Hui Gao*, Tianzhou Xu and Gangwei Wang

Optical Solitons for the Perturbed Nonlinear Schrödinger Equation with Kerr Law and Non-Kerr Law Nonlinearity

https://doi.org/10.1515/zna-2017-0400
Received November 6, 2017; accepted February 6, 2018; previously published online February 26, 2018

Abstract: This paper analyses the dynamics of soliton propagation through optical fibres for the perturbed nonlinear Schrödinger equation with Kerr law and non-Kerr law nonlinearity. Several integration schemes are used to construct solitons to the model. The two forms of nonlinearity that are studied in detail are power law and dual power law, while Kerr law and parabolic law emerge as special cases to these two laws.

Keywords: Optical Solitons; Perturbed Nonlinear Schrödinger Equation; Kerr Law; Integrability.

PACS codes: 02.30.Jr; 05.45.Yv; 02.30.Ik.

1 Introduction

In nonlinear optics, the nonlinear Schrödinger equation (NLSE) [1–6] can be used to describe the propagation of an optical pulse in nonlinear media including Kerr law and non-Kerr law media [7–10]. Several advances have been made in the area of the non-Kerr law media, such as power law, parabolic law, dual power law, and so on, which have attracted much attention.

The analytical solutions of the Schrödinger equations may give more insight into the described optical pulse transmission. So, an important issue with Schrödinger equations is to find their new analytical solutions and study the characteristics of the solutions. There are many relative methods, such as the modified mapping method and the extended mapping method, the modified trigonometric function series method, the bifurcation method and qualitative theory of dynamical systems, the modified \( \left( \frac{G'}{G} \right) \)-expansion method, and the dynamical systems approach, to study Schrödinger equations [11–15]. One type of the most significant solutions comprise the optical solitons, which hold their shapes in the process of transmission and collisions. Because of the perfect balance between dispersion or (and) diffraction and nonlinear effect in a nonlinear optical fibre, optical solitons are often utilised in optical fibre communication areas.

This paper is devoted to studying the NLSE with perturbation [3, 11]

\[
iq_x + aq_{xx} + bF(|q|^2)q = i(\alpha q_x + \lambda(|q|^m q)_x) + f(|q|^{2m}, q),
\]

where \( x \) represents the non-dimensional distance along the fibre, \( t \) represents time in dimensionless form, and \( a \) and \( b \) are real-valued constants. The dependent variable \( q(x, t) \) is a complex-valued function that represents the wave profile. \( \alpha \) is the inter-modal dispersion, \( \lambda \) represents the coefficient of self-steepening for short pulses (typically \( \leq 100 \) fs), and \( f \) is the higher order dispersion coefficient. The parameter \( m \) is a nonlinearity parameter [16]. \( F(|q|^2) \) is a real-valued function which represents the type of nonlinear media. When the right-hand side of the equation is zero, (1) is the simplified dimensionless form of the NLSE. It is a nonlinear partial differential equation that is not integrable, in general. The non-integrability is not necessarily related to the nonlinear terms in it.

Equation (1) is widely used in optics as a good model for optical pulse propagation in nonlinear fibres. In this paper, we consider four types of nonlinearity, namely Kerr law, power law, parabolic law, and dual power law. It is obvious that Kerr law and parabolic law are special cases of the other two laws. The three different effective integration schemes are the tanh function method, the Kudryashov method, and the trial solution method [17–19]. To the best of our knowledge, these methods have not been employed in solving the NLSE with perturbation.

Let

\[
q(x, t) = u(s)e^{iq(x,t)},
\]

*Corresponding author: Hui Gao, School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, PR China, E-mail: weil2345yi@126.com
Tianzhou Xu: School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, PR China, E-mail: xutianzhou@bit.edu.cn
Gangwei Wang: School of Mathematical and Statistical Sciences, Hebei University of Economics and Business, Shijiazhuang 050061, PR China; and School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, PR China, E-mail: pukai1211@163.com
where \( u(s) \) represents the shape of the pulse, \( s = x - vt \), and \( \phi = -kx + cot + \theta \). The function \( \phi(x, t) \) is the phase component of the soliton, \( k \) is the soliton frequency, while \( \omega \) is the wave number, \( \phi \) is the phase constant, and \( v \) is the velocity of the soliton.

