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Flow and Heat Transfer Analysis of an Eyring–Powell Fluid in a Pipe

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Abstract: The steady non-isothermal flow of an Eyring–Powell fluid in a pipe is investigated using both perturbation and numerical methods. The results are presented for two viscosity models, namely the Reynolds model and the Vogel model. The shooting method is employed to compute the numerical solution. Criteria for validity of perturbation solution are developed. When these criteria are met, it is shown that the perturbation solution is in good agreement with the numerical solution. The influence of various emerging parameters on the velocity and temperature field is also shown.

Keywords: Eyring–Powell Fluid; Heat Transfer; Perturbation and Numerical Methods; Viscosity Models.

1 Introduction

The last few years have seen an increased interest in flows of non-Newtonian fluids because of their various applications in industry and technology. Owing to the diverse rheological features of non-Newtonian fluids, several constitutive models have been proposed in the literature. The simplest of these is the second-grade model, which is capable of capturing the normal stress effect. However, this model is not capable of predicting shear-thickening effect. The third-grade model is the generalisation of second-grade model and is capable of predicting both shear-thickening and normal stress effects. This model has been studied in detail from the thermodynamics point of view by Fosdick and Rajagopal [1] in 1980. Since then, the model has been extensively used to discuss flows of non-Newtonian fluids in different scenarios. Both second- and third-grade fluids [2–10] fall into the category of differential type fluids. Another important class of non-Newtonian fluids is generalised non-Newtonian fluids (GNFs). Power-law, Eyring–Powell, Sisko, and Carreau are the common examples of generalised Newtonian fluids. GNF models are simple and capable of predicting shear-thinning and shear-thickening effects but lack the ability to predict normal stress effects. Because of their simplicity, GNF models are also used extensively in the literature to discuss flow and heat transfer in non-Newtonian fluids [11–20]. Viscosity is the property of fluid that has the tendency to decrease with increasing temperature. Several models have been proposed in the literature to account for the dependence of viscosity on the local fluid temperature. Among them, the Reynolds model and the Vogel model are quite popular. Massoudi and Christie [21] discussed flow and heat transfer in a pipe using the finite difference method. Later, Yurusoy and Pakdemirli [22] constructed approximate analytical solutions for the same problem using a perturbation method. Ellahi and Riaz [23] employed homotopy analysis to investigate the temperature-dependent viscosity effects in the flow of a third-grade fluid under the impact of magnetohydrodynamics (MHD) in a pipe. Hayat et al. [24] presented an analytical solution based on homotopy analysis for the pipe flow of a third-grade fluid and discussed the influence of variable viscosity and viscous dissipation on that fluid. Ellahi and Afzal [25] extended the work of Hayat et al. [24] by including the porous medium effect. Turkyilmazoglu [26] employed the spectral Chebyshev collocation numerical technique to predict the effects of thermal radiation on MHD transient permeable flow with variable viscosity on a rotating disc. Turkyilmazoglu [27] also provided exact solutions to heat transfer in fins with variable physical properties. The Reynolds and Vogel models have also been used by several authors to discuss the interaction between flow velocity and temperature in other scenarios [2, 6–8, 21–23, 27].

In 1944, Erying and Powell [28] developed the constitutive relation of Erying–Powell fluids. This model has gained popularity in the fields of engineering and technology, and several researchers have been using the Erying–Powell model to model isothermal hydrodynamic fluid phenomena inside channels and pipes and over stretching sheets with industrial and engineering applications. In the next paragraph, we present a brief review of these attempts.

Tanveer et al. [29] discussed the force convection peristaltic flow of an EP fluid with nanoparticles in a
curved channel. A numerical study of EP fluid flow over a stretching sheet under the influence of magnetic field was carried out by Akbar et al. [30]. A series of solutions of MHD Erying–Powell nano-fluid flow over a stretching surface were presented by Qayyum et al. [31]. Hayat et al. [32] investigated the effects of thermophoresis and Brownian motion on the flow of an Erying–Powell fluid inside a stretching cylinder. Hayat et al. [33] employed convective boundary conditions to investigate an Erying–Powell fluid flow over a moving surface. Recently, Hayat et al. [34] calculated the skin friction and heat transfer rate for an MHD EP fluid flow over a sheet by employing the homotopy approach.

