plasma volume. The toroidal, poloidal, and total magnetic field of the solution $U_8$ are plotted in Figure 3a–c. The magnetic field configurations described in these figures have helped in the formation of plasma and has a stabilizing effect on the hot magnetofluid inside the device. The plasma pressure is the main ingredient for the production of energy from nuclear fusion. Figure 4 shows the contour plot of high-pressure configurations corresponding to the magnetic flux shown in Figure 1a–d. The closed field lines described in Figure 1a–d show that high-temperature plasma can be created inside the vessel as shown in Figure 5.

6 Concluding Remarks

In this paper we have investigated the equilibrium properties of ideal MHD with incompressible flows in an axisymmetric case. The dynamical equilibrium variables are expressed in terms of a new dependent variable that has made the generalized GS equation similar to the form of the GS equation governing the magnetostatic case. We have constructed two types of constraints on the position variables to deal with the nonlinearities associated with the equilibrium equations. These constraints have been exploited to find exact solutions to several nonlinear cases of the whole MHD system. Previously, some cases of linear MHD and magnetostatic equilibria were
considered in [8, 9, 15, 16, 35, 50, 51, 53]. All magneto-fluid equations considered here are nonlinear and considered in the dynamic case. Also, a solution obtained in [52] can be recovered by the solutions presented here.

In magnetic confinement fusion devices, the hot plasma is confined away from material boundaries using magnetic fields. We have constructed magnetized equilibria with poloidal and toroidal components where the electric current creates the poloidal component of the magnetic field and serves to heat the plasma to temperatures at which nuclear fusion occurs. The high-pressure MHD equilibria constructed here can be balanced by the pressure of the magnetic field, and hence, the plasma can be controlled and shaped by magnetic fields. The obtained results may be helpful in the attempts of magnetic confinement fusion to create the physical conditions needed.

**Figure 3:** (a) Toroidal, (b) poloidal, and (c) total magnetic field of solution \( U_4 \) shown in Table 8. The function \( h(U_4) = U_4^2 \) is used with the aid of (14), (18), (24), (26), (30), and (31).

**Figure 4:** High-pressure configurations for plasma in axisymmetric confinement systems. The parameters of Figure 1 are used with the aid of (13), (14), (18), (30), and (31).

**Figure 5:** High-temperature configurations corresponding to magnetic surfaces described in Figure 1. The parameters of Figure 1 are used.
for the production of fusion energy by using the electrical conductivity of the plasma to contain it using magnetic fields. Also, they can be employed as starting points for stability and transport studies.

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References