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On the Dynamical Nature of Nonlinear Coupling of Logarithmic Quantum Wave Equation, Everett-Hirschman Entropy and Temperature

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Abstract: We study the dynamical behavior of the nonlinear coupling of a logarithmic quantum wave equation. Using the statistical mechanical arguments for a large class of many-body systems, this coupling is shown to be related to temperature, which is a thermodynamic conjugate to the Everett-Hirschman’s quantum information entropy. A combined quantum-mechanical and field-theoretical model is proposed, which leads to a logarithmic equation with variable nonlinear coupling. We study its properties and present arguments regarding its nature and interpretation, including the connection to Landauer’s principle. We also demonstrate that our model is able to describe linear quantum-mechanical systems with shape-changing external potentials.

Keywords: Logarithmic Quantum Wave Equation; Open Quantum Systems; Quantum Bose Liquid; Quantum Temperature.

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1 Introduction

Assuming three spatial dimensions (lower-dimensional cases can be studied by analogy), let us consider the logarithmic Schrödinger equation (LogSE)

\[ i\hbar \partial_t \Psi = (\hat{H} - b \ln(a^2 |\Psi|^2)) \Psi = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} - b \ln(a^2 |\Psi|^2) \right] \Psi, \]  

(1)

where the coupling

\[ b = b(\mathbf{r}, t) \]  

(2)

quantifies the strength of the nonlinear self-interaction, \( a \) is a constant parameter of the dimensionality length required to make the argument of the logarithm dimensionless, \( m \) is the particle’s mass, \( V_{\text{ext}} = V_{\text{ext}}(\mathbf{r}, t) \) is an external potential (sometimes dubbed the trap potential) and \( \nabla^2 = \nabla \cdot \nabla \) is the Laplacian (in the relativistic version of (1), the derivative part would be replaced by the d’Alembertian). The wavefunction \( \Psi = \Psi(\mathbf{r}, t) \) is assumed to be normalized to the number \( N \) and expressed as

\[ \langle \Psi | \Psi \rangle = \int \Psi^* \Psi d^3\mathbf{r} = N \geq 1, \]  

(3)

where \( \nu \) is the volume occupied by our system.

The simplest case of (2) expressed as

\[ b(\mathbf{r}, t) = b_0 = \text{const}, \]  

(4)

was historically the first case to be studied [1–4]. The corresponding models have been proven to be instrumental in dealing with extensions of quantum mechanics [3–5], physics of quantum fields and particles [1, 2, 6–11], optics and transport or diffusion phenomena [12, 13], classical hydrodynamics of Korteweg-type materials [14–19], nuclear physics [20, 21], theory of dissipative systems and quantum information [22–29], theory of quantum liquids and superfluidity [30–34] and theory of physical vacuum and classical and quantum gravity [35–38]. The mathematical properties of the logarithmic wave equation and its modifications and solutions were also extensively studied [5, 29, 30, 32, 34, 39–58], to mention just a few examples.

Notwithstanding the success of the models with a constant coupling (4), there remain a few questions that need to be addressed. First, what is the value of the coupling (2), is it a fundamental constant, or is it related to dynamical observables? Past studies [3, 4, 20, 21, 59] have shown that the nonlinear term’s effect must be rather small for a large class of conservative quantum and atomic systems. On the other hand, in the theory of superfluid He-4, which is an example of a system being in thermal contact with a reservoir of large heat capacity as to maintain constant temperature, this term plays a crucial role [31]. Therefore, it seems that this coupling can take different values, depending on the prevailing physical conditions, i.e. it can vary from system to system. In other words, its value cannot be a fundamental constant, but rather a dynamical function, while formula (4) can be
regarded as a first-order approximation or a limit value of this function.

Second, if this coupling is a nontrivial dynamical notion, then what is its physical meaning?

One possible idea, advocated in [23, 30], is to relate it to a certain kind of temperature $T_{\varphi}$, which is a thermodynamical conjugate to a quantum analogue of Shannon information entropy [60, 61], referred here as the Everett-Hirschman’s (EH) entropy [62, 63]. The latter being defined as

$$S_{\varphi} = -\langle \Psi | \ln(a^\dagger | \Psi \rangle) \langle \Psi | - \int \langle \Psi | | \ln(a^\dagger | \Psi \rangle \rangle, \quad (5)$$

where the Boltzmann constant is hereafter assumed to be $k_B = 1$, and we adopt the sign conventions of [60, 61]. This entropy can be used as a measure of the localization of a system, or as an inverse measure of its extendedness. More details can be found in [62].

