Stability and Spatiotemporal Bifurcations in Spatially Distributed Neural Networks with Nonlocal Delay

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Abstract: The stability of equilibria and bifurcations of neural networks in a real line with nonlocal delay are presented. A sufficient condition of stable equilibria is declared by the linear part. Eigenvalue analysis implies the existence of bifurcations, and by exploiting typical excitatory and inhibitory connectivity kernels in a neural network, the possible bifurcations are discussed according to various cases. It is an advantageous tool using a multiple-scale method to study the stability of bifurcated travelling waves or spots. As an illustration of our theory, the dynamics of a seashell continuous-time circular mask model are investigated. It is shown that both the shape and range of active function and synaptic weights can affect the dynamics of the model. Finally, the bifurcation set and the variety of bifurcated patterns of the seashell model are numerically revealed.

Keywords: Bifurcation; Pattern Formation; Nonlocal Delay; Neural Network; Spatiotemporal Interaction.

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1 Introduction

A pattern is a nonuniform macrostructure with some regularity in space or time. It is common in nature, and all kinds of patterns constitute a colourful world. Thus, it is of great significance for understanding the mysteries of nature to understand the causes and mechanisms of pattern formation. In 1952, the British mathematician A. M. Turing published a famous paper [1] to successfully illustrate how the patterns on the surface of certain organisms (such as the patterns of zebra) were produced. Turing patterns have stimulated great interest in the mechanisms of spatiotemporal activity and have led to a boom in the research on pattern dynamics. There are now numerous examples linking various spatial patterns [2, 3], travelling waves [4], and spirals [5, 6] revealed by experimental observation [7, 8], numerical analysis [9, 10], approximate analytical methods, and model building [11].

In particular, it is well known that the diversity of organisms is closely related to the activity of neurons. Therefore, it is very important to explore the pattern dynamics of neural networks and reveal their mechanisms for explaining the complex phenomena of the real nervous systems and revealing the mysteries of life. While, generally, cell membranes in neural networks are found at resting potentials, neurons will be excited when external stimuli and disturbances reach the neuronal discharge thresholds. Then, this excitation spreads around other cells to form a discharge array at different locations in the neural networks. Various states of a large number of neurons give rise to the pattern structures such as the helical wave or target wave in the heart tissue [12] and the stripe pattern in the muscle tissue [13]. Considerable research efforts have been devoted to the pattern formation of neural networks [5, 14–19].

From the perspective of dynamics, the neural network is a complex, nonlinear, dynamic system, that is, a multilevel system composed of a large number of nonlinear elements through a wide range of actions. The hierarchy has some emerging properties that the next level of the system does not have, forming the complexity and diversity of patterns. Study of the dynamics of neural networks is conducive to the in-depth understanding of the formation mechanism of different patterns as well as the diversity of states presented by the neurons in the network. This is also the motivation and value of the present article. For the results regarding the dynamics in neural networks, the readers are referred to the valuable work of Huang et al. [20–22].

The purpose of this paper is to study the stability and spatial-temporal bifurcations of the following spatially
distributed neural network model with nonlocal delay

\[ u(x, t) = \int_{0}^{\infty} \int_{-\infty}^{\infty} K(s, y, u(x - y, t - s), p) dy ds, \quad (1) \]

where \( p \in \mathbb{R} \) is a parameter.

This model represents a broad class of problems and has a wide range of applications. Actually, this system can be used to describe population models [23, 24]. Another interesting finding is that some reaction-diffusion models can be written in this form so that the existence of its travelling waves is definitely an important topic (see [25, 26]). At the same time, the crucial influence of nonlocal delay in reaction-diffusion systems has also attracted the attention of researchers. For example, Chen and Yu [27–29] reported a series of results regarding the impact of a spatiotemporal delay on the stability of equilibria and existence of Hopf bifurcation in some reaction-diffusion systems. Then, the global stability of steady states and the property of travelling waves in systems with nonlocal delay were proposed (see [30–32]). In addition, based on synaptic coupling, the model can be regarded as neural networks with nonlocal delay. During the past decade, there has been increasing interest in the relationship between connectivity and dynamics. However, most work only considered the spatial interactions without the time factor [19, 20, 33, 34]. As a result, it is necessary to study pattern dynamics caused by spatiotemporal weighting, as is done in this article.