However, from the above-cited literature, it is noted that the problem of flow and heat transfer in a pipe for EP fluids using the Reynolds and Vogel viscosity models has not been investigated. The purpose of this paper is to analyse this flow situation in detail.

1.1 Formulation of the Problem

Consider the two-dimensional, fully developed, steady-state, incompressible EP fluid inside a circular tube of radius $R$. It is assumed that the flow in the tube is due to the applied pressure gradient along the $z$-direction. It is also assumed that the temperature of the fluid is less than that of the tube. Thus, a uniform temperature is delivered to the pipe, and subsequent variations in temperature of the fluid inside the pipe are sought (Fig. 1).

The constitutive equation for the Erying–Powell model is given by

$$
T = -pI + S.
$$

(1)

For this model, the extra stress tensor is of the following form [28]:

$$
S = \left[ \mu + \frac{1}{K_2} \sinh^{-1}(K_2 \psi) \right] \mathbf{A}.
$$

(2)

where $\psi = \frac{1}{2} \text{tr}(\mathbf{A}')$.

Equation (1) was presented by Powell and Erying [28]. We apply the second-order approximation on $\sinh^{-1}$ in the form

$$
\sinh^{-1}(K_2 \psi) \equiv K_2 \frac{\psi}{6} - \frac{1}{6} (K_2 \psi)^3, |K_2 \psi| \ll 1.
$$

(3)

Equation (2) can then be written as

$$
S = \left[ \mu + A - B \left( \frac{dw}{dr} \right)^2 \right] \mathbf{A}, B \ll 1,
$$

(4)

where $A = \frac{K_1}{K_2}, B = \frac{K_1^3}{K_2^2}, A_1 = \mathbf{L} + \mathbf{L}'$, and $\mathbf{L}$ is the velocity gradient.

The appropriate velocity and temperature field for problem under investigation are

$$
V = [0, 0, w(r)],
$$

(5)

$$
\theta = \theta(r).
$$

(6)

Inserting the values of $A_1$ into (4), the following form of $S$ emerges:

$$
S = \left[ \mu + A - B \left( \frac{dw}{dr} \right)^2 \right] \left[ \begin{array}{ccc} 0 & 0 & \frac{dw}{dr} \\ 0 & 0 & 0 \\ \frac{dw}{dr} & 0 & 0 \end{array} \right].
$$

(7)

Using (8) in equation of motion without body force, one gets

$$
1 \frac{d}{dr} \left( \frac{wr}{dr} \frac{dw}{dr} \right) + \frac{1}{r} \frac{d}{dr} \left( r \left( A \frac{dw}{dr} - \frac{B}{6} \left( \frac{dw}{dr} \right)^2 \right) \right) = \frac{dp}{dz}.
$$

(9)

Three viscosity models are chosen for further analysis. The first one is the constant viscosity model for which $\mu = 1$. The second model is the Reynolds model for which $\mu = e^{-n \theta}$. The third one is the Vogel model for which $\mu = e^{ \frac{x}{r^{\alpha} \beta} }$. Assuming (5) and (6) hold, the energy equation becomes

$$
0 = \text{tr}(\mathbf{T} \mathbf{L}) - \text{div} \mathbf{q} + \rho r',
$$

(10)

where $\mathbf{q}$ is the heat flux vector and $r'$ is the radiative heating. The term $\text{tr}(\mathbf{T} \mathbf{L})$ in the present scenario is

$$
\text{tr}(\mathbf{T} \mathbf{L}) = \mu \left( \frac{dw}{dr} \right)^2 + A \left( \frac{dw}{dr} \right)^2 - B \left( \frac{dw}{dr} \right)^4.
$$

(11)

In view of (11), the energy equation (10) reduces to...
\[ \frac{\mu}{k} \left( \frac{dw}{dr} \right)^2 + A \left( \frac{dw}{dr} \right)^2 - B \frac{dw}{dr} - \frac{\text{div} \mathbf{q} + \rho \dot{r}}{6k} - \frac{\text{div} \mathbf{q} + \rho \dot{r}}{r} = 0. \] (12)

It is assumed that the heat flux vector \( \mathbf{q} \) is represented by Fourier's law with a constant thermal conductivity \( k \), i.e.
\[ \mathbf{q} = -k \nabla \theta, \] (13)
where \( \theta \) is temperature of the fluid.