In this framework, the logarithmic quantum wave equation can be viewed as a minimization condition, not for the energy of a system $\langle \Psi | \hat{H} | \Psi \rangle$, but for its quantum “internal energy”, which is given by

$$\mathcal{U}_{\varphi} = \langle \Psi | \hat{H} | \Psi \rangle + \Delta T_{\varphi} S_{\varphi}, \quad (6)$$

where $T_{\varphi}$ is referred as the EH or quantum temperature from now on; without loss of generality, one can assume it to be counted with respect to some reference value. The standard thermodynamic arguments yield the relation

$$\Delta T_{\varphi} = \left( \frac{\partial \mathcal{U}_{\varphi}}{\partial S_{\varphi}} \right)_V \propto b, \quad (7)$$

where the conventional thermodynamic notations are used. In the right-hand side of (6), the term $\langle \Psi | \hat{H} | \Psi \rangle$ comes from the system’s dynamics, whereas the other term determines the cost of the energy needed to obtain and handle information about a system

$$I_{\varphi} = -\log_2(a^\dagger | \Psi \rangle) = -\ln(a^\dagger | \Psi \rangle) / \ln 2. \quad (8)$$

These entropy considerations result in uncertainty relations that complement the Heisenberg relation [62–65]. The physical meaning of the EH conjugate temperature is discussed in the following sections, where it is related to conventional (thermal) temperature, whereas the information-handling cost of energy $\Delta T_{\varphi} S_{\varphi}$ lays the quantum-mechanical foundations for the Landauer’s principle.

Yet another interpretation of the coupling $b$ comes from the irreversible dynamics described by the Langevin equation. According to [22], in semiclassical approximation, this coupling is related to the friction coefficient. Given that the latter is essentially a macroscopic notion, it is unclear whether it has a well-defined analogue in the quantum picture of reality. Therefore, in what follows we will focus on an interpretation of the nonlinear coupling and logarithmic term in terms of $T_{\varphi}$ and EH entropy.

Third, while it appears that the considerations of the Everett-Hirschman entropy are an important step towards a better understanding of the temperature $\Delta T_{\varphi}$, we must also consider the laws governing its dynamical behavior.

These three questions are the main subject of this study. The rest of the paper is organized as follows. In Section 2, we enumerate the different ways of deriving wave equations with logarithmic nonlinearity, after which we focus on the statistical mechanical arguments and derive a relation between nonlinear coupling and temperature. In Section 3, we present a model where nonlinear coupling (hence temperature) becomes a dynamical value and introduce the basic notions and equations, which we will use in what follows. In Section 4, we analytically study some properties and solutions of the model. Discussion and conclusions are presented in Section 5, where we also outline the possible directions for future research.

### 2 Foundations

A wave equation with logarithmic nonlinearity can be introduced into physics using different independent approaches: dilatation covariance [1, 2], nonlinear generalization of quantum mechanics preserving energy additivity [3, 4], classical Korteweg fluids [16–19], irreversible Langevin dynamics [22], measurement and information entropy [23], effective nonlinearities in quantum systems [29], superfluidity of helium-4 [31], theory of physical vacuum, quantum gravity and superfluid-gravity correspondence [35, 36, 38], to mention only the examples known to the author. In this section, we will present another way by which this equation can manifest – one that underlines some of the above-mentioned approaches. This new method of deriving the logarithmic nonlinearity is based on the physically plausible arguments that are applicable to a large class of many-body systems.

Let us consider a many-body system of particles, whose average potential energy is larger than its kinetic energy (examples would be systems made of strongly interacting particles, or materials with suppressed kinetic...
degrees of freedom, such as cold Bose liquids [31] or melted thermal insulators in capillary tubes [16]). Then the probability of a microstate is given by a Boltzmann rule, in which the kinetic energy can be neglected in the leading approximation given by

\[ P \propto \exp(-\mathcal{E}/T) = \exp(-U/T), \]

where \( T \), \( \mathcal{E} \) and \( U \) are the temperature, energy and potential energy of a many-body system, respectively.

Generally such a system is described by a large number of linear Schrödinger equations; however, collective degrees of freedom are known to occur in many systems of this kind, which can substantially simplify the theory [66]. Therefore, if we want to effectively describe collective degrees of freedom, we must also add to a wave equation, (14), which is sometimes referred to as the cubic Schrödinger equation and is related to the Ginzburg-Landau theory. This equation, along with other higher-order polynomial nonlinear Schrödinger equations arising in the theory of condensed Bose systems, is one of the perturbative limits of the logarithmic quantum wave equation. This can be demonstrated by expanding a variational functional corresponding to (1) into the Taylor series in the vicinity of its potential’s nontrivial extrema, as described in [30, 36]. Indeed, the Taylor series expansion of a logarithmic term in the vicinity of \( |\Psi_0| \propto \exp(-U/T) \)

\[ b \ln(a^3 \Psi_0^2) \Psi = b(a_0^3 \Psi_0^2 - 1) \Psi + \ldots, \]

where the leading-order term can be recognized as the cubic or Gross-Pitaevskii nonlinearity. One can also show that the GP equation describes a special case of dilute Bose-Einstein condensates, in which the interparticle interaction potential can be approximated by a two-body potential of a contact (delta-singular) shape [66].

