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Non-linear Dynamics and Exact Solutions for the Variable-Coefficient Modified Korteweg–de Vries Equation

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Abstract: This paper presents some new exact solutions which contain soliton solutions, breather solutions and two types of rational solutions for the variable-coefficient-modified Korteweg–de Vries equation, with the help of the multivariate transformation technique. Furthermore, based on these new soliton solutions, breather solutions and rational solutions, we discuss their non-linear dynamics properties. We also show the graphic illustrations of these solutions which can help us better understand the evolution of solution waves.

Keywords: Breather Solutions and Rational Solutions; Multivariate Transformation Technique; Soliton Solutions; Variable-coefficient-modified Korteweg–de Vries (vc-mKdV) Equation.

1 Introduction

It is well known that non-linear evolution partial differential equations (NPDEs) play an essential role in describing non-linear science phenomena which had been modelled by integral order or fractional derivatives [1–5]. Among them, the variable-coefficient NPDEs may be better described in the real-world than the constant-coefficient NPDEs [6–9]. Therefore, the latter case has been attractive to many researchers in recent years [10–13].

In this paper, we consider the variable-coefficient-modified Korteweg–de Vries (vc-mKdV) equation [14], that is

\[ \frac{d\phi}{dt} + \omega(t)\frac{d\phi}{dx} + \sigma(t)\phi\frac{d\phi}{dx} + \varpi(t)\frac{d^3\phi}{dx^3} = 0, \]

where the coefficients \( \omega(t), \sigma(t) \) and \( \varpi(t) \) are arbitrary functions in variable time \( t \). Equation (1) can be used to study the dynamics hidden in the plasma sheath transition layer and inner sheath layer. Meanwhile, (1) also appears in ocean waves, fluid mechanics, ocean dynamics and plasma mechanics, and in other areas.

Recently, (1) was studied by many researchers. For example, the soliton solutions were obtained in [15, 16], and the new Jacobi elliptic function solutions and Riemann theta function periodic wave solutions were also obtained in [17, 18] and other works appeared in [19–21]. More recently, Pal et al. [22] derived the first-order rational, periodic solutions and second-order rational, periodic solutions. Zhang et al. [23] obtained the Painleve property, Backlund transformation, Lax pair and some new analytic solutions for (1). Meanwhile, Lax pair, auto-Backlund transformation and conservation law for (1) with external-force form can also be known from [24], and others [25–27].

The main aim of this paper is to construct soliton solutions, breather solutions, rational solutions and we further analyse their non-linear dynamical properties for the vc-KdV equation with the help of the multivariate transformation technique. The plan of this article is as follows: in Section 2, we show a multivariate transformation reducing vc-mKdV equation into constant-coefficient mKdV equation. In Section 3, some new soliton solutions, breather solutions and rational solutions are obtained, with the help of the multivariate transformation. Finally, the main conclusion is given.

2 Multivariate Transformation Technique

In this section, we apply the multivariate transformation technique to reduce (1) into constant-coefficient mKdV equation. In order to do this, we search for the following transformation [28]:

\[ \phi(x, t) = \Theta(t)\Xi[X(x, t), T(t)], \]  

where \( X(x, t) = \alpha(t)x + \beta(t) \).
Substituting (2) into (1) leads to
\[
\Theta_t \Xi_x + \sigma(t)\alpha(t) \Theta \Xi_x + \omega(t)\alpha^2(t)\Theta \Xi_x = 0.
\]
(3)

To get constant-coefficient mKdV equation from (3), first, we deal with the equation
\[
\Theta_t \Xi_x + \sigma(t)\alpha(t) \Theta \Xi_x + \omega(t)\alpha^2(t)\Theta \Xi_x = 0.
\]
(4)

Under these constraint conditions
\[
\frac{\sigma(t)\alpha(t) \Theta^2}{T_i} = 6, \quad \frac{\omega(t)\alpha^2(t)\Theta}{T_i} = 1.
\]
(5)

Hence, we can obtain the following constant-coefficient mKdV equation of the form
\[
\Xi_t + 6\Xi^2_x + \Xi_{xxx} = 0.
\]
(6)

Meanwhile, it is noted that \( \sigma(t) \) and \( \omega(t) \) satisfy the relation
\[
\alpha(t) = 6\sigma^2(t)\Theta \omega(t).
\]
(7)

By letting
\[
\Theta \frac{\alpha(t)}{\alpha(t)} X \Xi_x + \Theta \Xi_x = 0,
\]
one obtains
\[
\Theta = 0, \quad \alpha(t) = 0,
\]
(8)

By solving (8), we have got the following results:
\[
\Theta(t) = c_i, \alpha(t) = c_j, \beta(t) = -c_j \int \omega(t)dt,
\]
(9)

where \( c_i \) and \( c_j \) are free constants.

In the next section, we will use the multivariate transformation technique
\[
\phi(x, t) = c_i \Xi[X(x, t), T(t)],
\]
(10)

where \( X(x, t) = c_i x - c_j \int \omega(t)dt, T(t) = -\frac{c_i c_j}{6} \int \sigma(t)dt \).

To get soliton solution, breather solutions, rational solutions and analyse their non-linear dynamics properties.

3 Exact Solutions for vc-mKdV Equation

3.1 Soliton Solutions for vc-mKdV Equation

The one- and two-soliton solutions of (6) were obtained using Wronskian scheme. According to the transformation (10), we can obtain the new generalised one- and two-soliton solutions of vc-mKdV equation given by (1) as follows

\[
\phi_i(x, t) = \frac{\alpha_i c_i}{\sigma_i} \int \left( c_i x - c_j \int \omega(t)dt \right) + \frac{2}{3} k_i^2 \int \sigma(t)dt - \ln(K),
\]
(11)

where \( K = \pm 1 \), and \( k_i \) are non-constants. In view of (11), we see that the amplitude is \(-2c_i K k_i\), and velocity of the wave is \( \frac{1}{2k_i^2} \int \sigma(t) dt \). For the solution (11) non-linear dynamics process see Figures 1 and 2.

