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Exact Solutions of the Nonlocal Nonlinear Schrödinger Equation with a Perturbation Term

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Abstract: Analytical solutions of both the nonlinear Schrödinger equation (NLSE) and NLSE with a perturbation term have been attained. Besides, analytical solutions of nonlocal NLSE have also been obtained. In this paper, the nonlocal NLSE with a perturbation term is discussed. By virtue of the dependent variable substitution, trilinear forms of this equation is attained. Lax pairs and Darboux transformation of this equation are obtained. Via the Darboux transformation, two kinds solutions of this equation with the different seed solutions are attained.

Keywords: Darboux Transformation; Nonlocal Nonlinear Schrödinger Equation; PT Symmetry.

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1 Introduction

The nonlinear Schrödinger equation [1–9],

\[ i\mu \zeta + \frac{1}{2} \mu \zeta \zeta + \mu^2 \mu^* = 0, \]

has been used to characterise rogue waves generated by nonlinear energy transfer in an open ocean [2] and broadband optical pulse propagation in nonlinear fibres [3], where \( \bar{\mu} = -1 \), \( * \) is the complex conjugate, while \( \mu \) is the envelope of the wave field and depends on the variables \( \zeta \) and \( \xi \) [3]. Rogue waves and solitons are both the analysis solutions of (1) with different seed solutions. Physically speaking, a rogue wave is thought of as an isolated huge wave with an amplitude claimed to be much larger than the average wave crests around it in the ocean [4–6] and also seen in other fields [2, 7, 6]. A soliton is a solitary wave that preserves its velocity and shape after an interaction [4]; i.e. the soliton can be considered as a quasi-particle [8, 9].

Besides, nonlinear Schrödinger equation with a perturbation term has also been studied [10]:

\[ iP_T^\mu - P_{xx} - 2(PP' - B^3)P = 0, \] (2)

where \( B \) is a real constant and \( P \) is a complex function of the real variables \( x \) and \( \xi \). The trivial condensate solution of (2) is \( P = B \). While this solution is unstable with respect to small perturbations, the growth rate of instability is \( p_0 |B^2 - p^2|/4 \), where \( p \) is the wave number of perturbation. Relations of rogue waves and the nonlinear stage of modulation instability for (2) have been attained via the dressing method [10].

On the other hand, nonlocal nonlinear Schrödinger equation have been discussed [11]:

\[ iQ(X, T), -Q(X, T)x - 2Q(X, T)Q'(-X, T) = 0, \] (3)

where \( Q(X, T) \) is the complex-valued function of the real variables \( X \) and \( T \). Solutions of (3) have been attained by virtue of the inverse scattering transform [11–15]. It is remarkable that (3) turns out to be Lax integrable [12–15]. Some properties of (3) has also been proved as follows [11]: time-reversal symmetry: if \( Q(X, T) \) is a solution, so is \( Q(-X, -T) \); invariance under the transformation \( X \rightarrow -X \): If \( Q(X, T) \) is a solution, so is \( Q(-X, T) \); PT symmetry: if \( Q(x, t) \) is a solution, so is \( Q(-x, t) \); gauge invariance: if \( Q(X, T) \) is a solution, so is \( \exp(iI_X, T) \) with real and constant \( \theta_X \); complex translation invariance: If \( Q(X, T) \) is a solution, so is \( Q(X + i\lambda_x, T) \) for any constant and real \( \lambda_x \).

In this paper, we will work on the the nonlocal nonlinear Schrödinger equation with a perturbation,

\[ iq(x, t) - q(x, t)x + 2q(x, t)[-q'(-x, t)q(x, t) - A^2] = 0, \] (4)

where \( q(x, t) \) is a complex function of the real variables \( x \) and \( t \) and \( A \) is a real constant. We note that (4) also has the same properties as (3) as mentioned above; \( q(x, t) = A \) is a condensate solution of (4).

The paper is organised as follows. By virtue of the symbolic computation [16–20], in Section 2, Lax pairs, Darboux transforms, and \( N \)-order solutions of (4) will be obtained. In Section 3, trilinear forms of (4) will be attained. In Section 3 are the conclusions.

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2 Lax Pairs and Darboux Transformation of (4)

Lax pairs could transform a nonlinear evolution equation into a linear system. Thus, via the lax pairs, N-order solutions of (4) can be obtained. By virtue of the Ablowitz-Kaup-Newell-Segur [21] procedure, Lax pairs of (4) are

\[ \Phi(x, t) = U \Phi(x, t), \quad \Phi(x, t) = V \Phi(x, t) , \quad \text{where} \quad U = \begin{bmatrix} -\lambda & q(x, t) \\ -q(x, t) & \lambda \end{bmatrix}, \quad V = \begin{bmatrix} i[-q(x, t) & \lambda q(x, t) - A' - 2i\lambda^2] \\ 2i\lambda - q(x, t) & -i[q(x, t) - A'] \end{bmatrix}, \]

where \( \lambda \) is the complex spectrum parameter. Via the compatibility condition, \( U - V + UV - VU = 0 \), (4) can be attained.

