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Quasi-Periodic Solutions to the Mixed Kaup-Newell Hierarchy

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Abstract: The mixed Kaup-Newell (mKN) hierarchy, including the nonholonomic deformation of the KN equation, is obtained in the Lenard scheme. By the nonlinearity of the Lax pair, the mKN hierarchy is reduced to a family of mixed, finite-dimensional Hamiltonian systems (FDHSs) that separate its temporal and spatial variables. It turns out that the Bargmann map not only gives rise to the finite parametric solutions of the mKN hierarchy but also specifies a finite-dimensional, invariant subspace for the mKN flows. The Abel-Jacobi variables are selected to linearise the mKN flows on the Jacobi variety of a Riemann surface, from which some quasi-periodic solutions of mKN hierarchy are presented by using the Riemann-Jacobi inversion.

Keywords: Abel-Jacobi Variables; Forward and Backward Hamiltonian Systems; Mixed Kaup-Newell Equations; Quasi-Periodic Solutions.

1 Introduction

Infinite-dimensional integrable systems are described by integrable nonlinear evolution equations (INLEEs) with many applications in nonlinear waves, nonlinear optics, plasma physics, magnetic fluids, and so on [1–3]. With the isospectral nature [4], one emblematic character of INLEEs is that they are members of integrable hierarchies and can be formally expressed by the Lenard operator pair $K, J$, and Lenard gradients. INLEEs residing in the same hierarchy share the spatial part of Lax pairs, infinitely many conserved quantities, a pair of bi-Hamiltonian operators, and the commutable flows [5]. In fact, by selecting the kernel of $J$ as the initial values, one gets a class of positive-order INLEEs; if defining the kernel of $K$ as the starting points, one has a sequence of negative-order INLEEs. It is noted that the temporal part of Lax pairs of negative-order INLEEs admits negative powers of the spectral parameter with the singularity at zero. The negative-order INLEEs are therefore termed ‘the negative-order integrable systems’. The union of positive- and negative-order INLEEs will be recognised as the mixed soliton hierarchy [6, 7]. In this article, we make an endeavour to study the integrable structure and some explicit solutions associated with the mixed Kaup-Newell (mKN) hierarchy.

Most of negative-order INLEEs can only be realised in a form of nonlocal equations. The cases where the negative-order INLEEs can be thrown into the local form with physical backgrounds are therefore of special interest, such as the Camassa-Holm (CH) equation [8], the Degasperis-Procesi (DP) equation [9], and some others [10, 11], which allow not only the peaked soliton solutions (peakons) [8, 10, 12, 13] but also the modelling of wave-breaking [14]. Apart from the peakon systems which can be embedded into negative-order INLEEs [8–11], they are also related by the reciprocal transformations to some previously known negative-order INLEEs, like the CH and the negative-order Korteweg-de Vries (KdV) equations [15], the DP equation, and the negative-order Kaup-Kupershmidt equations [16], the 2CH and the negative-order Ablowitz-Kaup-Newell-Segur equations [10], in which the reciprocal transformation results in the key information of explicit solutions to the peakon systems. Very recently, Karasu-Kalkanli et al. [17] derived a sixth-order wave equation of KdV$^6$ which passes the the Painlevé test. Kupershmidt depicted the KdV$^6$ equation as the nonholonomic deformation of the KdV equation by means of a pair of Hamiltonian operators [6]. Zhou further extended it to the mixed hierarchy of integrable equations [7], which includes the nonholonomic deformation as its special member. The nonholonomic deformation and its generalisation are in fact the integrable couplings of negative- and positive-order flows. To our knowledge, the key step in solving the nonholonomic deformation or its generalisation of integrable equations is dependent upon the problem of managing negative-order flows. Based on the above observations, this article lays special emphasis on the study of negative-order flows.

To understand the dynamics of INLEEs in depth, it is indispensable to have detailed knowledge on the integrable structure and some explicit solutions for INLEEs. There exist several mature methods to obtain explicit solutions of INLEEs, such as the inverse scattering...
transformation [1–3] for N-soliton solutions, the algebro-geometric method [2,18–20] for quasi-periodic (finite-gap, or algebro-geometric) solutions, and some others in scope to include the Bäcklund and Darboux transformation [21,22], the Hirota bilinear transformation [23], the nonlinearisation of the Lax pair [24,25], and others. Referring to the theory of algebraic curves [26,27] and the nonlinearisation of the Lax pair [24,25], the mixed (backward and forward) finite-dimensional Hamiltonian systems (FDHSs) will be utilised for getting quasi-periodic solutions of the mKN hierarchy, which are the compound integrable systems of the positive-order Kaup-Newell (pKN) hierarchy labelled by the following KN equations [28]:

\[
\begin{align*}
\tilde{q}_s &= \frac{1}{2} \tilde{q}_{ss} - \frac{1}{2} \tilde{q}_t \tilde{r}_t - \frac{1}{2} \tilde{q}^2 \tilde{r}_s, \\
\tilde{r}_s &= -\frac{1}{2} \tilde{r}_{ss} - \frac{1}{2} \tilde{q}_t \tilde{r}^2 - \tilde{q} \tilde{r}_s,
\end{align*}
\]

(1)

which can be converted to the derivative nonlinear Schrödinger (dNLS) equation via \( \tilde{r} = -i \tilde{q} \) and \( t_s \rightarrow 2it_s \)

\[
i \tilde{q}_s = \tilde{q}_{ss} + i(\tilde{q}^2 \tilde{q}_s),
\]

(2)

and the negative-order Kaup-Newell (nKN) hierarchy characterised by the new negative-order INLEEs

\[
\begin{align*}
q_{\sigma_{s,\epsilon}} &= -4a_{\epsilon}(q + \tau q q_{\epsilon}), \\
r_{\sigma_{s,\epsilon}} &= 4a_{\epsilon}(q \tau r - \tau),
\end{align*}
\]

(3)

in view of \( \tilde{q} = gq \) and \( \tilde{r} = r \), where \( a_{\epsilon} \) is a constant.

The dNLS equation is a typical integrable model with several physical applications, which can be used for modelling wave processes, such as the propagation of ultrashort pulses in nonlinear optics, the Stokes waves in fluids of finite depth, and so on. To solve the dNLS equations (2), Kaup and Newell proposed a Lax pair displaying its Lax integrability and derived the one-soliton solution by the inverse scattering transformation [28]. From the KN spectral problem, the canonical structure, isospectral deformation, Riemann-Hilbert problem, and Hamiltonian structure were sequentially completed for the KN system [29–31]; moreover, N-soliton solutions, periodic solutions, and almost periodic solutions to the KN system have been obtained through various approaches [32–36]. Cao presented a (2+1)-dimensional derivative Toda equation in the context of the KN spectral problem and gave its finite genus solution [37]. Ma et al. [38] began with a different version of the KN spectral problem that can also yield the KN hierarchy, where the binary nonlinearisation of the Lax pair was discussed for the KN system of dNLS equation. With an algebraic curve of arithmetic genus \( n \), Geng et al. [39] arrived at the quasi-periodic solutions for the pKN hierarchy by the meromorphic solutions. Different from the above treatments, we set up a connection between the mixed FDHSs and the mKN hierarchy, from which some explicit solutions are obtained for the mKN hierarchy.

Let us now give a brief sketch of our treatment for the mKN hierarchy. Starting with the spectral problem [38], we deduce the nonholonomic deformation of KN equations and further specify its Lax integrability. From the nonlinearisation of the Lax pair [24], the mKN hierarchy is reduced to a family of backward and forward FDHSs, which separate the spatial and temporal variables. It follows from the Lax equation and the \( \epsilon \)-technique [40] that the resulting FDHSs are completely integrable in the Liouville sense. To solve mixed INLEEs by backward and forward FDHSs, one crucial point is to establish the relationship between FDHSs and mixed INLEEs. It is known that the involutive solutions of backward and forward FDHSs generate not only the finite parametric solutions of mKN hierarchy but also the finite-gap potential of negative and positive \( N \)-order stationary KN equations [41,42]. By a set of canonical bases of cycles on a hyperelliptic curve of Riemann surface, the Abel maps are defined to introduce the Abel-Jacobi variables, which straighten out the mKN flows on the Jacobi variety of a Riemann surface. Based on the compatibility of backward and forward Hamiltonian flows, the Riemann-Jacobi inversion is applied to the linearised mKN flows, from which some quasi-periodic solutions to the mKN hierarchy are derived in terms of the Riemann theorem.

This article is organised as follows. In Section 2, we deduce the nonholonomic deformation of the KN equations and establish its Lax integrability. The mKN hierarchy is decomposed into a sequence of backward and forward FDHSs, making the finite-dimensional reductions in Section 3; and the Liouville integrability of FDHSs is accomplished in Section 4. Section 5 focuses on the relationship between the mKN hierarchy and the mixed FDHSs. In Section 6, the algebraic geometrical data are processed for getting the quasi-periodic solutions of mKN hierarchy.