Using (13) in (12) and neglecting radiant heating term, i.e. \( \dot{r} = 0 \), we get
\[ \frac{\mu}{k} \left( \frac{dw}{dr} \right)^2 + A \left( \frac{dw}{dr} \right)^2 - B \frac{dw}{dr} + \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0. \] (14)

The equations to be solved are (9) and (14), subject to the following boundary conditions:
\[ w = 0, \theta = \theta_w \text{ at } r = R, \text{ and } \frac{dw}{dr} = \frac{d \theta}{dr} = 0 \text{ at } r = 0. \] (15)

Here, \( R \) is the radius of the pipe, \( \theta_w \) is the pipe temperature, \( w_o \) is reference velocity, and \( \mu_o \) is a reference viscosity.

Introducing the normalised quantities
\[ \bar{r} = \frac{r}{R}, \quad 
\bar{w} = \frac{w}{w_o}, \quad \bar{\theta} = \frac{\theta - \theta_m}{\theta_m - \theta_w}, \quad \bar{\mu} = \frac{\mu}{\mu_o}. \] (16)

Equation (14) takes the form
\[ \frac{d^2 \bar{\theta}}{d \bar{r}^2} + \frac{1}{\bar{r}} \frac{d \bar{\theta}}{d \bar{r}} + \lambda \left( \frac{dw}{dr} \right)^2 \left( 1 + M - M \chi \left( \frac{dw}{dr} \right)^2 \right) = 0, \] (17)
where \( \lambda = \mu_o w_o^2 / (k(\theta_m - \theta_w)) \) is called dimensionless non-Newtonian viscosity.

The boundary conditions on \( \bar{\theta} \) are
\[ \bar{\theta} = 0 \text{ at } \bar{r} = 1, \quad \frac{d \bar{\theta}}{d \bar{r}} = 0 \text{ at } \bar{r} = 0. \] (18)

The dimensionless version of the equation of motion (9) for different viscosity models are given as follows:

(i) **Constant viscosity model**
\[ \frac{1}{\bar{r}} \frac{d}{d \bar{r}} \left( \bar{r} \frac{d \bar{w}}{d \bar{r}} \right) + M \frac{d}{d \bar{r}} \left( \frac{d \bar{w}}{d \bar{r}} - \chi \left( \frac{d \bar{w}}{d \bar{r}} \right)^2 \right) = C, \] (19)

where \( M = \frac{A}{\mu \chi} = \frac{K_i w_o^2}{6 \mu_o R^2} \).

(ii) **Reynolds’ model**
\[ e^{-\sigma} \left( \frac{1}{\bar{r}} \frac{d}{d \bar{r}} \left( \frac{d \bar{w}}{d \bar{r}} \right) - \frac{d \bar{\theta}}{d \bar{r}} \right) - \frac{M}{\bar{r}} \frac{d \bar{w}}{d \bar{r}} + M \frac{d}{d \bar{r}} \left( \frac{d \bar{w}}{d \bar{r}} \right)^2 \left( 1 + M - M \chi \left( \frac{d \bar{w}}{d \bar{r}} \right)^2 \right) = C, \] (20)

where \( \chi = \frac{K_i w_o^2}{6 \mu R} \), \( C = (\partial p / \partial z)(R^2 / \mu_w) \), and \( \mu^* = \mu e^{-\sigma} \). For this case, \( \lambda \) in the energy equation (17) is defined by \( \lambda = \mu^* w_o^2 / (k(\theta_m - \theta_w)) \).

(iii) **Vogel’s model**
\[ e^{-\sigma} \left( \frac{1}{\bar{r}} \frac{d}{d \bar{r}} \left( \frac{d \bar{w}}{d \bar{r}} \right) - \frac{d \bar{\theta}}{d \bar{r}} \right) - \frac{M}{\bar{r}} \frac{d \bar{w}}{d \bar{r}} - \chi \left( \frac{d \bar{w}}{d \bar{r}} \right)^2 \left( \frac{d \bar{w}}{d \bar{r}} \right)^2 + \frac{1}{r} \frac{d \bar{\theta}}{d \bar{r}} = C, \] (21)

where \( \mu^* = \mu e^{-\sigma} \).