2.1 Darboux Transformation of (4)

In this section, Darboux transformation of (4) will be attained. We set that \( q(x, t) = q_0(x, t) \) is a seed solution of (4) and \( \Phi(x, t) = [\phi_1(x, t), \phi_2(x, t)]^T \) is a solution of Lax pairs (5) at \( \lambda = \lambda' \), while \( \lambda' \) is a complex spectrum parameter independent of \( x \) and \( t \). Then, \( \Psi(x, t) = [-\phi_1(x, t), -\phi_2(x, t)]^T \) is also a solution of Lax pairs (5) at \( q_0(x, t) \) and \( \lambda = \lambda' \). A Darboux matrix \( M \) has the form

\[ M = \lambda E - H \Lambda H^{-1}, \]

where \( E \) is the two-order identity matrix and the superscript \(-1\) denotes the inverse matrix. Therefore, the first-order solutions of (4) are attained as

\[ q_1(x, t) = q_0(x, t) + 2 \sum_{i=1}^{N} \frac{(\lambda_i - \lambda') \phi_1(x, t) \phi_2^*(-x, t)}{\phi_1(x, t) \phi_2(-x, t) - \phi_2(x, t) \phi_1^*(-x, t)} , \]

with \( \lambda_i \)'s as the different complex parameters and

\[ [\phi_{1,i}(x, t) \phi_{2,i}(x, t)]^T = M_1 [\phi_{1,i}(x, t) \phi_{2,i}(x, t)]^T , \]

where \( [\phi_{1,i}(\lambda), \phi_{2,i}(\lambda)] \) is a solution of Lax pairs (5) at \( \lambda = \lambda_i \) and \( q = q_{1,i} \) while \( \phi_{1,i}(x, t) \) and \( \phi_{2,i}(x, t) \) are all the functions of \( x \) and \( t \).

2.2 Solutions of (4)

In the following, we will attain some solutions with different seed solutions of (4) via (9).

Case 1. We will take \( q(x, t) = 0 \) as the seed solution, then the solutions of Lax pairs (5) at \( \lambda = \lambda_i \),

\[ \begin{bmatrix} \phi_{1,i}(x, t) \phi_{2,i}(x, t) \end{bmatrix} = [\exp(\eta_t) \exp(-\eta_t)] , \]

\[ \eta_t = -\lambda x - it(A' + 2\lambda^2) t + \delta_t , \]

where \( \delta_t \) is a complex constant. The first-order solutions of (4) are

\[ q_1(x, t) = \frac{2(\lambda - \lambda') \exp(-2itA't)}{\exp[2\delta_t + 2\lambda x + 2it\lambda] - \exp[-2\delta_t + 2\lambda x + 2it\lambda]} . \]

It can be found that solutions (11) has a singularity. To study those properties, we rewrite \( \lambda = Re(\lambda) + \text{Im}(\lambda) \) and \( \delta_t = Re(\delta_t) + \text{Im}(\delta_t) \), where Re(\( x \)) and Im(\( x \)) denote the real part and image part of \( x \), respectively. Singularity points of (11) are along the characteristic line

\[ \text{Re}(\delta_t) + 4\text{Re}(\lambda) - 2\text{Im}(\lambda)x = 0. \]

If we repeat the process for the expressions in Section 2.1, two-order solutions of (4) can be attained as follows:

\[ q_2(x, t) = q_1(x, t) + \frac{2(\lambda - \lambda') \phi_{1,i}(x, t) \phi_{2,i}^*(-x, t)}{\phi_{1,i}(x, t) \phi_{2,i}(-x, t) - \phi_{2,i}(x, t) \phi_{1,i}^*(-x, t)} , \]

where

\[ [\phi_{1,i}(x, t) \phi_{2,i}(x, t)]^T = M_1 [\exp(\eta_t) \exp(-\eta_t)]^T , \]
If we continue such process, we can derive the N-order solutions of (4).

**Case 2.** Plane wave seed solutions of (4) will be taken in the following. We take \( q_0(x, t) = c \exp[-2(A^2 + c^2)i\xi t] \) as the seed solution of (4), where \( c \) is a real constant. We introduce the following transformation:

\[
\lambda_j = -\frac{ic}{2} \left( \xi_j - 1 \right), \quad \xi_j = R_j \exp(i\beta_j), \quad (j = 1, 2, 3, \ldots, N)
\]  

(19)

where \( \beta_j \) is an angle and belongs to \([0, 2\pi]\) and \( R_j \) is a real constant.