2 The Nonholonomic Deformation of KN Hierarchy

We begin with the spectral problem [38]

\[
\varphi_{s,\epsilon} = U \varphi, \quad U = \lambda \sigma_{s,\epsilon} + \tilde{q} \sigma_s + \lambda \tilde{r} \sigma_{\epsilon}, \quad \varphi = (\varphi_s, \varphi_{\epsilon})^T,
\]

(4)
where $\lambda$ is a spectral parameter, $\bar{q}$ and $\bar{r}$ are two potentials, and

$$
\sigma_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

(5)

Solve the adjoint representation of (4)

$$
V^*_s = [U, V], \quad V = \begin{cases} a\sigma_3 + b\sigma_1 + \lambda c\sigma_2 = \sum_{j=0}^3 (a_j\sigma_3 + b_j\sigma_1 + \lambda c_j\sigma_2)\lambda^{-j}, & j \geq 0, \\ -a\sigma_3 - b\sigma_1 - \lambda c\sigma_2 = -\sum_{j=-1}^{3} (a_j\sigma_3 + b_j\sigma_1 + \lambda c_j\sigma_2)\lambda^{-j}, & j \leq -1, \end{cases}
$$

(6)

which are equal to

$$
a_i = -\frac{1}{2}\bar{q}i, \quad b_i = \bar{q}i, \quad c_i = \bar{r},
$$

$$
a_1 = \frac{1}{4}(\bar{q}i - \bar{q}^2i) + \frac{3}{8}\bar{q}^2i^2, \quad b_1 = \frac{1}{2}(\bar{q}i - \bar{q}^2i), \quad c_1 = -\frac{1}{2}(\bar{r}i + \bar{q}^2i),
$$

$$
a_2 = 2a_1\partial^{-1}(\bar{q}i\partial^{-1}\bar{q} + \partial^{-1}\bar{q}i),
$$

$$
b_2 = -4a_1(\partial^2\bar{q} + \partial^{-1}\bar{q}^2i + \bar{q}^{-1}\partial\bar{q}i),
$$

$$
c_2 = 4a_1(\partial^{-1}\bar{q}^{-1}(\partial\bar{q}i + \partial^{-1}\bar{q}i) - \partial^{-2}\bar{r}),
$$

(7)

where $\partial^{-1}$ is the inverse operator of $\partial = \partial/\partial x$ with the condition $\partial \partial^{-1} = \partial^{-1} \partial = 1$. Based on the recursive formula (7), we introduce the Lenard gradients $(g_j) (j \in \mathbb{Z})$

$$
Kg_{j+1} = Jg_j, \quad g_j = (c_{j+1}, b_{j+1}, a_{j+1})^T, \quad j \in \mathbb{Z},
$$

(9)

where

$$
J = \begin{pmatrix} -2\partial^{-1}\bar{q} & 2\partial\bar{q}^{-1}\bar{r} \\ -2\partial\bar{q}^{-1}\bar{q} & -2\partial^{-1}\bar{q} \end{pmatrix}, \quad K = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}.
$$

(10)

The recursive formula $g_{j+1} = K^{-1}Jg_j (j \leq -2)$ gives rise to the backward Lenard gradients $g_{-1}, g_{-2}, \ldots$ and so on; and $g_j = J^kKg_1 (j \geq 1)$ produces the forward Lenard gradients $g_0, g_1, \ldots$, and so on. By (8), we arrive at the first few members

$$
g_{-1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad g_0 = \begin{pmatrix} \bar{r} \\ \bar{q} \end{pmatrix}, \quad g_1 = \begin{pmatrix} 1/2(\bar{r}i + \bar{q}^2i) \\ 1/2(\bar{q}i - \bar{q}^2i) \end{pmatrix},
$$

(11)

$$
g_2 = \begin{pmatrix} 1/4\bar{q}^3i + 3/8\bar{q}^4i^2 + 3/8\bar{q}^{-1}i^2 \\ 1/4\bar{q}^3i - 3/8\bar{q}^2i + 3/8\bar{q}^{-1}i^2 \end{pmatrix}, \quad g_{-2} = \begin{pmatrix} 2a_1\partial^{-1}\bar{r} \\ -2a_1\partial^{-1}\bar{q} \end{pmatrix},
$$

(12)

$$
g_3 = \begin{pmatrix} 4a_1(\partial^{-1}\bar{r}^{-1}(\bar{q}i\partial^{-1}\bar{r} + \partial^{-1}\bar{r}i) - \partial^{-2}\bar{r}) \\ -4a_1(\partial^{-1}\bar{q} + \partial^{-1}\bar{q}^{-1}(\bar{q}i\partial^{-1}\bar{r} + \partial^{-1}\bar{r}i)) \end{pmatrix}.
$$

(13)

It is assumed that time-dependent $\varphi$ satisfies a differential equation given by the forward Lenard gradients

$$
\varphi_{i+1} = V^{(i)}\varphi, \quad n \geq 1,
$$

(14)

where

$$
V^{(i)} = \sigma(\bar{q}, \bar{r}, \lambda)[g_{i+1}] = g^{(i)}_1\sigma_1 + lg^{(i)}_2\sigma_2 + \lambda \partial^{-1}(\bar{q}g^{(i)}_n - \bar{r}g^{(i)}_n)\sigma_3,
$$

(15)

$$
g_i = (g^{(i)}_1, g^{(i)}_2, g^{(i)}_3) = \sum_{j=1}^{n} g_{j-i}(\lambda^{-j}),
$$

(16)

and $\sigma: \mathbb{R} \rightarrow s(t, \mathbb{R})$ is a linear operator. Taking into account the compatibility condition of the Lax pair (4) and (14)–(16), we come to a fundamental identity

$$
V^{(i)}[U, V^{(i)}] = U_i[(K - \lambda J)g_i],
$$

(17)

where

$$
U_i[\xi] = \frac{1}{d\xi} U(\bar{q} + \epsilon\xi, \bar{r} + \epsilon\xi), \quad \text{and the pKN hierarchy [28]}
$$

$$
(\bar{q}_n, \bar{r}_n)^T = Jg_n \triangleq X_n, \quad n \geq 1.
$$

(18)

It is clear to see that the first nontrivial member in (18) is the KN equation (1) with the Lax pair (4) and

$$
\varphi_{i+1} = V^{(i)}\varphi, \quad V^{(i)} = \begin{pmatrix} \lambda^2 - 1/2(\bar{q}i + \bar{q}^2i) & \lambda\bar{q} + 1/2(\bar{q}i - \bar{q}^2i) \\ \lambda\bar{r}^{-1} - 1/2(\bar{q}i + \bar{q}^2i) & -\lambda^2 + 1/2(\bar{q}i - \bar{q}^2i) \end{pmatrix}.
$$

(19)

On the other hand, it is supposed that $\varphi$ still satisfies another differential equation determined by the backward Lenard gradients

$$
\varphi_{i+1} = V^{(-i)}\varphi, \quad n \geq 2,
$$

(20)

where

$$
V^{(-i)} = \sigma(\bar{q}, \bar{r}, \lambda)[g_{i}] = g^{(i)}_1\sigma_1 + lg^{(i)}_2\sigma_2 + \lambda \partial^{-1}(\bar{q}g^{(i)}_{-n} - \bar{r}g^{(i)}_{-n})\sigma_3,
$$

(21)

and
g_ = (g_1^{(1)}, g_2^{(2)})^T = -\sum_{j=1}^{n-1} g_{j,j} \lambda_i^{-n_j}, \ n \geq 2. \quad (22)

The combination of (4) and (20)–(22) leads to the second fundamental identity

$$V_1^{(a)} - [U, V_1^{(a)}] = U, (K - \lambda_f)g_1$$, \quad (23)

together with the nKN hierarchy

$$\tilde{q}_i, \tilde{r}_i \bigg| = J g_{i,i} \neq X_{i,i}, \ n \geq 2,$$ \quad (24)

characterised by the nKN equations (3) in view of \( \tilde{q} = \tilde{q}_s \) and \( \tilde{r} = \tilde{r}_s \), which allow the Lax pair (4) and

$$\varphi_{t_i} = V^{(2)} \varphi, \ V^{(2)} = a_{i,1} \left( -\lambda_i^{-1} - 2\tilde{q}_r \lambda_i^{-1} \tilde{q} - \lambda_i^{-1} 2\tilde{q}^2 \right).$$ \quad (25)

To fix \( n = 1 \) in (14)–(16), the Lax representation \( \varphi_{t_i} = V^{(0)} \varphi \) becomes the spectral problem (4), which means that \( t_i \) is in fact the spatial variable \( x \). By analysing the construction of nonholonomic deformation of soliton equations, we arrive at the nonholonomic deformation of the KN equation [6]

$$\begin{align*}
\tilde{q}_{n,2,i} &= \frac{1}{2} \tilde{q}_{x x x} - \tilde{q}_{x x s} - \frac{1}{2} \tilde{q}_s^2 - 4 a_{i,1} \tilde{q}(1 + \tilde{q}^2 r), \\
\tilde{r}_{n,2,i} &= -\frac{1}{2} r_{x x x} - r_{x x s} - \frac{1}{2} r_s^2 + 4 a_{i,1} (r, q - 1),
\end{align*}$$ \quad (26)

which admits the Lax pair (4) and \( \varphi_{t_i} = (V^{(2)} + V^{(2)}) \varphi \). Its generalisation of mKN hierarchy reads [7]

$$\begin{align*}
\tilde{q}_{n,2,i} &= \mathcal{C}_1 \tilde{q}_{i,i} + \mathcal{C}_1 \tilde{r}_{i,i}, \ n \geq 2, \ m \geq 2, \\
K_{i,j} &= J g_{i,j}, \ j \geq 1, \text{ or } j \leq -3, \ j \in \mathbb{Z},
\end{align*}$$ \quad (27)

with the Lax pair (4) and \( \varphi_{t_i} = (\mathcal{C} V^{(0)} + \mathcal{C} V^{(0)} \varphi) \), where \( \mathcal{C}_1 \) and \( \mathcal{C}_1 \) are two constants.

### 3 The Mixed FDHSs

Assume that \( \lambda_i \) and \( \varphi = (p, q)^T, \ 1 \leq j \leq N \), are \( N \) distinct nonzero eigenvalues and the associated eigenfunctions. We consider \( N \) replicas of the spectral problem (4)

$$\begin{pmatrix}
p_j
\end{pmatrix} = \begin{pmatrix}
\lambda_j
\end{pmatrix} \begin{pmatrix}
\tilde{q}_j
\end{pmatrix} = \begin{pmatrix}
\lambda_j \tilde{q}
\end{pmatrix} \begin{pmatrix}
\tilde{r}_j
\end{pmatrix} = \begin{pmatrix}
-\lambda_j \tilde{r}
\end{pmatrix} \begin{pmatrix}
p_j
\end{pmatrix}, \ 1 \leq j \leq N. \quad (28)
$$