Solution for each case is presented below. The bars will be removed for simplicity.

(i) **Constant viscosity case**
In the present case, the governing equations are
\[ \frac{\frac{dw}{dr}}{r} + \frac{d^2 w}{dr^2} + \frac{M}{r} \frac{d}{dr} \left( \frac{dw}{dr} - \chi \left( \frac{dw}{dr} \right)^2 \right) = C, \] (22)

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} + \chi \left( \frac{d \bar{w}}{d \bar{r}} \right)^2 \left( 1 + M - M \chi \left( \frac{dw}{dr} \right)^2 \right) = 0. \] (23)

To find the solution of the above equations, we use the perturbation expansion for velocity and temperature fields as
\[ w = w_o + \epsilon w_1, \quad \theta = \theta_o + \epsilon \theta_1, \quad \text{and} \quad \chi' = \epsilon \eta, \] (24)

where \( \epsilon \) denotes perturbation parameter. The expansions defined in (24) are substituted into (22) and (23) and boundary conditions given in (18). This yields the following systems:

**System of \( O(\epsilon^0) \):**
\[ \frac{d^2 w_o}{dr^2} + \frac{d w_o}{dr} = \frac{C}{1 + M}, \] (25a)

\[ \frac{d^2 \theta_o}{dr^2} + \frac{1}{r} \frac{d \theta_o}{dr} + \frac{\chi}{1 + M} \left( \frac{dw}{dr} \right)^2 = 0. \] (25b)

\[ w_o = \theta_o = 0 \text{ at } r = 1, \quad \text{and} \quad \frac{d w_o}{dr} = \frac{d \theta_o}{dr} = 0 \text{ at } r = 0. \] (25c)

**System of \( O(\epsilon^1) \):**
\[ \frac{d^2 w_1}{dr^2} + \frac{d w_1}{dr} = \frac{\eta M}{1 + M} \left( \frac{d w_o}{dr} \right)^2 + \frac{3}{2} \frac{3}{r^2} \left( \frac{d w_o}{dr} \right)^2, \] (26a)

\[ \frac{d^2 \theta_1}{dr^2} + \frac{1}{r} \frac{d \theta_1}{dr} = -\frac{\lambda}{1 + M} \left( \frac{dw}{dr} \right)^2 \left( 2 + M \chi \left( \frac{dw}{dr} \right)^2 \right). \] (26b)
The solution of the leading-order system is

\[ w_0 = -\frac{C}{4(1+M)}(1-r^2), \]
\[ \theta_0 = \frac{\lambda C^2}{64(1+M)}(1-r^4). \]

Based on leading-order solution, the determining equations for the correction terms give

\[ w_i = -\frac{\eta C^3 M}{32(1+M)^3}(1-r^3), \]
\[ \theta_i = -\frac{\eta \lambda C^3 M}{576(1+M)^5}(1-r^5). \]

Substituting (27) and (28) into expansion given in (24), the final velocity and temperature expressions in the original parameters are

\[ w = -\frac{C}{4(1+M)}(1-r^2) - \frac{M \chi' C^3}{32(1+M)^3}(1-r^3), \]
\[ \theta = \frac{\lambda C^2}{64(1+M)}(1-r^4) + \frac{M \chi' \lambda C^3}{576(1+M)^5}(1-r^5). \]

For the validity of (29a) and (29b), the following criteria must be met:

\[ \frac{\chi' C^3}{32(1+M)^3} \ll \frac{C}{4(1+M)} \quad \text{and} \quad \frac{\chi' \lambda C^3}{576(1+M)^5} \ll \frac{\lambda C^2}{64(1+M)}. \]

Both inequalities in (30) reduce to

\[ \epsilon^T \ll \chi' M^3 \quad \text{and} \quad \frac{\chi' \lambda C^3}{8(1+M)^3} \ll 1. \]

Equation (31) represents the final criterion for a valid perturbation expansion. Note that these criteria are developed on the notion that the correction term should be much less than the leading term.