Substituting (2) into (1), and then equating the real and imaginary parts to zero, we obtain two equations. The imaginary part gives

\[
2a \left( \frac{1}{n} - 1 \right) v' + \frac{a}{n} v'' - (w + c k + ak^2) v' - \lambda k v^{2m-2} + bv^4 = 0. 
\]

Equation (3) gives the velocity of the wave, while (4) can be integrated to compute the soliton profile provided the functional is known. NLSE with perturbation will be considered for the following four forms of nonlinearity, as discussed in the following two sections.

2 Power Law

Power law nonlinearity is encountered in various materials including semiconductors [9]. This law is also applicable to higher order photon processes that dominate at different intensities. However, the power law solves the problem of small-K condensation in weak turbulence theory in nonlinear plasmas. We can see this law as a generalisation of the Kerr law nonlinearity.

For power law nonlinearity

\[
F(u) = u^\alpha. 
\]

Substituting (5) into (4) gives

\[
au'' - (w + c k + ak^2)u - \lambda k u^{2m-1} + bu^{2n+1} = 0, 
\]

where the parameter \( n \) dictates the power law nonlinearity. It is necessary to take \( 0 < n < 2 \) because of stability issues.

2.1 Application of the Kudryashov Method

In this subsection, we will apply the Kudryashov method for the NLSE with perturbation with power law nonlinearity. Balancing \( u'' \) with \( u^{2n+1} \) to obtain closed-form solutions gives \( N = \frac{1}{n} \). Using the transformation

\[
u = v^n,
\]

(6) becomes

\[
a \left( \frac{1}{n} - 1 \right) v' + \frac{a}{n} v'' - (w + c k + ak^2) v' - \lambda k v^{2m-2} + bv^4 = 0. 
\]

Case 1:

When \( m = \frac{1}{n} \). In this case, (8) takes the form

\[
a \left( \frac{1}{n} - 1 \right) v' + \frac{a}{n} v'' - (w + c k + ak^2) v' - \lambda k v + bv^4 = 0.
\]

Balancing \( v'' \) with \( v^4 \) in (9), we find that \( N = 1 \). The solution of (9) is of the form

\[
v(\xi) = A_0 + A Q(\xi),
\]

where \( A_i \) (\( i = 0, 1 \)) are some constants to be determined later, and \( Q(\xi) \) satisfies

\[
Q = \mu Q - Q^3.
\]

Substituting (10) into (9) with (11), collecting the coefficients of \( Q \), and solving the resulting system, we have

Case 1.1:

\[
A_0 = -\mu A_0, A_1 \neq 0, w = a \mu^2 - ak^2 n^2 - ckn^2 / n^2,
\]

where \( \mu \) is an arbitrary constant.

Consequently, (6) has the following exact soliton solutions:

\[
u(x, t) = \left\{ -\mu A_0 - A_1 \left[ \frac{1}{n} \left( 1 \pm \frac{\mu (x + 2ak + c)t}{2} \right) \right] \right\}^{1/n},
\]

and

\[
\nu(x, t) = \left\{ -\mu A_0 - A_1 \left( 1 \pm \frac{\mu (x + 2ak + c)t}{2} \right) \right\}^{1/2},
\]

where \( \mu \) is an arbitrary constant.

Figure 1 shows the kink soliton with power law nonlinearity. The parameter values are \( A_i = \mu = a = k = 0 = n = 1 \).
Case 1.2:

\[ A_0 = 0, A_1 \neq 0, w = \frac{k(-A_\lambda \lambda^2 + A_\lambda \lambda^2 n^2 - \alpha n - 2 \alpha \mu)}{\mu (2 + n)}, \tag{15} \]

where \( \mu \) is an arbitrary constant.

Consequently, we obtain the exact solutions of (6) as follows:

\[ u(x, t) = \left\{ -A_1 \left( 1 \pm \tanh \left( \frac{\pm \mu (x + (2a + \alpha) t)}{2} \right) \right) \right\}^{\frac{1}{n}}, \tag{16} \]

and

\[ u(x, t) = \left\{ -A_1 \left( 1 \pm \coth \left( \frac{\pm \mu (x + (2a + \alpha) t)}{2} \right) \right) \right\}^{\frac{1}{n}}, \tag{17} \]

where \( \mu \) is an arbitrary constant.