It is known from [24] that the functional gradient of \( \lambda_i \) with respect to \( \tilde{q} \) and \( \tilde{r} \) is

$$\nabla \lambda_i = (\partial \lambda_i / \partial \tilde{q}, \partial \lambda_i / \partial \tilde{r})^T = (-\tilde{q}^2, \lambda, \lambda p_r)^T, \quad (29)$$

which satisfies the Lenard eigenvalue equation

$$(K - \lambda_f)\nabla \lambda_i = 0. \quad (30)$$

Let us focus on the Bargmann (symmetric) constraint [24]

$$g_o = \sum_{i=1}^{N} \nabla \lambda_i, \quad (31)$$

which gives a Bargmann map from the eigenfunctions to the potentials

$$\tilde{q} = \langle \Lambda_p, p \rangle, \ \tilde{r} = \langle q, q \rangle, \quad (32)$$

where \( p = (p_1, p_2, ..., p_N)^T, q = (q_1, q_2, ..., q_N)^T, \Lambda = \text{diag} (\lambda_1, ..., \lambda_N) \), and the diamond bracket represents the inner product in \( \mathbb{R}^N \). Substituting (32) back into (4), (19), and (25), together with the symplectic structure \( \omega = dp \wedge dq \) in \( \mathbb{R}^{2N} \) and the Poisson bracket [5]

$$\{f(p, q), g(p, q)\} = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \quad (33)$$

we arrive at three FDHSs

$$p_i = \{p, H_o\}, \ q_i = \{q, H_o\}, \quad (34)$$

with

$$H_o = -\langle \Lambda_p, q \rangle - \frac{1}{2} \langle q, q \rangle \langle \Lambda_p, p \rangle, \quad (35)$$

$$p_i = \{p, H_i\}, \ q_i = \{q, H_i\}, \quad (36)$$

with

$$H_i = -\langle \Lambda^i p, q \rangle + \frac{1}{2} \langle q, q \rangle \langle \Lambda^i p, p \rangle + \frac{1}{8} \langle \Lambda^2 p, p \rangle \langle q, q \rangle + \frac{1}{2} \langle \Lambda^i p, p \rangle \langle \Lambda^i q, q \rangle + \langle \Lambda^i p, q \rangle \langle \Lambda^i q, p \rangle \langle \Lambda^2 q, q \rangle, \quad (37)$$

and

$$p_i = \{p, H_{i+1}\}, \ q_i = \{q, H_{i+1}\}, \quad (38)$$

with

$$H_{i+1} = -\langle \Lambda^i p, q \rangle - \frac{1}{2} \langle q, q \rangle \langle \Lambda^i p, p \rangle + \frac{1}{8} \langle \Lambda^2 p, p \rangle \langle q, q \rangle + \frac{1}{2} \langle \Lambda^i p, p \rangle \langle \Lambda^i q, q \rangle + \langle \Lambda^i p, q \rangle \langle \Lambda^i q, p \rangle \langle \Lambda^i q, q \rangle, \quad (39)$$

where we have restricted \( a_j \) to be a constant of motion \( 1 + (p, q) \) and the constant of motion \( H_{i+1} \) to be zero. It is noted that the spectral problems (4) and (19) appear in
the forward Lax representation (14)–(16), and the systems (34)–(37) are thus called the forward FDHSs and the system (38) and (39) will be termed the backward FDHS since the spectral problem (25) stays in the backward Lax representation (20).

To decompose the mKN hierarchy, we bring in a bilinear generating function

\[
G_j = \sum_{m=0}^{\infty} \frac{\nabla \lambda_j - \lambda_j - \lambda_j}{\lambda_j - \lambda_j} \left( -Q_j(q, q) \right) \equiv \left( Q_j(\Lambda p, p) \right),
\]

(40)

which satisfies

\[
(K - \lambda J) G_j = 0,
\]

(41)

where

\[
Q_j(\xi, \eta) = \sum_{i=0}^{\infty} \frac{\nabla \lambda_j - \lambda_j - \lambda_j}{\lambda_j - \lambda_j} \left( -Q_j(q, q) \right) \equiv \left( Q_j(\Lambda p, p) \right),
\]

(42)

under \(|\lambda| > \max \{ |\lambda_1|, |\lambda_2|, \ldots, |\lambda_N| \}\), or

\[
Q_j(\xi, \eta) = \sum_{i=0}^{\infty} \frac{\nabla \lambda_j - \lambda_j - \lambda_j}{\lambda_j - \lambda_j} \left( -Q_j(q, q) \right) \equiv \left( Q_j(\Lambda p, p) \right),
\]

(43)

in view of \(|\lambda| < \min \{ |\lambda_1|, |\lambda_2|, \ldots, |\lambda_N| \}\). By means of (15), (17) [or (21), (23)], (40), and (41), a direct calculation leads to a Lax matrix

\[
V_j = \sigma(q, \tilde{r}, \lambda) G_j \equiv \left( \begin{array}{cc} 1 - Q_j(\Lambda p, q) & Q_j(\Lambda p, p) \\ \lambda_j - Q_j(q, q) & 1 + Q_j(\Lambda p, p) \end{array} \right),
\]

(44)

together with an identity

\[
(V_j)_x - [U, V_j] = 0,
\]

(45)

which signifies that \(\det V_j\) is a generating function of integrals of motion for the forward FDHS (34) [43], in which the forward integrals of motion are derived by \(\det V_j\),

\[
\det V_j = F_j = 2(\Lambda p, q) + (\Lambda p, p)\langle q, q \rangle,
\]

(46)

\[
F_0 = 2(\Lambda p, q) + (\Lambda p, p)\langle q, q \rangle,
\]

(47)

\[
F_k = 2(\Lambda^{k+1} p, q) + (\Lambda^{k+1} p, p)\langle q, q \rangle - \langle \Lambda^{k+1} p, q \rangle^2,
\]

(48)

\[
F_k = 2(\Lambda^{k+1} p, q) + (\Lambda^{k+1} p, p)
\]

\[
+ \sum_{j=0}^{k-1} \langle \Lambda^{j+1} p, q \rangle - \langle \Lambda^{j+1} p, q \rangle^2, \quad k \geq 3,
\]

(49)

\[
\det V_j = F_j = F_0 - \sum_{k=1}^{\infty} \sum_{i=1}^{N} \frac{\lambda^i}{\lambda - \lambda_k} E_j \lambda^k, \quad \lambda \neq \lambda_k
\]

(50)

where

\[
F_0 = -(1 + \langle p, q \rangle)^2,
\]

(51)

\[
F_k = -2(1 + \langle p, q \rangle)\langle \Lambda^{-1} p, q \rangle + \langle p, q \rangle\langle \Lambda^{-1} q, q \rangle,
\]

(52)

in which \(F_0\) represents another integral of motion different from \(F_0\).

It follows from (35), (37), (46), and (47) that

\[
H_k = -\frac{1}{2} F_k^2, \quad H_k = -\frac{1}{2} F_1 - \frac{H_0^2}{2}.
\]

(53)

Followed by the forward integrals of motion (46)–(48), we define a class of forward FDHSs \((H_k, \omega, R^{2n})\)

\[
p_{k_1} = \{ p, H_k \}, \quad q_{k_1} = \{ q, H_k \}, \quad k \geq 2,
\]

(54)

where

\[
H_k = \frac{1}{2} F_k - \sum_{j=0}^{k-1} H_{k-j},
\]

(55)

Actually, (53) and (55) can be unified into

\[
F_k^2 = -(1 + H_k^2)^2,
\]

(56)

where

\[
H_k = \sum_{k=0}^{\infty} H_k \lambda^{-k-1}.
\]

(57)
Lemma 1: 
\[ \begin{align*}
\partial(\Lambda^k p, p) &= 2(\Lambda^k q, q) + \tilde{q} \partial^{-1} \tilde{q}(\Lambda^k q, q) \\
&\quad + \tilde{q} \partial^{-1} \tilde{r}(\Lambda^k p, p), \\
\partial(\Lambda^k q, q) &= -2(\Lambda^k q, q) - \tilde{r} \partial^{-1} \tilde{q}(\Lambda^k q, q) \\
&\quad - \tilde{r} \partial^{-1} \tilde{r}(\Lambda^k p, p).
\end{align*} \] 

According to Lemma 1, we have 
\[ g_k = (\langle \Lambda^k q, q \rangle, \langle \Lambda^k p, p \rangle), \quad k \geq 2. \] 

Similarly, inserting (32) and (59) back into (20)–(22), we get a sequence of backward FDHSs 
\[ p_{(k)\pm} = \{ p, H_{(k-1, m-1)} \}, \quad q_{(k)\pm} = \{ q, H_{(k-1, m-1)} \}, \quad n, m \geq 2, \] 

where 
\[ H_{(k-1, m-1)} = C H_{(k-1, m)} + C H_{(m-1, k)} \] 
is the mixed Hamiltonian and \( t_{(k, m)} \) is the flow variable. It will be seen that the Hamiltonian vector fields (34) and (61) are exactly mapped into the mKN vector fields (26) and (27).

4 The Liouville Integrability of Hamiltonian Systems

We suppose that 
\[ F_s = -2H_s, \quad H_s = H_0 + \sum_{k=0}^{\infty} H_s \lambda^k, \] 
is a generating function of \( \{ H_s \} (k \geq 1) \) with a supplementary definition \( H_0 = -\frac{F_0}{2} \). Let \( \tau_s, \tau_s, \tau_s(k \geq 0), \tau_s(k \geq 1), \tau_s, \tau_s, \tau_s(k \geq 0), \) and \( \tau_s(k \geq 1) \) be the flow variables of \( F_s, F_s, F_s(k \geq 0), F_s(k \geq 1), H_s, H_s, H_s(k \geq 0), \) and \( H_s(k \geq 1) \), respectively. Regarding \( F_s^s \) as the Hamiltonian in \( \mathbb{R}^{2n}, \omega^s \), we get a canonical Hamiltonian equation 
\[ \frac{d}{dr_j}(p_k) = \{ p_s, F_s^s \}, \quad \frac{d}{dr_j}(q_k) = \{ q_s, F_s^s \} = W(\lambda, \mu_j) \] 

where

\[ W(\lambda, \mu) = -\frac{2\mu}{\lambda - \mu} V_j - 2Q_s(\Lambda \mu, p) \sigma_1, \] 

Proposition 1: On \((\mathbb{R}^{2n}, \omega^s)\), the Lax matrix \( V \) satisfies a Lax equation 
\[ \frac{dV}{dr_j} = [W(\lambda, \mu), V], \quad \forall \lambda, \mu \in \mathbb{C}, \lambda \neq \mu. \] 

Besides 
\[ \{ F_s^s, F_s^s \} = 0, \] 

\[ \{ F_s^s, F_s^s \} = 0, \quad j, k = 0, 1, 2, \ldots, \] 

\[ \{ H_s, H_s \} = 0, \quad j, k = 0, 1, 2, \ldots \] 

Proof: We make the notation \( \xi_j^s = \lambda p_s q_s \sigma_1 - \lambda p_s q_s \sigma_1 + \lambda q_s \sigma_1, \) and then have 
\[ \frac{d}{dr_j}(\xi_j^s) = W(\lambda, \mu_j, \xi_j^s). \] 