The rest of the article is organized as follows. In Section 2, we investigate the system equilibria (1) and obtain a sufficient condition for the stability. For typical spatiotemporal interaction kernels, the types of bifurcations are elucidated by eigenvalue analysis. A multiple-scale method is used to compute the normal form in order to check the stability of the bifurcated patterns. Section 3 demonstrates the dynamics of a seashell model according to Section 2, and the patterns caused by different bifurcations are reported. Finally, the article is concluded in Section 4.

\section{Stability of Equilibrium and Bifurcations}

\subsection{Stability of Equilibrium}

In the beginning, without loss of generality, we assume \( K(s, y, 0, p) \equiv 0 \) and \( \bar{u} = 0 \) is the equilibrium of (1).

In fact, if \( u^* (\neq 0) \) is an equilibrium of (1), satisfying

\[ u^* = \int_{0}^{\infty} \int_{-\infty}^{\infty} K(s, y, u^*, p) dy ds, \quad (2) \]

let \( \bar{u} = u - u^* \); then (1) can be written as

\[ \bar{u}(x, t) = \int_{0}^{\infty} \int_{-\infty}^{\infty} K(s, y, u(x - y, t - s), p) - K(s, y, u^*, p) dy ds \]

\[ = \int_{0}^{\infty} \int_{-\infty}^{\infty} \frac{1}{n!} \left( \frac{\partial^n K(s, y, u, p)}{\partial u^n} \right) |_{u = u^*} \bar{u}^n dy ds. \quad (3) \]

Thus, the nonzero equilibrium is transferred into zero. Some transformation for problem (1) is necessary. Expanding \( K(s, y, u, p) \) with respect to \( u \) at \( u = 0 \), we obtain

\[ K(s, y, u, p) = K_1(s, y, p)u + K_2(s, y, p)\frac{u^2}{2!} + K_3(s, y, p)\frac{u^3}{3!} + \cdots, \quad (4) \]

where \( K_n(s, y, p) = \frac{\partial^n K(s, y, 0, p)}{\partial u^n}, n = 1, 2, 3, \ldots \).

Obviously, the associated linear operator is

\[ Lu = u - \int_{0}^{\infty} \int_{-\infty}^{\infty} K_1(s, y, p)u(x - y, t - s) dy ds. \quad (5) \]

Considering the ansatz \( u(x, t) = e^{\mu t} e^{ikx} \) in (5), the sign of \( \text{Re} \mu \) determines the stability of the equilibrium in the equation

\[ \int_{0}^{\infty} \int_{-\infty}^{\infty} K_1(s, y, p) e^{-\mu s} e^{-iky} dy = 1. \quad (6) \]

The first problem to be solved is the stability of zero equilibrium. Based on the eigenvalue analysis, we can give a sufficient condition.

\textbf{Theorem 1.} If \( c < 1 \), then the zero equilibrium is asymptotically stable, where \( c = \int_{0}^{\infty} \int_{-\infty}^{\infty} |K_1(s, y, p)| dy ds \).

\textit{Proof.} Assume \( \exists \mu = \sigma + i\omega \) satisfying the characteristic equation, where \( \sigma \geq 0 \); it then follows that

\begin{align*}
1 &= \int_{0}^{\infty} \int_{-\infty}^{\infty} K_1(s, y, p) e^{-\sigma i\omega s} e^{-iky} dy ds \\
&\leq \int_{0}^{\infty} \int_{-\infty}^{\infty} |K_1(s, y, p)| dy = c. \quad (7)
\end{align*}
This, however, contradicts \( c < 1 \). Thus, all \( \text{Re} \mu < 0 \), and the zero equilibrium is asymptotically stable.

### 2.2 Bifurcations

Considering the zero real part eigenvalue \( \mu = i \omega \) of (6), the bifurcation behavior of the system (1) is classified as follows:

I. \( \omega = 0 \) and \( k = 0 \): bifurcating to a spatiotemporally constant solution;

II. \( \omega = 0 \) and \( k \neq 0 \): bifurcating to a spatially periodic solution (Turing bifurcation);

III. \( \omega \neq 0 \) and \( k = 0 \): bifurcating to a temporally periodic solution (Hopf bifurcation);

IV. \( \omega \neq 0 \) and \( k \neq 0 \): bifurcating to travelling waves with wave speed \( \omega/k \) (Turing-Hopf bifurcation).