2.2 Application of the Trial Solution Method

In this subsection, we will apply the trial solution method to handle the NLSE with perturbation with power law nonlinearity. Equation (9) has the trial solution

\[ (v')^2 = A_0 + A_1 v + A_1 v^2 + A_1 v^3 + A_1 v^4. \tag{18} \]

Thus, we have

\[ v'' = \frac{A_1}{2} + A_1 v + \frac{3A_1}{2} v^2 + 2A_1 v^3. \tag{19} \]

Inserting (18) and (19) into (9), equating all coefficients of \( v \), and solving the resulting system, we get

\[
\begin{align*}
A_0 &= A_1 = 0, \\
A_2 &= \frac{n^2}{(n + 1)(n + 2)a} (an^2 k^2 + 3ank^2 + 2ak^2 + kcn^2) + 3kn\alpha + 2k\alpha + \nu n^2 + 2w + 3nw), \\
A_3 &= \frac{n^2}{(n + 1)(n + 2)a} (2kn\lambda + 2k\lambda), \\
A_4 &= \frac{n^2}{(n + 1)(n + 2)a} (bn + 2b),
\end{align*}
\tag{20}
\]

where \( w \) is an arbitrary constant.

Then (18) becomes

\[ (v')^2 = \frac{n^2}{(n + 1)(n + 2)a} ((an^2 k^2 + 3ank^2 + 2ak^2 + kcn^2 + 3kn\alpha + 2k\alpha + \nu n^2 + 2w + 3nw) v^4 + (2kn\lambda + 2k\lambda) v^3 + (bn + 2b) v^2). \tag{21} \]

Equation (6) has the following exact soliton solutions:

\[ u(x, t) = \left\{ -A_2 A_1 \left( \left( \frac{\sqrt{A_2 (x + t(2a + \alpha))}}{2} \right) \right) \right\}^{\frac{1}{n}}, \tag{22} \]

which are valid for

\[ A_2 > 0, \tag{23} \]

and

\[ u(x, t) = \left\{ -A_3 A_1 \left( \left( \frac{\sqrt{A_3 (x + t(2a + \alpha))}}{2} \right) \right) \right\}^{\frac{1}{n}}, \tag{24} \]

which are valid for

\[ A_2 < 0, A_3 > 0. \tag{25} \]

2.3 Application of the tanh Function Method

In this subsection, we apply the tanh function method for the NLSE with perturbation with power law nonlinearity. Using the formula

\[ u(s) = q \tanh^d (\mu s), \tag{26} \]
In (6), we get
\[
q(tanh^{\beta-1}(\mu s)(a\beta^2\mu^2 + a\beta\mu^2) + tanh^\beta(\mu s)(-2a\beta^2\mu^2) \\
- ak^2 - w - k\alpha) + tanh^{\beta-1}(\mu s)(a\beta^2\mu^2 - a\beta\mu^2) \\
- k\lambda q^{2n} tanh^{2n+1}(\mu s) + bq^{2n} tanh^{2n+1}(\mu s) = 0.
\] (27)

Equating all the exponents and the coefficients of \(\tanh\) via the balance method, we can obtain

**Case 1:**
\[
\begin{align*}
2m\beta + \beta &= \beta - 2, \\
2n\beta + \beta &= \beta + 2, \\
\alpha \beta^2 \mu^2 - a\beta \mu^2 + \lambda k q^{2n} &= 0, \\
2a\mu^2 + ak^2 + k\alpha + w &= 0, \\
a\beta^2 \mu^2 + a\beta \mu^2 + bq^{2n} &= 0.
\end{align*}
\] (28)

Solving the system (28) leads to
\[
\begin{align*}
q &= \left(\frac{a\mu^2 + a\mu^2}{n^2 - n} \right)^{1/2n} , \\
w &= -(2a\mu^2 + ak^2 + k\alpha),
\end{align*}
\] (29)

where \(\mu\) is an arbitrary constant.

Consequently, we obtain the exact solution of (6) as follows:
\[
u(x, t) = \left(\frac{a\mu^2 + a\mu^2}{n^2 - n} \right)^{1/2n} \tanh^\beta(\mu(x + (2ak + c)t)),
\] (30)

where \(\mu\) is an arbitrary constant.

**Case 2:**
\[
\begin{align*}
2m\beta + \beta &= \beta + 2, \\
2n\beta + \beta &= \beta + 2, \\
\alpha \beta^2 \mu^2 - a\beta \mu^2 + \lambda k q^{2n} &= 0, \\
2a\mu^2 + ak^2 + k\alpha + w &= 0, \\
a\beta^2 \mu^2 + a\beta \mu^2 + k\lambda q^{2n} &= 0.
\end{align*}
\] (31)

Solving the system (31) leads to
\[
\begin{align*}
q &= \left(\frac{a\mu^2 - a\mu^2}{n^2 - n} \right)^{1/2n} , \\
w &= -(2a\mu^2 + ak^2 + k\alpha),
\end{align*}
\] (32)

where \(\mu\) is an arbitrary constant.