### 3 The Model

Given that we want to upgrade the nonlinear coupling \( b \) (hence temperature, according to the previous section) to a dynamical value, we introduce an auxiliary field \( \sigma = \sigma(\mathbf{r}, t) \), and define the coupling as its scalar function: \( b = b(\sigma) \). Due to the expected gauge invariance of the model, we assume that the resulting wave equation must not depend on the field \( \sigma \) itself, but on its derivative, such as the gradient \( \nabla \sigma \). Given that the latter is a vector, whereas the coupling must be a scalar function, we assume that

\[ b = b(\bar{\nabla} \sigma, \bar{\nabla} \sigma \cdot \bar{\nabla} \sigma, \ldots), \]

where \( \bar{\mathbf{r}} = \mathbf{r} / r \) and \( r = \sqrt{\mathbf{r} \cdot \mathbf{r}} \) is an absolute value of the radius vector.

#### 3.1 Minimal Model

With assumptions (14) in hand, we keep only the terms that are linear with respect to \( \bar{\nabla} \sigma \). Thus, we introduce the simplest (“minimal”) variable-coupling model as
\[ i \hbar \partial_t \Psi = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} - \tilde{n} \cdot \tilde{\nabla} \ln(a^2 | \Psi|^2) \right] \Psi, \]  
\[ \nabla^2 \sigma = 4 \pi \kappa \rho_o, \]  
where \( \rho_o = \rho_o(r, t) \) is the energy density of the field \( \sigma \) and \( \kappa \) is a scale constant. It is natural to assume that this field’s distribution is correlated with our system; therefore, we can impose
\[ \kappa \rho_o = f(\rho), \]  
where \( f \) is a function that must be specified according to a particular model’s choice and \( \rho = |\Psi|^2 \) is the probability density of our system; in the case of many-body systems \( (N \gg 1) \), \( \rho \) would be an actual particle density. The exact form of the function \( f \) is generally unknown and the resulting model shown in (15)–(17) is not only nonlinear but also coupled. Therefore, further analytical studies could become complicated. Fortunately, some robust simplifications can be made in order to extract essential physical information.

Henceforth, we focus on the case of a trapless system, \( V_{\text{ext}} = 0 \); thus, one can assume a spatial isotropy of the auxiliary field. We thus set
\[ \sigma = \sigma(r, t), \]  
so that (15) and (16) can be respectively rewritten as
\[ i \hbar \partial_t \Psi + \frac{\hbar^2}{2m} \nabla^2 \Psi + \partial_t \sigma \ln(a^2 | \Psi|^2) \Psi = 0, \]  
\[ \nabla^2 \sigma = \nabla^2 \sigma = 4 \pi f(| \Psi|^2), \]  
where \( \nabla^2 = \partial^2_{rr} + \frac{2}{r} \partial_r \) is a radial part of Laplacian. These two equations must be supplemented with the normalization condition for \( \Psi \), boundary conditions for both \( \Psi \) and \( \sigma \) and a specific expression for a function \( f \) depending on the physical system in question.

### 3.2 Approximate Minimal Model

The minimal model shown in (19) and (20) contains the function \( f \), whose value must be specified depending on the physical system in question; otherwise, it is generally unknown. However, a certain class of dynamical systems must have a common \( f \), at least in the leading approximation.

For simplicity, let us impose here the time independence of the auxiliary field: \( \sigma = \sigma(r) \). Furthermore, let us assume that our system satisfies the condition \( \lim_{r \to \infty} \rho_o = \lim_{r \to \infty} |\Psi|^2 = 0 \), therefore its density can be formally represented as a decomposition of the Dirac delta-singular part and asymptotically vanishing extended part
\[ \rho_o = f(|\Psi|^2) \sim \delta(r) + \sum_{n=1} b_n, \]  
where \( \delta(r) \) is the Dirac’s delta function centered in the origin. This expression can be viewed as describing a point-like object’s density, plus an extended object’s density distribution represented by a Taylor series expansion with respect to \( 1/r \), which ensures that the field \( \sigma \) vanishes at spatial infinity.

Under these assumptions, (20) can be approximately written in series form and decoupled from (19)
\[ \nabla^2 \sigma = -4 \pi \left[ Q_o \delta(r) - \frac{b_o}{2 \pi r} + O(1/r^2) \right], \]  
where \( Q_o = -\frac{1}{\alpha} \int \nabla \sigma \cdot d\mathbf{S} \) and \( b_o \) are constants (the former being a Gauss law’s charge) and notation \( O(1/r^2) \) represents terms that decay faster than \( 1/r \) when \( r \to \infty \). Here, the constant \( Q_o \) labels the delta-singular part of the density \( \rho \), whereas the series coefficient \( b_o \) labels the leading-order term of the extended part; the chosen notation \( b_o \) is not a coincidence, as we will see below.