(ii) Reynolds’ model

In the present case, we apply the perturbation expansion on velocity, temperature fields, and \( \chi' \) as given in (24). The expansion for \( L \) is given by

\[ L \equiv \epsilon l. \]

Thus we can write to the first order of approximation

\[ e^{-\epsilon \theta} = 1 - \epsilon \theta \quad \text{and} \quad Le^{-\epsilon \theta} = -\epsilon. \]

The above expressions along with expansions (24) when substituted into (20) and (17) give the following systems at various orders of \( \epsilon \):

Zeroth-order system:

\[ \frac{d^2 w_0}{dr^2} + \frac{dw_0}{dr} = \frac{Cr}{(1+M)}, \]
\[ \frac{d^3 \theta_0}{dr^3} + \frac{1}{r} \frac{d \theta_0}{dr} + \lambda(1+M) \left( \frac{d w_0}{dr} \right)^2 = 0. \]

First-order system:

\[ \frac{d^2 w_1}{dr^2} + \frac{dw_1}{dr} = \frac{l \theta_0}{(1+M)} + \left( \frac{d w_0}{dr} + \frac{d^2 w_0}{dr^2} \right) + \eta \frac{M \eta}{(1+M)} \left( \frac{d w_0}{dr} \right)^2 + \frac{M \eta}{(1+M)} \left( \frac{d w_0}{dr} \right)^3 \]
\[ \frac{d^3 \theta_1}{dr^3} + \frac{1}{r} \frac{d \theta_1}{dr} = \lambda \left( \frac{d \theta_0}{dr} + M \eta \frac{d w_0}{dr} \right)^2 \]
\[ -2 \lambda(1+M) \frac{d w_0}{dr} \frac{d w_1}{dr} \]

\[ w_1 = \theta_1 = 0 \text{ at } r = 1 \quad \text{and} \quad \frac{dw_1}{dr} = \frac{d \theta_1}{dr} = 0 \text{ at } r = 0. \]

The solution of the leading terms of the equations is same as for the constant viscosity case. Using these solutions in (35a) and (35b) and solving the resulting equations subject to boundary conditions (35c), we get

\[ w_i = -\frac{\eta C^3 M}{32(1+M)^3}(1-r^3) - \frac{L \lambda C^3}{768(1+M)^3}(2-3r^2 + r^4), \]
\[ \theta_i = \frac{L \lambda C^3}{16,384(1+M)^3}(3-4r^4 + r^8) + \frac{\eta \lambda C^3 M}{576(1+M)^5}(1-r^4). \]

Combining zeroth-order and first-order solutions, one finds

\[ w = -\frac{C}{4(1+M)}(1-r^2) - \frac{\chi' C^3 M}{32(1+M)^3}(1-r^3) - \frac{L \lambda C^3}{768(1+M)^3}(2-3r^2 + r^4), \]
The criteria for valid perturbation expansion are given by
\[ c_i = \left( \frac{\lambda M C_i^2}{2(1 + M)^2} \right) \ll 1, \quad \text{and} \quad c_i = \left( \frac{\lambda M C_i^2}{9(1 + M)^3} \right) \ll 1. \] (38)

(iii) Vogel’s model
For this case, the dimensionless viscosity function is
\[ \mu = \exp \left( \frac{3(Y + 8)}{2} \right). \] (39)

The above expression of viscosity is more complicated compared to the Reynolds model of viscosity. As a result, the first-order equations of temperature and velocity profile would be very complex. To this end, we choose \( \lambda = \varepsilon \sigma \) and \( \chi = \varepsilon \eta \), and assume the following expansions for velocity and temperature:
\[ w = w_0 + \varepsilon w_1 \quad \text{and} \quad \theta = \theta_0 + \varepsilon \theta_1. \] (40)

Using all the above expressions in the original equations (energy and momentum) and the boundary conditions, one gets the following equations at the leading and first orders.