Consequently, we obtain the exact solution of (6) as follows:
\[
u(x, t) = \left(\frac{a\mu^2 - a\mu^2}{n^2 - n} \right)^{1/2n} \tanh^\beta(\mu(x + (2ak + c)t)),
\] (33)

where \(\mu\) is an arbitrary constant.

The special case with \(n = 1\) for the power law nonlinearity reduces to Kerr law nonlinearity. Kerr law nonlinearity comes from the fact that a light wave in an optical fibre faces nonlinear responses from the non-harmonic motion of electrons bound in molecules, caused by an external electric field. Power law nonlinearity and Kerr law nonlinearity of the solution process are similar, so we omit the latter here.

**3 Dual Power Law**

This model is used to describe the saturation of the nonlinear refractive index and solitons in photovoltaic-photo-refractive materials such as LiNbO\(_3\). For dual power law nonlinearity
\[
F(u) = u^n + hu^{2n}.
\] (34)

Substituting (34) into (4) gives
\[
au'' - (w + ck + ak^2)u - \lambda ku^{2n+1} + bu^{2n+1} + blu^{4n+1} = 0.
\] (35)

**3.1 Application of the Kudryashov Method**

In this subsection, we will apply the Kudryashov method for the NLSE with perturbation with dual power law nonlinearity. Balancing \(u''\) with \(u^{4n+1}\) to obtain closed-form solutions gives \(N = \frac{1}{2n}\). Using the transformation
\[
u = \sqrt[2n]{u},
\] (36)

(35) becomes
\[
\frac{a}{2n} \left(\frac{1}{2n} - 1\right)v'' + \frac{a}{2n} v^2 - (w + ck + ak^2)v^2 - \lambda kv^{2n+1} + bv^{2n+1} + blv^{4n+1} = 0.
\] (37)

**Case 1:**

When \(\frac{m}{n} = 1\). In this case, (37) takes the form
\[
\begin{align*}
\frac{a}{2n} \left( \frac{1}{2} - 1 \right) v'^2 + \frac{a}{2n} v'' - (w + \alpha k + \alpha k^2) v^2 \\
- \lambda k v^2 + b v^4 + b v^4 = 0.
\end{align*}
\]

(38)

Balancing \(vv''\) with \(v^4\) in (38), we find that \(N=1\). The solution of (38) has the form

\[
v(\xi) = A_0 + A_1 Q(\xi),
\]

(39)

where \(A_i (i=0, 1)\) are some constants to be determined later, and \(Q(\xi)\) satisfies

\[
Q_\xi = \mu Q - Q^2.
\]

(40)

Case 1.1:

\[
A_0 = 0, A_1 \neq 0, w = \frac{4bk^2ln^2 A_1^2 - A_1^2 b^2 \mu^2 - 2knc - kc}{2n+1},
\]

(41)

where \(\mu\) is an arbitrary constant.

Consequently, (35) has the following exact soliton solutions:

\[
u(x, t) = \left\{ -A_1 \mu \left[ 1 \pm \tanh \left( \frac{\mu(x + (2ak + \alpha) t)}{2} \right) \right] \right\}^{\frac{1}{n}},
\]

(42)

and

\[
u(x, t) = \left\{ -A_1 \mu \left[ 1 \pm \coth \left( \frac{\mu(x + (2ak + \alpha) t)}{2} \right) \right] \right\}^{\frac{1}{n}},
\]

(43)

where \(\mu\) is an arbitrary constant.

Case 1.2:

\[
A_0 = -\alpha A_1, A_1 \neq 0, w = \frac{4bk^2n^2 A_1^2 - A_1^2 b^2 \mu^2 - 2knc - kc}{2n+1},
\]

(44)

where \(\mu\) is an arbitrary constant.