Notice that in the original model (19) and (20), both \( Q_o \) and \( b_o \) would not be built-in parameters of a theory, but integration constants; therefore, their values would depend on boundary conditions and may vary from one system to another. Thus, the full model would allow us to reduce the number of parameters of a logarithmically nonlinear theory and make it more self-contained. However, within the frameworks of the approximation (22), the values of \( Q_o \) and \( b_o \) are unknown and have yet to be determined from other factors.

Furthermore, neglecting higher-order terms \( O(1/r^2) \), we can exactly solve (22). We obtain
\[ \sigma = \sigma_o + \frac{Q_o}{r} + b_o r, \]  
where \( \sigma_o \) is an additive constant, which can be set to zero due to the gauge invariance of the \( \sigma \)-field. Substituting this into (19), we obtain,
\[ i \hbar \partial_t \Psi + \frac{\hbar^2}{2m} \nabla^2 \Psi + \left( b_o - \frac{Q_o}{r^2} \right) \ln(a^2 | \Psi|^2) \Psi = 0, \]
thus confirming our earlier expectations that the nonlinear coupling is not generally constant. If the field’s charge $Q_a$ is nonzero, then at $r \to 0$, the coupling’s magnitude grows like $1/r^3$, whereas at large $r \to \infty$ the coupling tends to a constant, so that one asymptotically recovers (4).

Notice that the constant part of the coupling, $b_{oq}$, is induced not by the delta-singular part of the $a$-field’s distribution but by its extended part. This explains why the solutions of the conventional logarithmic equation (4) are applicable for describing non-singular extended objects, such as Q-balls and finite-size particles [6–11] and superfluid droplets [30, 31]. Additionally, the appearance of a new term, proportional to $Q_a$, indicates that the new model could also be instrumental in dealing with singular or point-like objects.

For the calculations that follow, we will make (24) dimensionless. This equation always contains three constants, $h$, $m$, and $a$, which are independent of the nonlinear coupling $b$. Therefore, from them, one can construct the following scales of length, time and mass, respectively: $a$, $ma^2/h$, $m$. Assuming $a > 0$ and $\tilde{r}^2 = r/a$, $t' = t/\tau$, $\tilde{\Psi} = a^{3/2} \Psi$, (25)

$$\tilde{\Psi} = \frac{\tilde{r}'}{a}, \quad t' = \frac{t}{\tau}, \quad \tilde{\Psi} = a^{3/2} \Psi,$$

where $\tau = 2ma^2/h$, we can write (24) in a dimensionless form given by

$$i\hbar \tilde{\Psi} + \nabla^2 \tilde{\Psi} + \left( \tilde{b}_0 - \tilde{q} \right) \ln(|\tilde{\Psi}|^2) \tilde{\Psi} = 0,$$

where $\tilde{b}_0 = b_{oq} r / h = 2mb_{oq} a^2 / h^2$ and $\tilde{q} = Q_a r / (ha^2) = 2mQ_a / h^2$. In the following sections we will omit primes, assuming that times, lengths, momenta and energies are measured in units of $\tau$, $a$, $h/a$ and $h/\tau$, respectively.

### 4 Properties and Solutions

In this section, we consider a stationary case and analytically derive the corresponding solutions. We begin by imposing a stationary ansatz

$$\Psi(\tilde{r}, t) = \exp(-i\tilde{\omega}t)\psi(\tilde{r}),$$

where $\tilde{\omega}$ is a frequency measured in units of $1/\tau$. Then (26) becomes an eigenvalue equation for the frequency

$$\nabla^2 \psi + \left( \tilde{b}_0 - \tilde{q} \right) \ln(|\psi|^2) \psi + \tilde{\omega} \psi = 0,$$  

where $\psi = \psi(\tilde{r})$ is a spatial wavefunction normalized to a number $N: \int |\psi|^2 d^3\tilde{r} = N > 1$.

Due to the symmetry of (28), it is convenient to work in spherical coordinates from now on. In doing so, the Laplacian can be decomposed into its radial and angular parts

$$\nabla^2 = \nabla_\rho^2 + \frac{1}{\rho^2} \nabla_\rho^2,$$

where $\nabla_\rho^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}$ is the Laplace-Beltrami operator on a sphere.

Here we also introduce the notion of effective external potential. Once a solution of (28) is known, the effective external potential for such a solution is given, in a dimensionless form, by the expression

$$\tilde{V}_{\text{eff}}(\tilde{r}) = \left( \tilde{q} \tilde{r} - \tilde{b}_0 \right) \ln(|\psi(\tilde{r})|^2),$$

where $\psi(\tilde{r})$ is a solution’s wavefunction. This potential indicates that a solution $\psi$ can be equivalently derived from the linear Schrödinger equation with external potential $V = \tilde{V}_{\text{eff}}(\tilde{r})$. In other words, an observer would not be able to empirically differentiate a nonlinear problem from a linear one, if based on the analysis of a solution $\psi$, alone. Note that the effective potential’s shape would vary from solution to solution for the same system; therefore, a nonlinear theory of type (19), (20) has the capacity to describe the linear systems with shape-changing external potentials depending on a state, e.g. those systems undergoing phase transitions as their temperature changes. Furthermore, this creates a framework for creating quantum-mechanical models where an external potential is not ab initio postulated but actually derived.