Let \( \hat{K}(\omega, k, p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_s(s, y, p) e^{-i\omega s} e^{-iyk} dy ds \); then, the conditions for different bifurcations are obtained easily. For case I, \( \hat{K}(0, 0, p) = 1 \) should lead to the bifurcation; the condition for case II is \( \hat{K}(0, k, p) = 1(k \neq 0) \); similarly, \( \hat{K}(\omega, 0, p) = 1(\omega \neq 0) \) is case III; and the condition for case IV is \( \hat{K}(\omega, k, p) = 1(\omega, k \neq 0) \).

Generally, similar to a single spatial interaction, one can choose \( K_1(s, y, p) \) as typical kernel functions to investigate the corresponding possible bifurcations. Figure 1 gives some typical spatiotemporal kernels, and the corresponding \( \hat{K}(\omega, k, p) \) are shown in Figure 2.

According to the discussion at the beginning of this subsection, this gives the bifurcations of a system (1) with different spatiotemporal kernels to check whether the figures in Figure 2 can intersect the plane \( \mu = 0 \) perpendicular to the vertical axis. Therefore, all four types of bifurcations are possible for kernels (a), (c), (f); kernel (d) causes bifurcations II, III, and IV; and no bifurcations appear in the system with (b), (e).

Additionally, to apply the multiple-scale method to investigate the stability of the bifurcated solutions, setting \( p = 0 \) is critical point of Turing-Hopf bifurcation, and expanding \( K_i(s, y, p)(i = 1, 2, \ldots) \) with respect to \( p \) at \( p = 0 \) and substituting (4) into (1), we can obtain

\[
\begin{align*}
0 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_0(s, y) u(x - y, t - s) dy ds + p \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_1(s, y) u(x - y, t - s) dy ds \\
&+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_2(s, y) u^2(x - y, t - s) dy ds + \cdots,
\end{align*}
\]

where \( K_0(s, y) = K_1(s, y, 0) \), \( K_1(s, y) = \frac{\partial K_0(s, y, 0)}{\partial p} \), \( K_2(s, y) = \frac{\partial^2 K_0(s, y)}{\partial p^2} \) \( (n = 2, 3, \ldots) \), and \( K_i(s, y) = K_i(s, -y) \), \( i = 0, 1, 2, \ldots \) as usual.

Processing the computation to simplify the problem (1) (the details are provided in the Appendix), the normal form of Turing-Hopf bifurcation is given by

\[
\begin{align*}
z_r &= z(a p_2 + \beta z + \gamma w w), \\
w_r &= w(a p_2 + \beta w w + \gamma z z),
\end{align*}
\]

where \( a = a_1 + i a_2 = \frac{K_{i}(0,0)}{M}, \beta = \beta_1 + i \beta_2 = \frac{2 K_{i}(0,0) (a_1 + a_2) + 3 K_{i}(0,0)}{M}, \gamma = \gamma_1 + i \gamma_2 = \frac{2(k_{i1}(0,0)(c_1 + c_2 + a_2) + 3 k_{i2}(0,0))}{M} \), \( M = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_0(s, y) se^{-i\omega s - ik y} dy ds \).

Let \( z = r_1 e^{i \theta_1}, w = r_2 e^{i \theta_2} \); one can transfer system (9) into the polar coordinate equation

\[
\begin{align*}
\frac{dr_1}{d\tau} &= r_1 [a_1 p_2 + \beta_1 r_1^2 + \gamma_1 r_1 r_2^2], \\
\frac{dr_2}{d\tau} &= r_2 [a_2 p_2 + \beta_2 r_2^2 + \gamma_2 r_1 r_2^2], \\
\frac{d\theta_1}{d\tau} &= a_1 p_2 + \beta_1 r_1^2 + \gamma_1 r_1 r_2^2, \\
\frac{d\theta_2}{d\tau} &= a_2 p_2 + \beta_2 r_2^2 + \gamma_2 r_1 r_2^2.
\end{align*}
\]

The stability of the Turing-Hopf bifurcated patterns is summarised in the following theorem.
Theorem 2. (i) The left-travelling waves $r_1 = 0$, $r_2 = \sqrt{-\alpha_1 p_2/\beta_1}$ (or right-travelling waves $r_1 = \sqrt{-\alpha_1 p_2/\beta_1}, r_2 = 0$) are stable if and only if $\gamma_1 < \beta_1 < 0$.

(ii) The spots $r_1 = r_2 = \sqrt{-\alpha_1 p_2/\left(\beta_1 + \gamma_1\right)}$ are stable if and only if $\beta_1 < -|\gamma_1| < 0$.