From (64), (65), and (71), after a direct but lengthy calculation, we obtain 
\[ \frac{dV}{dr_j} = \frac{d}{dr_j}(q_s, q_s) \sigma_1 - \sum_{k \geq 1} \frac{1}{\mu - \lambda_k} \frac{d\xi_j^s}{dr_j} \] 

\[ = -4(q, q)Q_s(\Lambda \mu, q) + Q_s(\Lambda q, q) \sigma_1 \] 

\[ - \sum_{k \geq 1} \frac{1}{\mu - \lambda_k} [W(\lambda, \mu), \xi_j^s], \] 

\[ = -4(q, q)Q_s(\Lambda \mu, q) + Q_s(\Lambda q, q) \sigma_1 \] 

\[ + 2Q_s(\Lambda \mu, q) \sigma_1 \sum_{k \geq 1} \xi_j^s \] 

\[ + \frac{2}{\lambda - \mu} \sum_{k \geq 1} \frac{\lambda_k - \lambda_k}{\mu - \lambda_k} [W(\lambda, \mu), \xi_j^s], \] 

\[ = 4(q, q)Q_s(\Lambda \mu, q) + Q_s(\Lambda q, q) \sigma_1 \] 

\[ + \frac{2}{\lambda - \mu} [V, \lambda V - \mu V - 2[V, \sigma_1 - \langle q, q \rangle \sigma_1] \] 

\[ = -2Q_s(\Lambda \mu, q) \sigma_1 - 2V, \sigma_1 - \langle q, q \rangle \sigma_1 \] 

\[ = 4(q, q)Q_s(\Lambda \mu, q) - Q_s(\Lambda q, q) \sigma_1 + [W(\lambda, \mu), V], \] 

\[ + 2Q_s(\Lambda \mu, q) \sigma_1 - 2V, \sigma_1 - \langle q, q \rangle \sigma_1 \] 

\[ = [W(\lambda, \mu), V]. \] 

By the Lax equation (66), it follows from (43) that \( \det V \) is the constant of motion along with the \( r_s \)-flow; resorting to the Poisson bracket, we have 
\[ \{ F_s^s, F_s^s \} = \frac{dF_s^s}{dr_j} = \frac{d}{dr_j} \det V = 0. \]
Substituting the expressions (44) and (49) into (72) results in the formula (68). It follows from the Leibniz rule of Poisson bracket that
\[
\{H^+_\mu, H^-_\nu\} = \frac{1}{4} \{ F^+_\mu, F^-_\nu \} = 0, \tag{73}
\]
\[
\{H^+_{\mu}, H^-_{\nu}\} = \frac{1}{4(1+H^-_{\mu})} \{ F^+_{\mu}, F^-_{\nu}\} = 0, \tag{74}
\]
\[
\{H^-_{\mu}, H^+_{\nu}\} = \frac{1}{4(1+H^-_{\mu})} \{ F^-_{\mu}, F^+_{\nu}\} = 0, \tag{75}
\]
\[
\{H^-_{\mu}, H^-_{\nu}\} = \frac{1}{4} \{ F^-_{\mu}, F^-_{\nu}\} = 0, \tag{76}
\]
and inserting (57) and (63) into (73)–(76) leads to the identity (70).

It is shown from (43) that \( \{ F \}(k \in \mathbb{Z}) \) are the integrals of motion for the forward FDHS (34). Actually
\[
\frac{dF^+_{\mu}}{dt} = \{ F^+_{\mu}, H^-_{\nu} \} = -2(1+H^-_{\mu}) \{ H^+_{\nu}, H^-_{\nu}\} = 0, \tag{77}
\]
\[
\frac{dF^-_{\mu}}{dt} = \{ F^-_{\mu}, H^+_{\nu} \} = -2(H^-_{\nu}, H^+_{\nu}) = 0, \tag{78}
\]
which implies that \( \{ F \} \) and \( \{ H \} \) are two sets of involutive conservation integrals for the FDHSs (34), (36), (38), (54), and (60).

**Proposition 2:** The backward integrals of motion \( \{ F_{m,1}, F_{m,2}, \ldots, F_{m,N} \} \) \( (m \geq 1) \) given by (51) and (52) are functionally independent in a dense open subset of \( (\mathbb{R}^{2N}, \omega) \).

**Proof:** Recalling (49), it is apparent that
\[
F_m = \sum_{j=1}^{N} \lambda_j^{-m-1} E_j, \quad m \geq 1. \tag{79}
\]
Let \( P_0 = (p_1, \ldots, p_N, q_1, \ldots, q_N)^T \) be a given point in \( \mathbb{R}^{2N} \), and \( p_i = 0, q_i \neq 0 \) \( (i = 1, 2, \ldots, N) \). By (45), we calculate the Jacobi determinant of \( \{ E \} \) with regard to \( p \) at \( P_0 \)
\[
\frac{\partial (E_1, E_2, \ldots, E_N)}{\partial (p_1, p_2, \ldots, q_N)} \bigg|_{P_0} = \begin{vmatrix}
\frac{\partial E_1}{\partial p_1} & \frac{\partial E_1}{\partial p_2} & \cdots & \frac{\partial E_1}{\partial p_N} \\
\frac{\partial E_2}{\partial p_1} & \frac{\partial E_2}{\partial p_2} & \cdots & \frac{\partial E_2}{\partial p_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial E_N}{\partial p_1} & \frac{\partial E_N}{\partial p_2} & \cdots & \frac{\partial E_N}{\partial p_N}
\end{vmatrix} = 2^N \prod_{j=1}^{N} \lambda_j^{-q_j} \neq 0. \tag{80}
\]
which implies the linear independence of one-form differentials \( dE_1, dE_2, \ldots, dE_N \) over a dense open subset of \( \mathbb{R}^{2N} \) [5]. It is assumed that there are \( N \) constants \( \gamma_1, \gamma_2, \ldots, \gamma_N \) such that
\[
\sum_{k=1}^{N} \gamma_k dF_{-m-1} = 0. \tag{81}
\]
Based on the linear independence of \( \{ dE \} \) and (79), we have
\[
\sum_{k=1}^{N} \gamma_k d_0 = 0, \quad 1 \leq j \leq N, \tag{82}
\]
where, obviously, the determinant of coefficients \( \gamma_k \)
is the Vandermonde determinant, which means that \( \gamma_1 = \gamma_2 = \cdots = \gamma_N = 0 \). As a result, \( \{ F \}(m \leq k \leq m+N-1, m \geq 0) \) are functionally independent in a dense open subset of \( (\mathbb{R}^{2N}, \omega) \).

Similar to the recipe managed as Proposition 2, we also come to the functional independence of forward integrals of motion \( \{ F \} \) \( (m \leq k \leq m+N-1, m \geq 0) \).

**Proposition 3:** The backward integrals of motion \( \{ F_{m,1}, F_{m,2}, \ldots, F_{m,N} \} \) \( (m \geq 1) \) described by (46)–(48) are functionally independent over a dense open subset of \( (\mathbb{R}^{2N}, \omega) \).

It follows from (53), (55), and (63) that
\[
H_{-m} = -\frac{1}{2} F_{-m}, \quad m \geq 1, \tag{83}
\]
\[
\begin{pmatrix}
\frac{dE_0}{dt} \\
\frac{dE_1}{dt} \\
\vdots \\
\frac{dE_N}{dt}
\end{pmatrix}
= \begin{pmatrix}
H_0 & 1 & 0 & \cdots & 0 \\
H_1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
H_{N-1} & 1 & 0 & \cdots & 0 \\
H_N & 1 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\frac{dH_0}{dt} \\
\frac{dH_1}{dt} \\
\vdots \\
\frac{dH_{N-1}}{dt} \\
\frac{dH_N}{dt}
\end{pmatrix}. \tag{84}
\]

**Corollary 1:** \( \{ H_1, H_{m-1}, \ldots, H_{m-N} \} \) \( (m \geq 1) \) are functionally independent of \( (\mathbb{R}^{2N}, \omega) \).

**Corollary 2:** \( \{ H_1, H_{m-1}, \ldots, H_{m-N} \} \) \( (m \geq 0) \) are functionally independent of \( (\mathbb{R}^{2N}, \omega) \).

**Proposition 4:** The backward FDHSs described by (38) and (60) are completely integrable in the Liouville sense.

**Proposition 5:** The forward FDHSs given by (34), (36), and (54) are completely integrable in the Liouville sense.

Based on Propositions 4 and 5, it is shown that \( (H_j, \omega_i, \mathbb{R}^{2N}) \), \( i \geq 1 \) and \( (H_j, \omega_j, \mathbb{R}^{2N}) \), \( j \geq 0 \), reduced from the mKN equations, are compatible in \( (\mathbb{R}^{2N}, \omega) \). It means that there exists a smooth function associated with flow variables.
$t_{j+1}$ and $t_{j}$ giving an involutivity solution for the associated backward and forward FDHSs [5].

5 Relation between the mKN Hierarchy and the Mixed FDHSs

In this section, we specify the relationship between the mKN hierarchy and the backward and forward FDHSs. Define two generating functions of the Lenard gradients $\{g_j\} (k \in \mathbb{Z})$ by

$$ g_j^k = \sum_{k=0}^{N} g_{j-1}^k, \quad (85) $$

$$ g_j^k = \sum_{k=0}^{N} g_{j}^k, \quad (86) $$

which satisfy the Lenard eigenvalue equations

$$ (K - \lambda J)g_j^k = 0, \quad (87) $$

$$ (K - \lambda J)g_j^k = 0, \quad (88) $$

in view of the recursive formula (9). By suitably choosing integral constants, $g_j$ is viewed as the kernel of $f_1$ and $g_j^k$ as the kernel of $K$. Let us apply $K^{-1}J$ and $J^{-1}K$ to the Bargmann constraint (31) $k+1$ and $k$ times, and then obtain

$$ \sum_{j=1}^{N} \lambda_j^{j-1}\nabla j = g_{j-1}^k + c_2 g_{j-1}^k + \cdots + c_{N}^k g_{j-1}^N, \quad k \geq 1, \quad (89) $$

$$ \sum_{j=1}^{N} \lambda_j^{j-1}\nabla j = g_{j}^k + c_2 g_{j-1}^k + \cdots + c_{N}^k g_{j}^N, \quad k \geq 1, \quad (90) $$

where $c_2, c_3, \ldots, c_N$ and $c_2, c_3, \ldots, c_N$ are two sets of integration constants. Taking into account the constrained condition $|\lambda| < \min(|\lambda_j|, |\lambda_j|, \ldots, |\lambda_j|)$, by (85) and (89) we have

$$ G_j = c_j^k g_j^k, \quad c_j^k = -1 - \sum_{k=0}^{N} c_{k}^j \lambda_j^{j+1}. \quad (91) $$