Another instrumental value to be introduced is a radial density of the Everett-Hirschman entropy (5), which is measured in units $1/a$ and expressed as

$$\tilde{S}_\psi^{(r)} = -\int \ln(|\psi|^2) r^2 \sin \theta d\theta d\varphi,$$

where the integral is taken over the sphere; then the entropy (5) is simply $S_\psi = \int_0^{2\pi} S_\psi^{(r)} dr$. The $S_\psi$-conjugate temperature in this case is given by

$$\Delta T_\psi = \frac{\tilde{b}_0 - \tilde{q}}{r^2},$$

when written in our units of energy $h/r$.

Furthermore, when dealing with the analytical solutions of (28), one must distinguish between different cases of nonlinear couplings’ parameters that occur.
4.1 Case $\tilde{b}_0 = 0, \tilde{q} = 0, 1$

In this case, the normalized spherically-symmetric solution of (28) can be written as

$$\psi_o(r) = \exp\left(-\frac{\pi}{2N^{2/3}}r^2\right),$$

(33)

and both the $\tilde{\omega}$ and coupling $b_0$ are no longer arbitrary, but become eigenvalues, which are respectively expressed as

$$\tilde{\omega} := \tilde{\omega}_0 = \tilde{b}_0(3 - \tilde{q}) = \frac{\pi(3 - \tilde{q})}{N^{2/3}},$$

(34)

$$\bar{b}_0 := \bar{b}_0(0) = \frac{\pi}{N^{2/3}},$$

(35)

where the subscript $(0)$ denotes the ground state.

Equations (34) and (35) indicate that for this solution to exist, the original parameters $m$, $b_0$, and $a$ must not be independent, but must obey a constraint $2mb_0a^2 = \pi\hbar^2/N^{2/3}$ instead. For the model’s applications, this can be helpful because it decreases the number of free parameters. Notice also that the (eigen)value of $b_0$ depends on the combination of other parameters, namely $maN^{2/3}$, which could explain the empirical non-observability of the logarithmic nonlinear effects in some systems and their dominance in others; further discussion of this can be found in the concluding section.

For the solution (33)–(35), the effective external potential (30) can be evaluated as

$$\tilde{V}_{\text{eff}}^{(0)}(r) = \frac{1}{4}\tilde{Q}_{\text{eff}}^2(r^2 - \tilde{q}_0),$$

(36)

where $\tilde{Q}_{\text{eff}} = 2\tilde{b}_0(0) = 2\pi/N^{2/3}$ and $\tilde{q}_0 = \tilde{q}/\tilde{b}_0(0) = \tilde{q}N^{2/3}/\pi$. This formula indicates that most of the physical properties of the solution (33)–(35) must be identical to those of a quantum harmonic oscillator of the dimensionless frequency $\tilde{Q}_{\text{eff}}$. This correspondence between the LogSE’s ground states and the quantum harmonic oscillators has been pointed out in [29]. However, excited states are not likely to be interpreted in terms of the oscillator (36), because the corresponding expression for an effective external potential would certainly be more complex.

Furthermore, the entropy density (31) for the solution (33)–(35) appears to be

$$S^{(0)}_\omega = \frac{4\pi^2}{N^{2/3}}\kappa^4 \exp\left(-\frac{\pi}{N^{2/3}}\kappa^2\right),$$

(37)

and the integral Everett-Hirschman entropy (5) is simply

$$S_\omega = \frac{3}{2}N,$$

(38)

which results in the following relation between $S_\omega$ and the frequency’s eigenvalue for this solution that does not contain the normalization number $\tilde{\omega}_0 S_{\omega}^{1/2} = \pi(3/2)^{1/2}(3 - \tilde{q})$, where the frequency $\tilde{\omega}_0$ is defined in (34).

In this case, the quantum temperature (32) becomes

$$\Delta T_\omega = \frac{\pi}{N^{2/3}} \left(1 - \frac{\tilde{q}_0}{\kappa^2}\right),$$

(39)

The value $\Delta T_\omega$ is always positive-definite if $\tilde{q}_0 > 0$, which corresponds to a non-negative value of the charge $Q$. If $\tilde{q}_0 > 0$ then the sign of $\Delta T_\omega$ changes when crossing the radius $\sqrt{\tilde{q}_0}$ and becomes negative at $r < \sqrt{\tilde{q}_0}$. Possible reasons for and the implications of this behavior are discussed in the concluding section.