And the two-soliton solution in the form

\[
\phi_2(x, t) = 2c_i \left( \int \left( c_i x - c_j \int \omega(t)dt \right) \right)
\]
(12)

where \( k_i \) \((i = 1, 2)\), \( l_j \) \((j = 1, 2, 3, 4)\) are constants.

The two-soliton solution (12) processes two types of soliton solutions, namely the soliton–soliton interaction and the soliton–anti-soliton interaction, as shown in Figures 3 and 4.

3.2 Breather Solutions for vc-mKdV Equation

The breather solutions of constant-coefficient mKdV equation were also obtained using the Wronskian scheme. Applying the same idea, the breather solution can be expressed as

\[
\phi_2(x, t) = -2c_i \left( \int \left( c_i x - c_j \int \omega(t)dt \right) \right),
\]
(14)

with
Figure 1: Plots of the one-soliton solution (11) with parameters $c_1 = 1$, $c_2 = 2$, $K = 1$, $k = 2$, $w(t) = 1$, $\sigma(t) = 2t$. (a) 3D-one-soliton solution and (b) 2D-curse at $t = 0$.

Figure 2: Plots of the one-soliton solution (11) with parameters $c_1 = 1$, $c_2 = 2$, $K = 1$, $k = -2$, $w(t) = 1$, $\sigma(t) = 2t$. (a) 3D-one-anti-soliton solution and (b) 2D-curse at $t = 0$.

Figure 3: Plots of the soliton–soliton interactions (12) with parameters $c_1 = 1$, $c_2 = 2$, $K = 1$, $k_1 = -0.7$, $k_2 = 0.2$, $w(t) = 1$, $\sigma(t) = 2t$, $l_1 = l_2 = -1$, $l'_1 = l'_2 = 1$. (a) 3D-plot and (b) 2D-density plot.
For the breather solutions some non-linear dynamics behaviour is depicted in Figures 5–7.

Figure 5a,b shows an oscillating wave moving along the curve line. This point is different from the constant-coefficient mKdV equation.

When $3k^2_1 = k^2_2$, there is a “stationary” breather and the oscillating wave moving along a straight line as seen in Figure 6.
in Figure 6a, b. However, the upper-lower amplitude is different from the constant-coefficient mKdV equation. When $|k_1|$ is small enough and $|k_2| \gg |k_1|$ so that the oscillation dominates, we can obtain a splindle-like wave (Fig. 7b).

### 3.3 Rational Solutions for vc-mKdV Equation

In this subsection, we will give two non-trivial rational solutions. The non-trivial rational solutions of constant-coefficient KdV–mKdV equation were also obtained using Wronskian scheme. Applying the same idea as above and Galilean transformation, the rational solutions can be expressed in the form

$$\phi(x, t) = c_x c_1 - \frac{4c_x^2}{4c_x^2(X - 6c_x^2T) + 1},$$

with

$$X(x, t) = c_x x - c_1 \int \omega(t) dt, \quad T(t) = -\frac{c_x^3}{6} \int \sigma(t) dt,$$

where $c_x$ is a non-zero constant.

It is a non-singular travelling wave moving with constant speed $6c_x^2c_1$, constant amplitude $-3c_x c_1$ and the asymptotic line is $\phi = c_x c_1$. It is depicted in Figures 8 and 9.
Figure 9: Shape and motion of the rational solution (16) with parameters $c_1 = 0.5$, $c_2 = 2$, $c_2 = 1$, $w(t) = 1$, $a(t) = t$. (a) 3D-plot for breather solution and (b) 2D-curve at $t = 0$.

Figure 10: Plots of the rational solution (21) with parameters $c_1 = 1$, $c_2 = -0.5$. (a) 3D-plot for rational solution and (b) 2D-density plot.

And another rational solution can be given by

$$
\phi(x, t) = c_0c_1 - \frac{12c_0c_1}{4c_0^2} \left[ (X - 6c_0^2T)^4 + \frac{3}{2c_0^2} (X - 6c_0^2T)^2 - \frac{3}{16c_0} - 24(X - 6c_0^2T) \right],
$$

with

$$
X(x, t) = c_0x - c_1 \int \omega(t) \, dt, \quad T(t) = -\frac{c_1^2c_2}{6} \int \sigma(t) \, dt.
$$

To understand better the asymptotic behaviour of (18), we rewrite the solution (18) in the following coordinate system

$$
(L = c_0x - c_1 \int w(t) \, dt + c_1^2c_2^2c_3 \int a(t) \, dt, \quad t),
$$

and it gives

$$
\phi_\tilde{r}(x, t) = c_0c_1 - \frac{12c_0c_1}{4c_0^2} \left[ L^4 + \frac{3L^2}{2c_0^2} - 24L \right],
$$

It is described in Figures 10 and 11.
From Figures 10 and 11, we see that the rational solution (21) possessing display shape, motion and density plot was overlapped by the above wave trajectory curve.

4 Conclusion

In this paper, based on the multivariate transformation technique we have searched soliton solutions, breather solutions and rational solutions to the vc-mKdV equations. These soliton solutions, breather solutions and rational solutions were first obtained by us. Meanwhile, we analysed some of their non-linear dynamical properties and non-linear evolution processes by showing graphics in order to demonstrate different physical phenomena which could help us to better understand the evolution of solution waves.

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References

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