Similarly, under $|\lambda| > \max(|\lambda_j|, |\lambda_j|, \ldots, |\lambda_j|)$, $G_j$ is of the compact form

$$ G_j = c_j^k g_j^k, \quad c_j^k = 1 + \sum_{k=0}^{N} c_{k}^j \lambda_j^{j+1}, \quad (92) $$

in view of (86) and (90). Specially, from the rapidly decaying condition of $q$ and $\bar{r}$ as $x \to \infty$, the combination of (42) and (92) yields $F^i_+ = \det \sigma(\bar{q}, \bar{r}, \lambda)[G_j^i] = -(c_j^i)^2$, which, together with (56), gives rise to

$$ c_j^i = 1 + H_j^i, \quad H_j^i = c_j^i, \quad k \geq 0. \quad (94) $$

Lemma 2:

$$ \frac{d}{dr_j} = \frac{d}{dr_j} = -\frac{1}{2} \frac{d}{dr_j}, \quad k \geq 1, \quad (95) $$

$$ \frac{d}{dr_j} = \frac{d}{dr_j} = \frac{1}{2} \frac{d}{dr_j}, \quad k \geq 0. \quad (96) $$

Proof: Let $f$ be an arbitrary, smooth function. Recalling (63), and the Poisson bracket (33), it is clear that

$$ \frac{d}{dt} \{f, F_j^i\} = \{f, -2H_j^i\} = -2(1 + H_j^i)^2, \quad (97) $$

which results in the formula (95). From the Leibniz rule of Poisson bracket, it follows from (56) and (94) that

$$ \frac{d}{dt} \{f, F_j^i\} = \{f, -(1 + H_j^i)^2\} = -2(1 + H_j^i)[f, H_j^i] $$

$$ = -2c_j^i \frac{df}{dt}, \quad (98) $$

which gives rise to the formula (96). Inserting (49) and (63), into (97), and (44) and (57) into (98), the comparison of coefficients of $\lambda$ indicates that (95), (96), and (96) hold true.

Proposition 6: Let $(p(x, t_{(n+1)}), q(x, t_{(n+1)}))$, $(n \geq 1)$, be the involutive solution of the forward FDHS (34) and the backward FDHSs (38) and (60). Then

$$ \bar{q} = \langle \lambda p(x, t_{(n+1)}), q(x, t_{(n+1)}) \rangle $$

$$ \bar{r} = \langle q(x, t_{(n+1)}), q(x, t_{(n+1)}) \rangle, \quad (99) $$

solve the $n$th nKN equation (24).

Proof: According to the backward FDHSs (38) and (60), one gets

$$ \langle \lambda p, p \rangle_{t_{(n+1)}} = -2(\langle \lambda^{-1} p, p \rangle + \langle \lambda p, p \rangle \langle \lambda^{-1} p, q \rangle), $$

$$ \langle q, q \rangle_{t_{(n+1)}} = 2(\langle \lambda^{-1} q, q \rangle + \langle q, q \rangle \langle \lambda^{-1} p, q \rangle). \quad (100) $$

By the forward FDHSs (34), one has

$$ \partial(\lambda^{-1} p, p) = 2(\langle \lambda^{-1} p, p \rangle + \langle \lambda p, p \rangle \langle \lambda^{-1} p, q \rangle), $$

$$ \partial(\lambda^{-1} q, q) = -2(\langle \lambda^{-1} q, q \rangle + \langle q, q \rangle \langle \lambda^{-1} p, q \rangle). \quad (101) $$

Substituting (100) and (101) back into (24), it turns out that (99) are the finite parameter solutions of the $n$th nKN equation (24) with the help of the recurrence chain (9).
Proposition 7: Let \((p(x, t_{nm}), q(x, t_{nm}))^T, (n \geq 1)\), be an involutive solution of the forward FDHSs \((H_0, \omega^*, \mathbb{R}^{2n})\) and \((H_0, \omega^*, \mathbb{R}^{2n})\). Then
\[
\tilde{q} = \langle \Lambda p(x, t_{nm}), p(x, t_{nm}) \rangle, \quad \tilde{r} = -\langle q(x, t_{nm}), q(x, t_{nm}) \rangle,
\]
(102)
satisfy the \(n^\text{th} \) pKN equation (18).

Proof: On one hand, by (64) and (65), a direct calculation yields
\[
\frac{d\langle \Lambda p, p \rangle}{dr_j} = -4Q_j(\Lambda^2 p, p) + \langle \Lambda p, p \rangle Q_j(\Lambda p, q),
\]
(103)
\[
\frac{d\langle q, q \rangle}{dr_j} = 4Q_j(\Lambda p, q) + Q_j(\Lambda q, q),
\]
(104)
in view of \(Q_j(\Lambda^2 \xi, \eta) = \lambda Q_j(\Lambda^{-1} \xi, \eta) + (\Lambda^{-1} \xi, \eta)\), \((j \in \mathbb{Z})\). On the other hand, resorting to (86), (92), (96), (103), and (104), we obtain
\[
\frac{d}{dr_j} \tilde{q} = \frac{1}{2c_j} \frac{d}{dr_j} \tilde{r} = -\frac{1}{2c_j} (-2Kg_j) = Kg_j = \sum_{k=1} \tilde{c}_j g_k \lambda^k,
\]
which implies that (102) are the finite parameter solutions to the \(n^\text{th} \) pKN equation (18).

Proposition 8: Let \((p(x, t_{nm}), q(x, t_{nm}))^T\) be an involutive solution of FDHSs \((H_0, \omega^*, \mathbb{R}^{2n})\) and \((H_{(0 \sim n-m)}, \omega^*, \mathbb{R}^{2n}), n, m \geq 2\). Then
\[
\tilde{q} = \langle \Lambda p(x, t_{nm}), p(x, t_{nm}) \rangle, \quad \tilde{r} = -\langle q(x, t_{nm}), q(x, t_{nm}) \rangle,
\]
(105)
are the finite parameter solutions of mKN equations (26) and (27).

Proof: Let \(h_k^{(n-m)}\) and \(h_k^{(n)}\) be the flow operators of initial value problems to the forward FDHSs \((H_{(0 \sim n-m)}, \omega^*, \mathbb{R}^{2n})\) and the backward FDHSs \((H_{(0 \sim n-m)}, \omega^*, \mathbb{R}^{2n})\), that is, \((p(t), q(t)) \Rightarrow h_k^{(n-m)}(p_0, q_0)\) and \((p(t), q(t)) \Rightarrow h_k^{(n)}(p_0, q_0)\). It follows from Propositions 4 and 5 that
\[
\langle \Lambda p, p \rangle_{(n-m)} = \langle \Lambda p, p \rangle_{(n)} = \langle \Lambda p, p \rangle_{(n-m)} + \phi \Lambda H_{(n-m)} + \phi \Lambda H_{(n)}
\]
(106)
and
\[
\langle q, q \rangle_{(n-m)} = C_{(q, q)}_{(n-m)} + C_{(q, q)}_{(n)},
\]
(107)
which means that the expression of (105) solves the mKN hierarchy by Propositions 6 and 7.

A solution is said to be a finite-gap potential if it solves the high-order stationary soliton (Novikov) equation. The Bargmann map has been used to generate finite-gap potentials for a number of positive-order INLEEIs known so far. To obtain quasi-periodic solutions for the mKN hierarchy, it is pointed out that the Bargmann map produces not only the finite-gap potential for the positive \(N\)-order stationary KN equation (or the positive-order Novikov equation) but also the finite-gap potential for the negative \(N\)-order stationary KN equation (or the negative-order Novikov equation).

Proposition 9: Let \((p(x), q(x))^T\) be a solution of the forward FDHSs (34). Thus,
\[
\tilde{q} = \langle \Lambda p(x), p(x) \rangle, \quad \tilde{r} = -\langle q(x), q(x) \rangle,
\]
(108)
are the finite-gap potentials to the negative \(N\)-order stationary KN equation
\[
X_{-N} + \tilde{c}_j X_{-N + j} + \cdots + \tilde{c}_j X_{-1} = 0,
\]
(109)
where \(\tilde{c}_j\) are some constants of integration given by
\[
\tilde{c}_j = \sum_{k=0}^{j} a_{-k} c_{-j+k}, \quad j = 2, 3, \ldots, N,
\]
(110)
with a supplementary definition \(c_{-1} = 1\), and
\[
a_{-1} = 1, \quad a_j = (-1)^j \sum_{i_j \leq j \leq i_{j-1}} \lambda_j^{-1} \lambda_{j-1}^{-1} \cdots \lambda_1^{-1}, \quad j = 1, 2, \ldots, N.
\]
(111)

Proof: By using an auxiliary polynomial
\[
a(\lambda^{-1}) = \prod_{j=1}^{N} (\lambda^{-1} - \lambda_j^{-1}) = a_{1} \lambda^{-1} + a_{2} \lambda^{-N+1} + \cdots + a_{N-1} \lambda^{-1} + a_{-1},
\]
(112)
we achieve from (89) that
\[
0 = \sum_{j=1}^{N} a(\lambda^{-1}) \nabla \lambda_j = \sum_{j=1}^{N} (\lambda_j^{-N} + a_{-1} \lambda_j^{-N+1} + \cdots + a_{N-1} \lambda_j^{-1} + a_{-1}) \nabla \lambda_j
\]
\[
= (g_{-N} + c_{-1} g_{-N+1} + \cdots + c_{-N+1} g_1) + a_{1} (g_{-N+1} + c_{-2} g_{-N+2} + \cdots + c_{-N+2} g_{-1})
\]
\[
+ \cdots + a_{N-1} (g_2 + c_{-2} g_3 + \cdots + c_{-N+2} g_{-1}) + a_{-1} g_1
\]
\[
= g_{-N} + \tilde{c}_j g_{-N+1} + \cdots + \tilde{c}_j g_{-1} + \tilde{c}_j g_1 + a_{-1} g_0.
\]
(113)

Applying the Lenard operator \(J\) on (113), we get the negative \(N\)-order stationary KN equation.
In a way similar to the treatment conducted in [25, 44], we arrive at the following:

**Proposition 10:** Let \((p(x), q(x))'\) be a solution of the forward FDHSs (34). Thus,
\[
\tilde{q} = (\lambda p(x), p(x)), \quad \tilde{r} = (-q(x), q(x)),
\]
are the finite-gap potentials to the positive N-order stationary KN equation
\[
X_N + \bar{c}_1 X_{N-1} + \cdots + \bar{c}_1 X_1 = 0,
\]
where \(\bar{c}_j\) are some constants of integration determined by
\[
\bar{c}_j = \sum_{j < k \leq N} a_{j-k} c_{k-j}, \quad j = 2, 3, \ldots, N,
\]
with a supplementary definition \(c_0 = 1\), and
\[
a_0 = 1, \quad a_j = (-1)^j \sum_{j < \lambda < \lambda^*} \lambda \lambda^* \ldots \lambda_j, \quad j = 1, 2, \ldots, N.
\]

6 Straightening Out of mKN Flows

Only for the convenience of description, we designate
\[
V_i^{II} = \left( \begin{array}{cc} V_i^{11} & V_i^{12} \\ V_i^{21} & -V_i^{12} \end{array} \right), \quad \det V_i = -V_i^{12} V_i^{21} - V_i^{11} V_i^{22}.
\]

It is seen from (42) and (118) that \(\det V_i, V_i^{12}, \text{ and } V_i^{22}\) are the rational polynomial functions of \(\lambda\) with simple poles at \(\{\lambda_j\} (j = 1, 2, \ldots, N)\). As a consequence, we define
\[
\det V_i = -\frac{b(\lambda)}{a(\lambda)} = -\frac{\lambda \rho(\lambda)}{a(\lambda)},
\]
\[
V_i^{12} = Q_i(\lambda p, p) = \langle \lambda p, p \rangle \frac{m(\lambda)}{a(\lambda)},
\]
\[
V_i^{22} = -\lambda Q_i(q, q) = -\lambda \langle q, q \rangle \frac{m(\lambda)}{a(\lambda)},
\]
with
\[
a(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_j), \quad m(\lambda) = \prod_{j=1}^{N-1} (\lambda - \mu_j), \quad n(\lambda) = \prod_{j=1}^{N-1} (\lambda - \nu_j),
\]
where \(\mu_1, \mu_2, \ldots, \mu_{N-1}\) and \(\nu_1, \nu_2, \ldots, \nu_{N-1}\) are two sets of elliptic variables for the FDHSs (34), (36), (38), (54), and (60). By (118)–(121), a direct calculation yields
\[
V_i^{11} \bigg|_{a \rightarrow a_j} = \sqrt{\frac{\rho(\mu_j)}{a(\mu_j)}}, \quad V_i^{22} \bigg|_{a \rightarrow a_j} = \sqrt{\frac{\rho(\nu_j)}{a(\nu_j)}}, \quad 1 \leq k \leq N-1.
\]

Resorting to (65), (66), and (120)–(122), we arrive at the Dubrovin-type equations
\[
\begin{align*}
1 \frac{d\mu_k}{d\lambda} &= -\mu_m(\lambda), \\
4 \sqrt{\rho(\mu_j)} \frac{d\mu_k}{d\lambda} &= a(\lambda) - \mu_m(\lambda), \quad 1 \leq k \leq N-1.
\end{align*}
\]

Applying the Lagrange interpolation formula on (123) yields
\[
\begin{align*}
\sum_{j=1}^{N-1} \mu_j^{N-1-j} \frac{d\mu_k}{d\lambda} &= \frac{\lambda^{N-j}}{a(\lambda)}, \\
\sum_{j=1}^{N-1} \nu_j^{N-1-j} \frac{d\nu_k}{d\lambda} &= \frac{\lambda^{N-j}}{a(\lambda)},
\end{align*}
\]

Let us introduce a hyperelliptic curve of Riemann surface \(\Gamma\)
\[
\xi^2 = \rho(\lambda),
\]
with the usual holomorphic differentials
\[
\tilde{w}_j = \frac{\lambda^{N-j} d\lambda}{4 \sqrt{\rho(\lambda)}}, \quad 1 \leq j \leq N-1.
\]

It is noted that \(\deg \rho(\lambda) = 2N\), and the Riemann surface \(\Gamma\) is of genus \(N-1\) with two infinite points \(\infty_1\) and \(\infty_\infty\), which are not the branch points and can be expressed as \((0, -1)\) and \((0, 1)\) in the local coordinate \(\lambda = z^2\). For the same reason, \(\lambda = \sqrt{\rho(\lambda)}\) and \(\lambda = -\sqrt{\rho(\lambda)}\) on the upper and lower sheets of \(\Gamma\). Let \(P_s(\lambda) = 0, s = 1, 2\) be a fixed point on \(\Gamma\), and \(P_s(\lambda) = (\lambda, \xi = \pm \sqrt{\rho(\lambda)})\). Introduce two sets of quasi-Abel-Jacobi variables
\[
\tilde{\phi}_j = \sum_{s=1}^{N} \int_{P_s(\lambda)}^{P_s(\lambda)} \omega_j, \quad \tilde{\psi}_j = \sum_{s=1}^{N} \int_{P_s(\lambda)}^{P_s(\lambda)} \psi_j, \quad 1 \leq j \leq N-1.
\]

These, together with (124), give rise to
\[
\frac{d \tilde{\phi}_j}{d\lambda} = \frac{\lambda^{N-j}}{a(\lambda)} \frac{d \tilde{\psi}_j}{d\lambda} - \frac{\lambda^{N-j}}{a(\lambda)} \frac{d \tilde{\phi}_j}{d\lambda}, \quad 1 \leq j \leq N-1.
\]

Select two sets of canonical basis of cycles \(a_1, a_2, \ldots, a_{N-1}\), and \(b_1, b_2, \ldots, b_{N-1}\) on \(\Gamma\) with the intersection numbers \(a_i^* a_j = b_i^* b_j = 0, a_i^* b_j = \delta_{ij}\) (\(i, j = 1, 2, \ldots, N-1\)). By the canonical basis of cycles, we define
\[ A = (A_j)_{j=1}^{N-1}, \quad C = (A_j^{-1})_{j=1}^{N-1}, \quad A_j = \int_{a_j}^{b_j} \omega_i, \quad 1 \leq i, j \leq N-1, \]

which leads to the normalised holomorphic differential

\[ \omega = (\omega_1, \omega_2, \ldots, \omega_{N-1})^T, \quad \omega_j = \sum_{i=1}^{N-1} C_i \omega_i, \quad 1 \leq j \leq N-1, \]

and \((2N-1)\) periodic vectors

\[ \delta_j = \int_{a_j}^{b_j} \omega_i, \quad 1 \leq j \leq N-1, \]

with the components

\[ \delta_{\omega} = \int_{a\omega}^{b\omega} \omega_i, \quad \delta_{\omega} = \int_{a\omega}^{b\omega} \omega_i, \]

where \(\delta = (\delta_j)_{j=1}^{2N-1}\) is a unit matrix, and \(B = (B_j)_{j=1}^{2N-1}\) is a symmetric matrix with a positive-definite imaginary part.

To progress further, the Riemann theta function on \(\Gamma\) is defined as [26, 27]

\[ \theta(\xi | B) = \sum_{n \in \mathbb{Z}^{N-1}} \exp(2\pi i (Bz, \xi) + 2\pi i (\xi, \xi)), \quad \xi \in \mathbb{C}^{N-1}, \]

where \(\delta(\xi | B) = \sum_{n \in \mathbb{Z}^{N-1}} \exp(2\pi i (Bz, \xi) + 2\pi i (\xi, \xi))\), \(\xi \in \mathbb{C}^{N-1}\).

Consider the Abel map \(A: \text{Div}(\Gamma) \to J(\Gamma)\) given by

\[ A(P) = \int_{\tau}^{P} \omega_i, \quad A = \left( \sum_{i=1}^{N-1} P_{i} \right) = \sum_{i=1}^{N-1} A_i(P_i), \]

where \(\text{Div}(\Gamma)\) is the divisor group of \(\Gamma\), and \(J(\Gamma)\) is the Jacobi variety

\[ J(\Gamma) = \mathbb{C}^{N-1} / \mathcal{T}, \quad \mathcal{T} = \text{Span}_{\mathbb{Z}} \{ \delta_1, \ldots, \delta_{N-1}, B_1, \ldots, B_{N-1} \}. \]

With two specific divisors \(\sum_{j=1}^{N-1} \mu_j \) and \(\sum_{j=1}^{N-1} \nu_j \), we come to the Abel-Jacobi variables

\[ \phi = A \left( \sum_{k=1}^{N-1} \mu_k \right), \quad \psi = A \left( \sum_{k=1}^{N-1} \nu_k \right), \]

which can be used to straighten out the mKN flows on \(J(\Gamma)\).