Equations for leading term solution:
\[ O(\varepsilon) : \quad \frac{d^2 w_0}{dr^2} + \frac{1}{r} \frac{dw_0}{dr} + \sigma \left( \frac{1}{a} + M \right) \left( \frac{dw_0}{dr} \right)^2 = 0, \] (41b)
\[ w_0 = \theta_0 = 0 \quad \text{at} \ r = 1 \quad \text{and} \quad \frac{dw_0}{dr} = \frac{d\theta_0}{dr} = 0 \quad \text{at} \ r = 0. \] (41c)

Equations of the correction term:
\[ O(\varepsilon) : \quad \frac{1}{r} \frac{d^3 w_1}{dr^2} = \frac{X}{Y^2(1 + Ma)} \theta_1 \left( \frac{dw_0}{dr} + \frac{r^2 d^2 w_0}{dr^2} \right), \] (42a)
\[ + \frac{1}{1 + Ma} \left( \frac{1}{a} + M \right) \left( \frac{dw_0}{dr} + 3r \frac{d^2 w_0}{dr^2} \right), \]
\[ \frac{d^2 \theta_1}{dr^2} + \frac{1}{r} \frac{d\theta_1}{dr} = \sigma \left( \frac{1}{a} \right) \left( \frac{Y^2}{\theta_0 + \eta M \left( \frac{dw_0}{dr} \right)^2} \right), \] (42b)
\[ - 2\sigma \left( \frac{1}{a} \right) \frac{d^2 w_0}{dr^2} \frac{dw_0}{dr} \]

The leading term solution is given by
\[ w_0 = -\frac{\delta}{4(1 + Ma)} (1 - r^2), \] (43a)
\[ \theta_0 = \frac{\sigma \delta}{64a(1 + Ma)} (1 - r^2). \] (43b)

Using the above solutions in (43a) and (43b), the correction term solution turns out to be
\[ w_1 = \frac{\eta M \delta}{32(1 + Ma)} (1 - r^2) - \frac{X \sigma \delta}{768aY^2(1 + Ma)} (2 - 3r^2 + r^4), \] (44a)
\[ \theta_1 = \frac{X \sigma \delta}{16,384Y^2(1 + Ma)} (3 - 4r^2 + r^4) + \frac{\eta M \sigma \delta}{192(1 + Ma)} (1 - r^2). \] (44b)

The final form of velocity and temperature based on the expansions given in (40) are
\[ w = \frac{\delta}{4(1 + Ma)} (1 - r^2) - \frac{X \sigma \delta}{768aY^2(1 + Ma)} (2 - 3r^2 + r^4), \] (45a)
\[ \theta = \frac{\lambda \delta}{64a(1 + Ma)} (1 - r^2) + \frac{X \lambda^2 \delta}{16,384Y^2(1 + Ma)} (3 - 4r^2 + r^4) \]
\[ - \frac{\chi M \lambda \delta}{192(1 + Ma)} (1 - r^2). \] (45b)

In the present case, the criteria of valid perturbation expansions are
\[ c_i = \left( \frac{\lambda M \delta}{8(1 + Ma)} \right) \ll 1 \quad \text{and} \quad c_i = \left( \frac{3X \lambda \delta}{256 a Y^2(1 + Ma)} \right) \ll 1. \] (46)

2 Numerical Solution

The energy and momentum equations in all three cases were also solved numerically using the shooting technique [35, 36]. The results presented graphically in next section are produced using both solutions. A brief outline of the method for solution of the constant velocity case is given below.

For application of the shooting method, the boundary value problem for the constant viscosity case (17)–(19) is converted into a first-order initial value problem as
subject to the initial conditions
\[ w(0) = s, \quad f(0) = 0, \quad \theta(0) = t, \quad g(0) = 0. \]

In (50), the constants \(s\) and \(t\) are the unknown missing conditions, and such missing conditions are chosen so that the boundary conditions at unity are satisfied. The Newton–Raphson iterative scheme [37, 38] is employed to calculate the missing conditions, and the whole numerical scheme is summarised in flow chart. It is remarked that the numerical solution obtained via the shooting method is independent of all criteria developed for valid perturbation expansion in the previous section.