In this case we can take the wave functions as

\[
\phi_{ni}(x, t) = [h_n \exp(k_n + w_n t + i\beta_n) + i h_n \exp(-k_n x - w_n t)] \exp(i(A^2 + c^2)t),
\]

(20)

\[
\phi_{ni}(x, t) = [h_n \exp(-k_n x - w_n t + i\beta_n) + i h_n \exp(k_n x + w_n t)] \exp[-i(A^2 + c^2)t],
\]

(21)

where \( k_n \) and \( w_n \) are both complex constant independent \( x \) and \( t \). By virtue of Lax pairs (5), relations of \( \beta_n, k_n, w_n, h_n, \) and \( h_n \) are obtained as

\[
k_n = \frac{i c \cos[\phi_{ni}(x, t) + \phi_{ni}(x, t)]}{2 R_n},
\]

\[
w_n = \frac{i c \cos[-2i\beta_n][R_n \exp(4i\beta_n) - 1]}{2 R_n^2},
\]

\[
h_n = R_n h_n.
\]

Thus, the first-order solutions of (4) are

\[
q_1(x, t) = q_0(x, t) - 2ic \left( R_1 - \frac{1}{R_1} \right) \cos \beta_n \frac{\phi_{ni}(x, t)\phi_{ni}(x, t) - \phi_{ni}(x, t)\phi_{ni}(x, t)}{\phi_{ni}(x, t)\phi_{ni}(x, t) - \phi_{ni}(x, t)\phi_{ni}(x, t)}.
\]

(22)

Second-order solutions of (4) are

\[
q_2(x, t) = q_0(x, t) - 2ic \left( R_2 - \frac{1}{R_2} \right) \cos \beta_n \frac{\phi_{ni}(x, t)\phi_{ni}(x, t) - \phi_{ni}(x, t)\phi_{ni}(x, t)}{\phi_{ni}(x, t)\phi_{ni}(x, t) - \phi_{ni}(x, t)\phi_{ni}(x, t)}.
\]

(23)

where \( \phi_{ni}(x, t) \) and \( \phi_{ni}(x, t) \) are

\[
[M_{\phi_{ni}(x, t) \phi_{ni}(x, t)}] = M_{\phi_{ni}(x, t) \phi_{ni}(x, t)},
\]

(24)

\[
H_1 = \left[ \begin{array}{cc} \phi_{ni}(x, t) - \phi_{ni}(x, t) \\ \phi_{ni}(x, t) - \phi_{ni}(x, t) \end{array} \right].
\]

(25)

If we continue such process, we can derive the N-order solutions of (4).

### 3 Trilinear Forms of (4)

Multi-linear forms of the nonlinear evolution equations can help people to attain their solutions, especially to the bilinear forms of those [22]. In the following, we will obtain the trilinear forms of (4) by virtue of the independent variable substitution. With the aid of the transformation,

\[
q(x, t) = \frac{f(x, t)}{f(x, t)},
\]

(26)

where \( f(x, t) \) is a real function of the variables \( x \) and \( t \), while \( g(x, t) \) is a complex function of the variables \( x \) and \( t \) of (4) have the following trilinear forms:

\[
[iD^2 f(x, t)] g(x, t) = \frac{\phi(x, t) f(x, t)}{f(x, t)},
\]

(27)

\[
f(x, t) g(x, t) + 2 f(x, t) f(x, t) A^2 - \frac{\phi(x, t) f(x, t)}{f(x, t)} = \phi(x, t) f(x, t) f(x, t),
\]

(28)

where \( \phi \) is a complex constant and the operator \( D^m_D^n \) is defined as [22]

\[
D^m_D^n a(x, t) \cdot b(x', t')
\]

(29)

\[
\begin{bmatrix} D^m \frac{\partial}{\partial x} - D^n \frac{\partial}{\partial t} \end{bmatrix} a(x, t) b(x', t')
\]

\[\quad \left|_{\alpha x, \beta t} \right.,
\]

with \( m \) and \( n \) both as the integers, \( a(x, t) \) as the function of \( x \) and \( t \), and \( b(x', t') \) as the function of formal variables \( x' \) and \( t' \).

### 4 Conclusions

In this paper, we have studied the nonlocal nonlinear Schrödinger equation (4) with a perturbation term. Our main results are displayed as follows: (a) Trilinear forms (3) of (4) have been attained, by virtue of the independent
variable transformation. (b) Lax pairs (5) and Darboux transformation (Section 2.1) have been both obtained. Via the Darboux transformation, two kinds of the \( N \)-order solutions of (4) with different seed solutions have been attained. From the above discussion, we have found that some properties of (4) are different from both (3) and (2).

References