**Lemma 3:** Let \(s_k = \sum_{j=1}^{N} \lambda_k \). By \(|l| < \min \{ 1, |\nu_1|, |\nu_2|, \ldots, |\nu_{N-1}| \} \)

\[ \frac{1}{a(\lambda)} = \sum_{k=0}^{N} A_k \lambda_k, \]

where

\[ A_k = 0(k \geq 1), \quad A_k = (-1)^k \prod_{j=1}^{N} A_j^{-1}, \quad A_1 = A_{s_1}, \]

\[ A_2 = \frac{1}{2} A_0 (s_2 + s_2^2), \quad A_3 = A_1 \left( \frac{1}{3} s_3 + \frac{1}{2} s_2 s_2 + \frac{1}{6} s_2^3 \right), \]

\[ A_4 = \frac{1}{k} \left( A_2 s_2 + \sum_{i+j+k=4} A_i A_j A_k \right), \quad k \geq 4. \]

**Theorem 1:** The Abel-Jacobi variables \(\phi\) and \(\psi\) straighten out \(H_{\omega}\)-flow \((k \geq 1)\) as

\[ \frac{d\phi}{dt_{k-j}} = \Omega_j, \quad \frac{d\psi}{dt_{k-j}} = -\Omega_j, \quad k = 1, 2, \ldots, \]

where

\[ \Omega_1 = \frac{1}{2} C_{-1} A_1, \quad \Omega_2 = \frac{1}{2} (C_{-2} A_0 + C_{-1} A_1), \]

\[ \Omega_3 = \frac{1}{2} (C_{-3} A_0 + C_{-2} A_1 + C_{-1} A_2), \]

\[ \Omega_4 = \frac{1}{2} (C_{-4} A_0 + C_{-3} A_1 + \ldots + C_{-1} A_4), \quad 4 \leq k \leq N-1, \]

\[ \Omega_k = \frac{1}{2} (C_{-k} A_0 + C_{-k+1} A_1 + \ldots + C_{-1} A_k), \quad k \geq N. \]

**Proof:** Let us denote \( C = (C_1, C_2, \ldots, C_{N-1}) \) and \( \phi = (\phi_1, \phi_2, \ldots, \phi_{N-1})^T \). From (95), (128), (134), and Lemma 3, a direct computation yields

\[ \frac{d\phi}{dt_k} = -\frac{1}{2} \frac{d\psi}{dt_k} = -\frac{1}{2} \left( C_1, C_2, \ldots, C_{N-1} \right) \left( \frac{d\phi_1}{dt_k}, \frac{d\phi_2}{dt_k}, \ldots, \frac{d\phi_{N-1}}{dt_k} \right)^T, \]

\[ = \frac{1}{2} \sum_{k=1}^{N} C_k \lambda_k \sum_{k=1}^{N} C_k \lambda_k \sum_{k=0}^{N} A_k \lambda^k, \]

\[ = -\sum_{k=1}^{N} \Omega_k \lambda_k. \]

In addition,

\[ \frac{d\phi}{dt_k} = \left( \phi_k, \sum_{k=1}^{N} H_k \right)^T \lambda_k = \frac{1}{2} \sum_{k=1}^{N} \frac{d\phi}{dt_k} \lambda_k. \]

It follows from (139) and (140) that (137) is true by comparing coefficients of the same powers of \(\lambda\). The formula (137) can be verified in a way similar to that of (137).

**Lemma 4:** Let \( S_k = \sum_{j=1}^{N} \lambda_k \) and \( \tilde{R}(\lambda^{-1}) = \prod_{j=1}^{N} (1 - \lambda \lambda^{-1}) \).

Then

\[ \frac{1}{\sqrt{\tilde{R}(\lambda^{-1})}} = \sum_{k=0}^{N} \Lambda_k \lambda^{-k}, \quad |\lambda| > \max \{ \lambda_1, \lambda_2, \ldots, \lambda_{2N} \}. \]
\[ \Lambda_{-k} = 0(k \geq 1), \quad \Lambda_0 = 1, \quad \Lambda_1 = \frac{1}{2} S_i, \]
\[ \Lambda_k = \frac{1}{2k}(S_i + \sum_{i+j+k=1} S_i S_j), \quad k \geq 2. \] 

(142)

**Theorem 2:** The Abel-Jacobi variables \( \phi \) and \( \psi \) straighten out \( H_k \)-flow \((k \geq 0)\) as
\[ \frac{d\phi}{dt_{k+1}} = \Omega_k, \quad \frac{d\psi}{dt_{k+1}} = -\Omega_k, \quad k \geq 0, \]

(143)

where
\[ \Omega_0 = \frac{1}{2} C_1, \quad \Omega_k = \frac{1}{2}(\Lambda_k C_1 + C_k), \]
\[ \Omega_k = \frac{1}{2}(\Lambda_k C_1 \cdots + \Lambda_k C_k + C_{k+1}), \quad 2 \leq k \leq N - 2, \]
\[ \Omega_k = \frac{1}{2}(\Lambda_k C_1 \cdots + \Lambda_k C_{N-2} C_{N-1}), \quad k \geq N - 1. \]

(144)

**Proof:** By virtue of (93) and (119), it is clear that
\[ \sqrt{R(\lambda)} = c_r a(\lambda). \]

(145)

Recalling (96), (128), (134), and (145) and Lemma 4, we obtain
\[ \frac{d\phi}{dt} = \frac{1}{C_1} \frac{d\phi}{dt_{1/k}} = -\frac{C}{C_1} \frac{d\phi}{dt_{1/k}} \]
\[ \frac{d\psi}{dt} = \frac{1}{C_1} \frac{d\psi}{dt_{1/k}} = \frac{C}{C_1} \frac{d\psi}{dt_{1/k}} \]
\[ = -\frac{1}{2C_1}(C_1, C_1, \ldots, C_{N-1}) \left( \frac{\lambda^{N-1}}{a(\lambda)}, \ldots, -\frac{\lambda}{a(\lambda)} \right)^T \]
\[ = -\frac{1}{2C_1}(C_1, C_1, \ldots, C_{N-1}) \left( \lambda^{N-1}, \ldots, -\lambda \right)^T \]
\[ = -\frac{1}{2C_1} \sum_{j=1}^{N-1} C_j \lambda^{j-1} = -\frac{1}{2} \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} \Omega_i \lambda^{j-1}. \]

(146)

By the Poisson bracket and (57), we arrive at
\[ \frac{d\phi}{dt_{k+1}} = \{\phi, H_k\} = \sum_{k=0}^{N} \{\phi, H_k\} \lambda^{-k-1} = \sum_{k=0}^{N} \frac{d\phi}{dt_{k+1}} \lambda^{-k-1}. \]

(147)

It is apparent that (143), holds true by comparing the same power of \( \lambda \) between (146) and (147). The proof of (143) is a repetition of the verification (143).

As a concrete application of Theorems 1 and 2, we derive
\[ \phi = \sum_{k=0}^{N} \Omega_k t_{k+1}, \quad \psi = -\sum_{k=0}^{N} \Omega_k t_{k+1}, \]
\[ \phi = \sum_{k=0}^{N} \Omega_k t_{k+1}, \quad \psi = -\sum_{k=0}^{N} \Omega_k t_{k+1}, \]

(148)

and the Abel-Jacobi solutions for the nKN, pKN, and mKN flows:

the \( k \)-th nKN flow: \( \phi = \phi_0 + \Omega_0 x + \Omega_k t_{k+1}, \)
\[ \psi = \psi_0 - \Omega_0 x - \Omega_k t_{k+1}, \quad k \geq 1, \]

(149)

the \( k \)-th pKN flow: \( \phi = \phi_0 + \Omega_0 x + \Omega_k t_{k+1}, \)
\[ \psi = \psi_0 - \Omega_0 x - \Omega_k t_{k+1}, \quad k \geq 1, \]

(150)

the mKN flow: \[ \begin{aligned}
\phi &= \phi_0 + \Omega_0 x + (C_n \Omega_{n+1} + C_m \Omega_{m+1}) t_{n+m}, \\
\psi &= \psi_0 - \Omega_0 x - (C_n \Omega_{n+1} + C_m \Omega_{m+1}) t_{n+m}, \\
n, m &\geq 2.
\end{aligned} \]

(151)

### 7 Quasi-Periodic Solutions of the mKN Hierarchy

Equations (149)–(151) give explicit solutions of the nKN equation (26), the pKN equation (18), and the mKN equation (27) in the Abel-Jacobi variables, respectively. To write out solutions in the potentials \( \tilde{q} \) and \( \tilde{r} \), we need to set up the link between \( \tilde{q}, \tilde{r} \) and \( \{\mu \}, \nu \}, \) and then complete the procedure of Riemann-Jacobi inversion.

**Lemma 5:** For any \( \lambda \neq \lambda_j \), \((1 \leq j \leq N)\), the relationship between the eigenfunctions and the symmetric functions of elliptic variables is of as follows:
\[ \frac{\langle \Lambda^p, \mu \rangle}{\langle \Lambda, \mu \rangle} = \sum_{j=1}^{N} \lambda_j - \sum_{j=1}^{N} \mu_j, \]
\[ \frac{\langle \Lambda^q, \mu \rangle}{\langle \Lambda q, \mu \rangle} = \sum_{j=1}^{N} \lambda_j - \sum_{j=1}^{N} \mu_j, \]
\[ \frac{\langle \Lambda^q, \nu \rangle}{\langle \Lambda q, \nu \rangle} = \sum_{j=1}^{N} \nu_j - \sum_{j=1}^{N} \lambda_j - \sum_{j=1}^{N} \nu_j, \]

(152)

(153)

(154)

(155)

**Proof:** By means of (120), we have
\[ Q_n(\Lambda, \mu) = \sum_{j=1}^{N} \lambda_j \mu_j \prod_{j=1}^{N} (\lambda_j - \lambda), \]
\[ = \sum_{j=1}^{N} \lambda_j \mu_j \left( \lambda^{N-1} - \sum_{j=1}^{N} \lambda_j \lambda^{N-2} + \sum_{i+j=N-1} \lambda_j \lambda_i \right) \]
\[ \quad + \cdots + (-1)^{N-1} \prod_{j=1}^{N} \lambda_j \]
\[ = \langle \Lambda, \mu \rangle \left[ \lambda^{N-1} - \sum_{j=1}^{N} \mu_j \lambda^{N-2} + \sum_{i+j=N-1} \mu_j \lambda_i \right] \]
\[ \quad + \cdots + (-1)^{N-1} \prod_{j=1}^{N} \mu_j, \]

(156)
Taking a look at the terms of $\lambda^{N-2}$ and $\lambda^{N-3}$ in (156), we come to the formula (152) and (153) with the aid of $\sum_{i=0}^{N} \lambda_{i} = \sum_{j=1}^{N} \lambda_{j} - \lambda_{1}$. Analogous to the recipe above, (154) and (155) are thus derived from (121).

Followed by the integrable FDHSs (34), Lemma 5, and (46), we obtain

$$\partial \ln(A(p, p)) = 2\left(\sum_{i=1}^{N} \lambda_{i} - \sum_{j=1}^{N} \mu_{j}\right) + e^{2\Pi_{i=1}^{N} v_{1}_{i} + F_{0}}, \tag{157}$$

$$\partial \ln(A(q, q)) = 2\left(\sum_{j=1}^{N} \lambda_{j} - \sum_{j=1}^{N} \lambda_{j}\right) - e^{2\Pi_{j=1}^{N} v_{1}_{j} - F_{0}}, \tag{158}$$

As for the step of the Riemann-Jacobi inversion, we turn to the Riemann theorem [26, 27], which asserts that there exist two vectors of Riemann constant $\gamma \in \mathbb{C}^N$ such that

- $f^{0}(\lambda) = \theta(A(P(\lambda)) - \phi - M^{(0)})$ has $N - 1$ simple zeros at $\mu_{1}, \ldots, \mu_{N-1}$
- $f^{0}(\lambda) = \theta'(A(P(\lambda)) - \psi - M^{(0)})$ has $N - 1$ simple zeros at $\nu_{1}, \ldots, \nu_{N-1}$.