4.2 Case $\tilde{b}_0 = 0, \tilde{q} = 1$

In this case, the normalized solution of (28) and a corresponding frequency eigenvalue can be respectively written as

$$\psi_o(r) = \exp\left(\kappa r - \frac{1}{2}\tilde{b}_0 r^2\right),$$

(40)

$$\tilde{\omega} := \tilde{\omega}_0 = 2\tilde{b}_0 - \tilde{k}^2,$$

(41)

where the constant $\tilde{k}$ is a solution of the transcendental equation

$$\sqrt{\frac{\pi}{2\tilde{b}_0}} \left[\tilde{b}_0 \left(\tilde{b}_0 - \tilde{k}^2\right)\right]^{1/2} + \tilde{k} = \frac{N\tilde{b}_0^2}{2\pi} - \tilde{k},$$

(42)

while $\tilde{b}_0$ remains a free parameter.

For the solution (40), the effective external potential (30) becomes

$$\tilde{V}_{\text{eff}}^{(0)}(r) = \frac{2\tilde{k}}{r} + \tilde{b}_0(0) \left(r - \frac{\tilde{k}}{\tilde{b}_0}\right)^2 - \tilde{b}_0 - \tilde{k}^2,$$

(43)

thus indicating that most of the physical properties of the system in a ground state would be identical to those of a particle trapped in the harmonic potential of a frequency $2\tilde{b}_0$ crossed with the Coulomb-type potential with a strength constant $2\tilde{k}$, $\tilde{n}$.
The entropy density (31) for the solution (40) can be evaluated as
\[ \tilde{s}_\psi^{(0)} = 4\pi r^3 (\tilde{b}_0 r - 2\tilde{k}) \exp(2\tilde{k} r - \tilde{b}_0 r^2), \]
and the Everett-Hirschman entropy is given as
\[ S_\psi = N (3\tilde{b}_0^2 - 4\tilde{k}^2) - 4\pi \tilde{k}. \]
In this case, the quantum temperature (32) becomes
\[ \Delta \tilde{T}_\psi = \tilde{b}_0 - \frac{1}{r^2}, \]
thus, similar to the previous case, \( \Delta \tilde{T}_\psi \) is positive at large \( r \), changes sign when crossing the radius \( \sqrt{\tilde{b}_0} \), and becomes negative at small \( r \).

### 4.3 Case \( \tilde{b}_0 = 0, \tilde{q} = 0 \)

In this case, we recover the logarithmic Schrödinger equation with a constant nonlinear coupling,
\[ i\partial_t \psi + \nabla^2 \psi + \tilde{b}_0 \ln(|\psi|^2) \psi = 0, \]
discussed after (4) above.

For a stationary case (27), the normalized spherically-symmetric ground-state solution of (47) and a corresponding frequency eigenvalue can be respectively written as
\[ \psi_0(\vec{r}) = \left(\frac{\tilde{b}_0}{\pi}\right)^{3/4} \sqrt{N} \exp\left(-\frac{1}{2} \tilde{b}_0 r^2\right), \]
\[ \tilde{\omega} := \tilde{\omega}^{(0)} = 3\tilde{b}_0 \left[ \frac{1}{2} \ln\left(\frac{\tilde{b}_0 N^{2/3}}{\pi}\right) \right], \]
which describes a Gaussian-shaped spherical wave.

For the solution (48), the effective external potential (30) reads as
\[ \tilde{V}^{(0)}(\vec{r}) = \tilde{b}_0^2 r^2, \]
which makes this case similar to that of (36): most of physical properties of the solution (48) must be determined by a quantum harmonic oscillator of the mass 1/2 and the dimensionless frequency \( 2\tilde{b}_0 \).

The entropy density (31) for this solution appears to be
\[ \tilde{s}_\psi^{(0)} = \frac{4\tilde{b}_0^{3/2} N}{\sqrt{\pi}} r^3 \left[ \tilde{b}_0 r^2 - \frac{3}{2} \ln\left(\frac{\tilde{b}_0 N^{2/3}}{\pi}\right) \right] e^{-\tilde{b}_0 r^2}, \]
hence the integrated Everett-Hirschman entropy is simply
\[ S_\psi = \frac{3}{2} N \left[ 1 - \ln\left(\frac{\tilde{b}_0 N^{2/3}}{\pi}\right) \right] = \frac{9}{2} \frac{\tilde{\omega}^{(0)}}{\tilde{b}_0}, \]
where the frequency \( \tilde{\omega}^{(0)} \) is defined in (49). This results in relation between \( S_\psi \) and frequency’s eigenvalue for this solution, which is given by
\[ \tilde{\omega}^{(0)} - \frac{\pi (9 N - 2 S_\psi)}{2 N^{5/3}} \exp\left(\frac{2 S_\psi}{3 N} - 1\right) = 0. \]

The conjugate quantum temperature (32) becomes a constant in this case
\[ \Delta \tilde{T}_\psi = \tilde{b}_0, \]
which is positive-definite.