### 3 Results and Discussion

The purpose of this section is to illustrate the effects of various involved parameters on velocity and temperature using perturbation solution obtained in the previous sections. The numerical solution of the problem obtained via the shooting method is also shown in all plots via filled dots. Figure 2 highlights the effects of the material parameter \(M\) of the EP fluid on both velocity and temperature. First, we observe that for the selected values of \(M\), there is excellent agreement between both perturbation and numerical solutions. This obviously proves the validity of the perturbation solution for the selected set of parameters. Further, it is observed that the rheological parameter \(M\) causes a deceleration in the flow velocity. Similarly, the temperature of the fluid also decreases on increasing the parameter \(M\). The effect of the parameter \(L\) is shown in Figure 3. This parameter characterises the dependence of viscosity on temperature in the Reynolds model. Smaller values of \(L\) correspond to weak dependence of viscosity on temperature, and vice versa. It is noted that
both temperature and velocity increase with increasing $L$. The impact of the driving pressure gradient on the flow velocity and temperature fields is shown in Figure 4. Here, it is seen that the flow accelerates and its temperature rises with increasing magnitude of the driving pressure gradient. Figures 5 and 6 highlight the effects of the rheological parameter $\chi^*$ of the EP fluid on both velocity and temperature, which are found to increase with increasing $\chi^*$. However, this increase is not significant when $M=0.5$. A rapid increase is noted in the velocity and temperature profiles with increasing $\chi^*$ when $M=0.01$. Therefore, it is concluded that an increase in the parameter $\chi^*$ has a small effect on both the velocity and temperature fields when $M$ is relatively large, i.e. $M=0.5$. However, the effect is more when $M$ is relatively small, i.e. $M=0.01$. It is further observed that, in Figure 6, the perturbation solution deviates from the numerical solution for $\chi^*=80$. This deviation is due to the fact that the criteria given by (38) are not satisfied. In fact, for this case, (38) gives $c_{r_1}=0.75$ and $c_{r_2}=0.67$. Clearly, both these values are not much less than unity. Figures 7–10 are plotted for Vogel’s model. The equation governing the velocity and temperature fields for Vogel’s model has three characteristic parameters, namely, $X$, $Y$, and $\theta_\infty$, in addition to the rheological parameters $\chi^*$ and $M$. 

**Figure 2:** Effects of $M$ on the velocity and temperature profiles for $C=-1, \lambda=L=1$, and $\chi=0.5$.

**Figure 3:** Effects of $L$ on the velocity and temperature profiles for $C=-1, \lambda=\chi^*=1$, and $M=0.03$.

**Figure 4:** Effects of $C$ on the velocity and temperature profiles for $M=0.01, \lambda=L=1$, and $\chi^*=0.25$. 
Figure 5: Effects of $\chi^*$ on the velocity and temperature profiles for $C = -1, \lambda = \theta = \delta = X = Y = 1$, and $M = 0.5$.

Figure 6: Effects of $\chi^*$ on the velocity and temperature profiles for $C = -1, \lambda = \theta = \delta = X = Y = 1$, and $M = 0.1$.

Figure 7: Effects of $M$ on the velocity and temperature profiles for $C = -1, \lambda = \theta = \delta = X = Y = 1$, and $\chi^* = 0.1$.

Figure 8: Effects of $X$ on the velocity and temperature profiles for $C = -1, \lambda = \theta = \delta = M = Y = 1$, and $\chi^* = 0.2$. 
Figure 7 shows the effects of $M$ on velocity and temperature for Vogel’s model. Both these variables follow a decreasing trend with increasing $M$.

The effects of the parameter $X$ on velocity and temperature are shown in Figure 8. The effect of $X$ is to decrease both the velocity and temperature. In contrast, an increase in the Vogel’s model parameter $Y$ accelerates the flow and raises the temperature of the fluid (Fig. 9). A similar trend is noted while increasing the parameter $\theta_w$.

4 Conclusion

Steady-state flow of an EP fluid in a pipe was studied both analytically and numerically for the case where the viscosity is dependent on the local fluid temperature. The flow and energy equations were modelled for two viscosity models, namely the Reynolds model and the Vogel model. The solution was obtained via perturbation and numerical techniques. The validity of perturbation solution was checked by comparing it with the corresponding numerical solution. The influence of various emerging parameters was shown through several plots. It was found that characteristic parameters of the Reynolds and Vogel models could affect the velocity and temperature of the EP fluid in similar way as they do affect the velocity and temperature fields of a third-grade fluid [22]. The rheological parameter $M$ of the EP fluid decreases the flow velocity and temperature. In contrast, the rheological parameter $\chi^*$ enhances both these quantities. It is concluded that, although the shooting method gives better solution over a wider range of the involved parameters, it does not provide analytical expressions. Such analytical expressions may be helpful in future for calculating the velocity and temperature profiles efficiently once experimental data for the dimensionless material parameters of the EP fluid are available.

References