To make the function $f^{0}(\lambda)(m=1,2)$ single valued, $\Gamma$ is cut along by all $a_k$ to form a simply connected region whose boundary is designated by $\gamma$. To recover solutions such that $M = \sqrt[4]{R(\lambda)}$, there exist two vectors of Riemann constant $\gamma$ whose boundary is designated by $\gamma$. To recover solutions that we need to calculate not each $\mu_j$ and $\nu_j$ ($1 \leq j \leq N - 1$) but the symmetric functions of elliptic variables. The negative and positive power sums of elliptic variables are obtained by virtue of the calculation of residues of $f^{0}(\lambda)$ at $\mu_j$ and $\nu_j$.

$$\sum_{i=1}^{N} \lambda_j = l_i(\Gamma) - 2\sum_{j=1}^{N-1} \lambda_j \ln f^{0}(\lambda),$$

$$\sum_{j=1}^{N} \nu_j = l_j(\Gamma) - 2\sum_{j=1}^{N-1} \nu_j \ln f^{0}(\lambda), \tag{159}$$

where $l_i(\Gamma) = \sum_{j=1}^{N} \lambda_j \omega_j$ is a constant independent of $\phi$ and $\psi$ (see [18] for details).

By the local coordinate $z = \lambda^\nu$, in the neighborhood of infinity, the affine equation of Riemann surface $\Gamma$ (125) is put into the form

$$\tilde{\xi}^2 - \tilde{R}(\xi) = 0, \quad \tilde{\xi} = z^\nu \xi, \tag{160}$$

the two infinities become two zeroes on the upper ($s=2$) and lower ($s=1$) sheets

$$\infty_s = (z, (-1)^s \sqrt{R}(z))|_{z=0} = (0, (-1)^s), \quad s = 1, 2. \tag{161}$$

As a consequence, it follows from (161) that

$$\lambda^{-N} \sqrt{R(\lambda)} = (-1)^s \sqrt{R(z)}. \tag{162}$$

**Proposition 11:** In the neighborhood of $\infty_s$ ($s=1,2$), under the local coordinate $\lambda = z^\nu$, the normalised holomorphic differential $\omega$ can be expanded as

$$\omega = \frac{(-1)^{s-1}}{2} \sum_{k=0}^{\nu - 1} \Omega_k z^k dz. \tag{163}$$

**Proof:** Resorting to (130), (126), and (141), a direct calculation gives

$$\omega = C \bar{\omega} = (C_{1}, \ldots, C_{N+1})(\frac{\lambda^{N-2}}{4\sqrt{R(\lambda)}}, \ldots, \frac{1}{4\sqrt{R(\lambda)}})^T d\lambda$$

$$= \frac{1}{4\lambda} \sum_{j=1}^{N} C_{j} \lambda^{-j} d\lambda = \sum_{j=1}^{N} C_{j} \lambda^{-j} d\lambda = (-1)^{s-1} \frac{1}{2} \sum_{k=0}^{\nu - 1} \Omega_k z^k dz. \tag{164}$$

**Corollary 3:** Near $\infty_s$ ($s=1, 2$), $A(\lambda(\xi))$ has the asymptotic expansion

$$A(\lambda(\xi)) = -\chi_s + (-1)^s \sum_{k=0}^{\nu - 1} \frac{1}{2(k+1)} \Omega_k z^k dz, \tag{165}$$

where $\chi_s = \int_{\omega_s}^\nu \omega$ is a constant of integration.

After these preparations, the rest of this section is directed to calculating the residues of the inversion formula. Only for succinctness in writing, let $\xi_j$ be the $j$th component of $\Theta$, $\partial_\xi_\theta = \partial/\partial_\xi_\theta$, $\partial_\theta = \partial^2/\partial_\xi_\theta \partial_\xi_\theta$, and so on. Following the Einstein summation convention, at $\lambda = \infty_s$, the asymptotic expansion of $f^{0}(\lambda)$ in the local coordinate $z = \lambda^\nu$ reads

$$f^{0}(\lambda) = \theta^{(-)}(\psi + M_2 + \chi_s) + \frac{1}{2} (-1)^s \Omega_0 \partial_\theta^{(-)} z$$

$$+ \frac{1}{8} \Omega_0 \Omega_0 \partial_\theta^{(-)} z^2 + 2(-1)^s \Omega_0 \partial_\theta^{(-)} z^2$$

$$+ \frac{1}{48} (8(-1)^s \Omega_0 \partial_\theta^{(-)} z^2 + 6\Omega_0 \Omega_0 \partial_\theta^{(-)} z^3)$$

$$+ (-1)^s \Omega_0 \Omega_0 \partial_\theta^{(-)} z^3 + o(z^3), \tag{166}$$

which gives

$$\frac{d \ln f^{0}(z)}{dz} = (\frac{1}{2\theta_s^{(-)} \Omega_j \partial_\theta^{(-)}) z}$$

$$- \left((-1)^s \frac{1}{2\theta_s^{(-)} \Omega_j \partial_\theta^{(-)} z} + \frac{1}{4\theta_s^{(-)} \Omega_0 \Omega_0 \partial_\theta^{(-)} z}ight) z$$

$$- \left((-1)^s \frac{1}{2\theta_s^{(-)} \Omega_j \partial_\theta^{(-)} z} + \frac{1}{4\theta_s^{(-)} \Omega_0 \Omega_0 \partial_\theta^{(-)} z}ight) z.$$
The combination of (159) and (167) yields the trace formulae

$$\sum_{k=1}^{N-1} \psi_k = I_1(\Gamma) - \frac{1}{2} \frac{\partial \ln \theta(-\epsilon)}{\partial \theta(\epsilon)}$$

$$\sum_{k=1}^{N-1} \psi_k = I_2(\Gamma) - \frac{1}{2} \frac{\partial \ln \theta(-\epsilon)}{\partial \theta(\epsilon)} - \frac{1}{4} \frac{\partial^2 \ln \theta(-\epsilon)}{\partial \theta(\epsilon)}.$$ (168)

Continuing a similar calculation as (166) and (167), we derive the trace formulae

$$\sum_{k=1}^{N-1} \mu_k = I_1(\Gamma) + \frac{1}{2} \frac{\partial \ln \theta(-\epsilon)}{\partial \theta(\epsilon)}$$

$$\sum_{k=1}^{N-1} \mu_k = I_2(\Gamma) + \frac{1}{2} \frac{\partial \ln \theta(-\epsilon)}{\partial \theta(\epsilon)} - \frac{1}{4} \frac{\partial^2 \ln \theta(-\epsilon)}{\partial \theta(\epsilon)}.$$ (169)

For the sake of simplifying presentations, let us introduce the symbols

$$\Theta = 2 \sum_{k=1}^{N} \beta_k + F_0 - 2 I_1(\Gamma), \quad \alpha_s = \phi_0 + M_1 + \chi_s,$$

$$\beta_s = \psi_0 + M_2 + \chi_s, \quad s = 1, 2.$$ (170)

1. The nKN equations (24) have quasi-periodic solutions

$$\partial \ln \bar{q}(x, t_{k+1}) = -\partial \ln \frac{\theta(\Theta x + \Omega x + \Omega_{k+1} + \alpha_s)}{\theta(\Theta x + \Omega_{k+1} + \alpha_s)}$$

$$+ \frac{\partial(-\Theta x - \Omega x + \Omega_{k+1} + \alpha_s)}{\partial(\Theta x + \Omega_{k+1} + \alpha_s)} + \frac{\partial(-\Theta x - \Omega x + \Omega_{k+1} + \alpha_s)}{\partial(\Theta x + \Omega_{k+1} + \alpha_s)}.$$

(171)

2. The pKN equations (18) have quasi-periodic solutions

$$\partial \ln \bar{q}(x, t_{k+1}) = -\partial \ln \frac{\theta(\Theta x + \Omega x + \Omega_{k+1} + \alpha_s)}{\theta(\Theta x + \Omega_{k+1} + \alpha_s)}$$

$$+ \frac{\partial(-\Theta x - \Omega x + \Omega_{k+1} + \alpha_s)}{\partial(\Theta x + \Omega_{k+1} + \alpha_s)} + \frac{\partial(-\Theta x - \Omega x + \Omega_{k+1} + \alpha_s)}{\partial(\Theta x + \Omega_{k+1} + \alpha_s)}.$$

(172)

3. The mKN equations (27) have quasi-periodic solutions

$$\partial \ln \bar{q}(x, t_{k+1}) = -\partial \ln \frac{\theta(\Theta x + \Omega x + \Omega_{k+1} + \alpha_s)}{\theta(\Theta x + \Omega_{k+1} + \alpha_s)}$$

$$+ \frac{\partial(-\Theta x - \Omega x + \Omega_{k+1} + \alpha_s)}{\partial(\Theta x + \Omega_{k+1} + \alpha_s)} + \frac{\partial(-\Theta x - \Omega x + \Omega_{k+1} + \alpha_s)}{\partial(\Theta x + \Omega_{k+1} + \alpha_s)}.$$

(173)

$$\partial \ln \bar{r}(x, t_{k+1}) = -\partial \ln \frac{\theta(\Theta x + \Omega x + \Omega_{k+1} + \alpha_s)}{\theta(\Theta x + \Omega_{k+1} + \alpha_s)}$$

$$+ \frac{\partial(-\Theta x - \Omega x + \Omega_{k+1} + \alpha_s)}{\partial(\Theta x + \Omega_{k+1} + \alpha_s)} + \frac{\partial(-\Theta x - \Omega x + \Omega_{k+1} + \alpha_s)}{\partial(\Theta x + \Omega_{k+1} + \alpha_s)}.$$

(174)

For the sake of simplifying presentations, let us introduce the symbols

$$\Theta = 2 \sum_{k=1}^{N} \beta_k + F_0 - 2 I_1(\Gamma), \quad \alpha_s = \phi_0 + M_1 + \chi_s,$$

$$\beta_s = \psi_0 + M_2 + \chi_s, \quad s = 1, 2.$$ (175)

By the algebro-geometric construction of mKN flows, together with (157), (158), (168), and (169), we obtain some quasi-periodic solutions for the nKN, pKN, mKN hierarchy as follows:

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