### 4.4 Case \( \tilde{b}_0 = 0, \tilde{q} \neq 0 \)

In this case, (26) becomes
\[ i\partial_t \psi + \nabla^2 \psi - \frac{\tilde{q}}{r^2} \ln(|\psi|^2) \psi = 0, \]
where a decomposition (29) is implied.

While this equation looks more complicated to solve than the logarithmic equation with a constant nonlinear coupling (47), it has certain features that make it easy to study. Most significantly, this equation allows a separation of the angular variables from others, which paves the way for us to drastically decrease the dimensionality of the problem in a general case. Assuming the stationary ansatz (27), where
\[ \psi(\vec{r}) = R(r) Y(\theta, \varphi), \]
and using the decomposition (29), we can separate (55) into its radial and angular parts respectively given by
\[ \nabla^2 Y - \frac{\tilde{q}}{r^2} \ln(|Y|^2) Y + L^2 Y = 0, \]
where \( L \) is a separation constant whose eigenvalue follows from the last equation. The latter resembles a differential equation for spherical harmonics, but contains a nonlinear term, making its solutions a separate topic of research. This equation indicates that the system’s total angular
momentum acquires a nonlinear correction, which can manifest in those systems for which $Q_\sigma \neq 0$.

Furthermore, (55) can be viewed either as a short-distance limit $r \ll \sqrt{\sigma / b_0} = Q_\sigma / (b_0 a^2)$ of (26), or as a large-charge limit, $Q_\sigma \to \infty$, thereof. Thus, it is no longer necessary to assume that $b_0$ is unnaturally small to fit the existing experimental data for physical systems, for which the model (4) is a priori inapplicable. It is sufficient to assume that models with a large value of $Q_\sigma$ are more relevant for those systems.

Given that the separation of angular variables from others is possible in this case, we need not restrict ourselves to a spherically symmetric ansatz to find the analytical solutions. Instead, we will search for a solution for the radial wavefunction $R(r)$ given by (57) and normalized as
\[ \int_0^\infty |R|^2 r^2 dr = N. \]

Then the solution of (57) can be written as
\[ R_{\text{nl}}(r) = \frac{\exp(-\tilde{\mu}^2 r - L^2 / 2)}{r^2} \quad \text{if } \tilde{q} = 1, \]
\[ \exp(-L^2 / 2) \quad \text{if } \tilde{q} \gg 1, \]

where $\tilde{\mu}^2 = (4N)^{-1/3} \exp(-L^2 / 3)$, and a subscript 0 refers to a lowest value of a main quantum number. It is clear that on a real semi-axis, a physically suitable solution is the one for which
\[ \tilde{q} = 1, \]

and this can be regarded as a constraint for the original value $Q_\sigma$: $Q_\sigma \to \hbar^2/(2m)$.

For the solution (59)–(61), the effective external potential (30) reads
\[ V_{\text{eff}}^{(0)}(r) = \frac{2 \tilde{\mu}^2 - L^2}{r^2}, \]

thus indicating that most of physical properties of this solution would be identical to those of a particle in the Coulomb-type potential with a strength constant $2\tilde{\mu}^2$, in the presence of the standard “centrifugal” potential $L^2/r^2$.

Furthermore, the entropy density (31) for the solution (59)–(61) appears to be
\[ S_\psi = 8 \tilde{\mu}^8 N r \left(1 + \frac{L^2 + S_\psi}{2\tilde{\mu}^2 r}\right) \exp(-2\tilde{\mu}^2 r), \]

where we denote the constant
\[ S_\psi = -\int |Y|^2 \ln(|Y|^2) \sin^2 \theta d\theta d\phi, \]

an integration being taken over a sphere. The integral EH entropy (5) reads
\[ S_\psi = N(L^2 + S_\psi + 3), \]

while the conjugate quantum temperature (32), in this case, becomes
\[ \Delta \bar{T}_\psi = -\frac{1}{r^2}, \]

the implications of which are discussed below.

5 Discussion

We have studied the dynamical behavior of nonlinear coupling $b$ in a logarithmic quantum wave equation. Using statistical mechanics arguments, we have shown that this coupling is related to the thermal temperature of many-body systems, which satisfy the following conditions: (i) their potential energy must be larger than the kinetic energy, and (ii) they must allow an effective description in terms of collective degrees of freedom governed by a wavefunction. One example of such systems would be strongly interacting Bose liquids, where the logarithmic model has been proven to very instrumental [30, 31].

Furthermore, the nonlinear coupling has been associated with a certain kind of quantum temperature: a thermodynamical conjugate to the Everett-Hirschman entropy, which allows us to relate thermal temperature, EH quantum temperature and nonlinear coupling, as discussed in Section 2.