Consequently, (35) has the following exact soliton solutions:

\[
u(x, t) = \left\{ -\alpha A_1 \mu \left[ 1 \pm \tanh \left( \frac{\mu(x + (2ak + \alpha) t)}{2} \right) \right] \right\}^{\frac{1}{n}},
\]

(45)

and

\[
u(x, t) = \left\{ -\alpha A_1 \mu \left[ 1 \pm \coth \left( \frac{\mu(x + (2ak + \alpha) t)}{2} \right) \right] \right\}^{\frac{1}{n}},
\]

(46)

where \(\mu\) is an arbitrary constant.

\[
\begin{align*}
A_0 = 0, \frac{m}{n} = 2. \text{ In this case, (37) takes the form}
\end{align*}
\]

\[
\begin{align*}
\frac{a}{2n} \left( \frac{1}{2} - 1 \right) v'^2 + \frac{a}{2n} v'' - (w + \alpha k + \alpha k^2) v^2 \\
- \lambda k v^2 + b v^4 + b v^4 = 0.
\end{align*}
\]

(47)

Balancing \(vv''\) with \(v^4\) in (47), we find that \(N=1\). The solution of (47) has the form

\[
v(\xi) = A_0 + A_1 Q(\xi),
\]

(48)

where \(A_i (i=0, 1)\) are some constants to be determined later, and \(Q(\xi)\) satisfies

\[
Q_\xi = \mu Q - Q^2.
\]

(49)

Case 2.1:

\[
A_0 = 0, A_1 \neq 0, w = \frac{4ak^2n^2 + 4kncn^2 - a \mu^2}{-4n^2},
\]

(50)

where \(\mu\) is an arbitrary constant.

Consequently, (35) has the following exact soliton solutions:

\[
u(x, t) = \left\{ -\mu A_1 \left[ 1 \pm \tanh \left( \frac{\mu(x + (2ak + \alpha) t)}{2} \right) \right] \right\}^{\frac{1}{n}},
\]

(51)

and

\[
u(x, t) = \left\{ -\mu A_1 \left[ 1 \pm \coth \left( \frac{\mu(x + (2ak + \alpha) t)}{2} \right) \right] \right\}^{\frac{1}{n}},
\]

(52)

where \(\mu\) is an arbitrary constant.

Case 2.2:

\[
A_0 = -\alpha A_1, A_1 \neq 0, w = \frac{4bk^2n^2 A_1 - A_1^2 b^2 \mu^2 - 2knc - kc}{2\mu(n+1)},
\]

(53)

where \(\mu\) is an arbitrary constant.

Consequently, (35) has the following exact soliton solutions:

\[
u(x, t) = \left\{ -\mu A_1 \left[ 1 \pm \tanh \left( \frac{\mu(x + (2ak + \alpha) t)}{2} \right) \right] \right\}^{\frac{1}{n}},
\]

(54)
and
\[
u(x, t) = \left( -\mu A_1 - A_2 \left( \frac{1 \pm \coth \left( \frac{\mu (x + (2ak + \alpha)t)}{2} \right)}{-2} \right) \right)^{\frac{1}{2}},
\]
which are valid for
\[
A_j > 0,
\]
and
\[
u(x, t) = \left( -\mu A_1 - A_2 \sqrt{\frac{-A_1 (x + t(2ak + \alpha))}{2}} \right)^{\frac{1}{2}},
\]
which are valid for
\[
A_j < 0, \quad A_k > 0.
\]

3.2 Application of the Trial Solution Method

In this subsection, we will apply the trial solution method to handle the NLSE with perturbation with dual power law nonlinearity.

Case 1:
Equation (38) has the trial solution as follows:
\[
(v')^2 = A_1 + A_2 v^2 + A_3 v^3 + A_4 v^4.
\]
Thus, we have
\[
v'' = \frac{A_1}{2} + A_2 v^2 + \frac{3A_3}{2} v^3 + 2A_4 v^4.
\]

Inserting (56) and (57) into (38), equating all coefficients of \(v\), and solving the resulting system, we get
\[
\begin{align*}
A_0 &= A_1 = 0, \\
A_2 &= \frac{4n^2}{(n+1)(2n+1)} (2an^2 + 2kn^2 + 2kn' \alpha + ak^2 \\
&\quad + 3kn' \alpha + 2wn^2 + k \alpha + w + 3nw), \\
A_3 &= \frac{4n^2}{(n+1)(2n+1)} (-2kn\lambda + k\lambda - b - 2bn), \\
A_4 &= \frac{4n^2}{(n+1)(2n+1)} (-ln - lb),
\end{align*}
\]
where \(w\) is an arbitrary constant.