In view of the dynamical nature of nonlinear coupling, a combined quantum-mechanical and field-theoretical model is proposed in Section 3, which leads to a logarithmic equation with variable nonlinear coupling. By studying the behavior of the latter, we achieve a deeper understanding of the Everett-Hirschman entropy and its thermodynamical conjugate. Considering this model in a next-to-leading order approximation with respect to the auxiliary field $\sigma$ (see Section 3.2), we analytically obtained a number of stationary solutions and established a number of profound properties, which are presented in Section 4. Below, we present a discussion of the results obtained.

First, the logarithmic model reveals a connection between the EH conjugate temperature and the conventional (thermal) one. The latter is still a notion that is well-defined at a classical level only (e.g. as measured by bringing a system into a thermal equilibrium
with a calibrated system), but its quantum analogue is unknown. This problem manifests itself most strikingly when dealing with cold quantum gases and liquids, such as Bose-Einstein condensates. Strictly speaking, one cannot measure the temperature of a condensate in a classical way – experimental condensates are energetically isolated, so no thermal equilibrium can be achieved with a calibrated system without affecting the condensate’s state. Thus, the standard method of measurement consists of switching a trap off, using a laser upon the condensate’s atoms and measuring the scattered light to deduce their temperature from experimental profiles of density and momentum distributions, assuming that the energy spectrum is also known. However, this method of measurement presumes that the relations between temperature and the above-mentioned distributions and spectra are derived from some theoretical model, which must be thus presumed to be a priori valid for that particular condensate. Besides, the detrapping measurement is unlikely to be reliable for strongly interacting quantum Bose liquids, which tend to confine their atoms into a droplet, in the absence of trapping potentials [30, 31, 34].

In this regard, a conjecture that the thermal temperature is related to quantum temperature \( T_q \) and nonlinear coupling \( b \), at least for a large class of systems, has been discussed in Section 2. Aside from solving the above-mentioned issue with the fundamental (quantum-mechanical) definition of the thermal temperature, the relation (12) also lays the quantum-mechanical foundations for the Landauer’s principle [69]. Specifically, it is the term \( \Delta T_q \) from (6) which is responsible for information-handling cost of energy, including the energy cost for the information eraser. The latter has been confirmed in experiments with different nanoscale systems [70–72].

Second, in the model with the variable nonlinear coupling, the nature of the quantum EH temperature (6) becomes clearer. For the solution described in Section 4.3, the value \( \Delta T_q \) is positive semi-definite, whereas in Section 4.4, it is negative semi-definite; meanwhile, in Sections 4.1 and 4.2, it changes its sign at a certain radius. The common feature of the cases in Sections 4.1 and 4.2 is that \( \Delta T_q \) tends toward negative values at small distances from the origin and to a positive constant at large distances. Moreover, the common feature of all cases is that \( \Delta T_q \) tends to negative values whenever the term \( Q/N^2 \) in the nonlinear coupling predominates over the constant one. Analyzing these features together, one can hypothesize that the EH temperature can serve as a means of differentiating phases, e.g. those related to the microscopic and macroscopic scales of radius: \( \Delta T_q \) is negative in the microscopic scale and it is positive in the macroscopic one (up to a sign convention adopted in the definition of \( S_q \)).

Third, in the minimal model of Section 3, the coupling constant \( b_0 \) is no longer a predefined parameter of a theory, cf. (4). Instead, it becomes one of the integration constants of the evolution equations, such as (19) and (20); therefore, its value can vary from one system to another. Thus, the full model allows us to reduce a number of parameters of the theory and make it more self-consistent and self-sufficient. However, within the frameworks of the approximation (22)–(24), the values of \( Q \) and \( b_0 \) are unknown and have yet to be determined from other considerations.

Nevertheless, even the approximate minimal model, cf. Section 3.2, offers an explanation as to why the constant \( b_0 \) is negligible for some systems but crucial for others, as mentioned in the Introduction. To illustrate this, let us compare the cases described in Section 4. For the solutions of (24), which are described in Sections 4.2 and 4.3, the constant \( b_0 \) remains a free parameter, which can take any value, either defined ad hoc or fitted from an experiment; an example of the latter procedure can be found in [31]. However, for the solution in Section 4.1, the constant \( b_0 \) becomes an eigenvalue, i.e. a function of other constants of the model and quantum numbers (if one considers the excited states). Specifically, it is small if the combination \( ma^4N_0^3 \) is large. Given that neither of those three constants are fundamental nor universal for all quantum systems, the value of \( b_0 \) can vary between systems. Moreover, it can also vary for the different solutions of the same system, because of the above-mentioned eigenvalue structure and a Hilbert space associated with it.

Finally, the effective external potentials computed in Sections 4.2 and 4.4 illustrate a possibility that some fundamental interactions, such as gravity, could emerge as a nonlinear quantum-mechanical phenomenon based on a concept of the quantum information entropy, cf. (6) and the evolution equations of a logarithmic type. This conjecture is supported by other studies, which suggest that the most probable candidate for such a phenomenon is a background superfluid of a logarithmic type [35, 36, 38].

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References