Then (56) becomes
\[
(v')^2 = \frac{4n^2}{(n+1)(2n+1)} \left( 2an^2 + 2kn^2 + 2kn' \alpha + ak^2 \\
+ 3kn' \alpha + 2wn^2 + k \alpha + w + 3nw \right) v^2 \\
+ (2kn\lambda + k\lambda - b - 2bn) v^3 + (-ln - lb) v^4.
\]

Equation (35) has the following exact soliton solutions:
\[
\begin{align*}
u(x, t) &= \begin{cases} \\
-\frac{A_j A_k \sech \left( \sqrt{\frac{A_j (x + t(2ak + \alpha))}{2}} \right)}{A_j^2 - A_k A_k} \left( 1 \pm \tanh \left( \sqrt{\frac{A_j (x + t(2ak + \alpha))}{2}} \right) \right)^{\frac{1}{2}},
\end{cases}
\end{align*}
\]

Figure 2 shows the periodic solution with dual power law nonlinearity. The parameter values are \(A_j = A_k = 2, A_j = a = k = \alpha = n = 1\).

Case 2:
Equation (47) has the trial solution as follows:
\[
(v')^2 = A_1 + A_2 v^2 + A_3 v^3 + A_4 v^4.
\]
Thus, we have
\[
v'' = \frac{A_1}{2} + A_2 v^2 + \frac{3A_3}{2} v^3 + 2A_4 v^4.
\]

Inserting (64) and (65) into (47), equating all coefficients of \(v\), and solving the resulting system, we get
\[
\begin{align*}
A_0 &= A_1 = 0, \\
A_2 &= \frac{4n^2}{(n+1)(2n+1)} (2an^2 + 2kn^2 + 2kn' \alpha \\
&\quad + ak^2 + 3kn' \alpha + 2wn^2 + k \alpha + w + 3nw), \\
A_3 &= \frac{4n^2}{(n+1)(2n+1)} (b + 2bn), \\
A_4 &= \frac{4n^2}{(n+1)(2n+1)} (ln + lb - kn\lambda - k\lambda),
\end{align*}
\]

Figure 2: Optical soliton with dual power law nonlinearity.
where \( w \) is an arbitrary constant.

Then (64) becomes

\[
(\nu')^2 = \frac{4n^2}{(n+1)(2n+1)a} (2an'k^2 + 3an'k\alpha + ak^2 + 3kn\alpha + 2wn^2 + k\alpha + w + 3nw)\nu' + (b + 2bn)\nu^3 + (ln + lb - kn\lambda - k\lambda)\nu^4.
\]

Equation (35) has the following exact soliton solutions:

\[
\begin{align*}
\nu(x, t) &= -\frac{A_2 A_3}{A_1 \left(1 \pm \tanh \left(\frac{\sqrt{A_1} (x + t(2ak + c))}{2}\right)\right)}^{\frac{1}{2}}, \\
&= -\frac{A_2 A_3}{A_1 \left(1 \pm \tanh \left(\frac{\sqrt{-A_1} (x + t(2ak + c))}{2}\right)\right)}^{\frac{1}{2}},
\end{align*}
\]

which are valid for

\[ A_2 > 0, \quad (69) \]

and

\[
\begin{align*}
\nu(x, t) &= \frac{A_2 A_3}{A_1 \left(1 \pm \tanh \left(\frac{\sqrt{-A_1} (x + t(2ak + c))}{2}\right)\right)}^{\frac{1}{2}}, \\
&= \frac{A_2 A_3}{A_1 \left(1 \pm \tanh \left(\frac{\sqrt{A_1} (x + t(2ak + c))}{2}\right)\right)}^{\frac{1}{2}},
\end{align*}
\]

which are valid for

\[ A_2 < 0, \quad A_3 > 0. \quad (70) \]

The special case with \( n = 1 \) for dual power law nonlinearity reduces to the parabolic law nonlinearity. Dual power law nonlinearity and parabolic law nonlinearity of the solution process are similar, so we omit the latter here.

### 4 Discussion and Conclusions

The dynamics of soliton propagation through optical fibres for the perturbed NLSE with two forms of nonlinear optical fibres was studied in detail in this paper. They are the power law nonlinearity and the dual power law nonlinearity. For the first case, the Kudryashov method was applied, which led to new kink solitons. The trial solution method and the tanh function method for this law gave new periodic solutions. For dual power law, only two algorithms were applied, which were the Kudryashov method and the trial solution method, and both these also retrieved new kink solitons and new periodic solutions. In using these methods, the parameter relationships for \( m \) and \( n \) were required to allow integrability of the perturbed NLSE. These results, which have not been reported previously, are immensely helpful for additional investigation of the perturbed NLSE, which is under way. However, we are considering whether there is still an effective method to investigate the perturbed NLSE with nonlinear optical fibres for future study. These results will be expected to eventually change the future of optical fibre communication technology.

**Conflict of interests:** The authors declare no conflicts of interest regarding the publication of this paper.

### References