3 Applications

Next, we examine the results of the above method using a seashell pigmentation model [35], which describes the neurosecretory system of aquatic mollusks generating diversity of shell structures and pigmentation patterns. The continuous-time circular mask model of seashell is an integral equation:

$$
\begin{align*}
    u(x, t) &= \int_0^\infty \int_{-\infty}^\infty \frac{2\lambda_e e^{-\frac{y^2 + h^2}{\sigma_i^2}}}{\pi \sigma_i^2} f_e(u(x - y, t - s))dyds \\
    &\quad - \int_0^\infty \int_{-\infty}^\infty \frac{2\lambda_i e^{-\frac{y^2 + h^2}{\sigma_i^2}}}{\pi \sigma_i^2} f_i(u(x - y, t - s))dyds,
\end{align*}
$$

where $f_e(u) = \frac{\sigma_1}{1 + e^{-\sigma_1 u - \sigma_2}}, f_i(u) = \frac{\sigma_2}{1 + e^{-\sigma_1 u - \sigma_2}}$, and all parameters are positive. This is a two-layer network model where the subscripts $e$ and $i$ represent the synaptic parts of the excitatory and inhibitory neurons. The system does show some steady states (see Fig. 3) for different parameters.

Since all parameters are positive, the spatiotemporal weighting declares lateral inhibitory if $\frac{\nu_e}{\lambda_e} < \frac{\nu_i}{\lambda_i}$ or lateral excitatory if $\frac{\nu_e}{\lambda_e} > \frac{\nu_i}{\lambda_i}$. A deep analysis can be carried out. For lateral inhibitory, the bifurcation case II appears for $\sigma_e < \sigma_i$ as $\lambda_e < \lambda_i$, and, similarly, III for $\sigma_e > \sigma_i$ and IV for $\sigma_e < \sigma_i$ as $\lambda_e > \lambda_i$. However, the same results are right.

Figure 3: Equilibrium solutions of system (11) with $\nu_e$ variation.
Black curve: $\alpha_e = 3, \alpha_i = 2, \nu_i = 1, \theta_e = 0.5, \theta_i = 0.6$; red curve: $\alpha_e = 1.5, \alpha_i = 1, \nu_i = 10, \theta_e = 0.25, \theta_i = 0.25$; blue curve: $\alpha_e = 2, \alpha_i = 3, \nu_i = 10, \theta_e = 0.05, \theta_i = 1$. Especially, the blue curve demonstrates the co-existence of multiple steady states as $\nu_e > 2.4$.
with the opposite relationship between $\lambda_e$ and $\lambda_l$ for lateral excitatory. Hence, the patterns of (11) can be revealed by Figure 4.

For the purpose of simplifying the calculation, we choose the parameters, making $\hat{u} = \theta_e = \theta_l$ be an equilibrium, that is, it demands $\frac{a_e - a_l}{2} = \theta_e = \theta_l$. In this case, taking $\nu_e$ as the bifurcation parameter,

$$K_1(s, y, p) = \frac{a_e \nu_e}{2} \frac{\lambda_e e^{-\frac{s^2 + p^2}{\sigma_e^2}}}{\frac{\lambda_i e^{-\frac{s^2 + \nu_e^2}{\sigma_i^2}}}{2}},$$

$$K_0(s, y) = -\frac{a_i \nu_i \lambda_e}{2 \pi \sigma_i^2},$$

$$K_1(s, y) = \frac{a_e \lambda_i e^{-\frac{s^2 + \nu_e^2}{\sigma_i^2}}}{2 \pi \sigma_i^2},$$

$$K_2(s, y) = 0,$$

$$K_3(s, y) = -\frac{1}{2 \pi \sigma_e} \left[ \frac{a_e \nu_e \lambda_e}{\sigma_e} \frac{e^{-\frac{s^2 + \nu_e^2}{\sigma_e^2}}} {\frac{\lambda_i \nu_i}{\sigma_i^2}} \frac{e^{-\frac{s^2 + p^2}{\sigma_i^2}}}{2} \right].$$

With regard to this model, $c = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K_1(s, y, p)| \, ds \, dy$ is the bifurcation pattern in Theorem 1; thus, $|a_e \nu_e - a_i \nu_i| < 4$ is the sufficient condition of asymptotically stable equilibrium $\hat{u}$. Obviously, we need to study bifurcated patterns as $|a_e \nu_e - a_i \nu_i| > 4$.

Furthermore, if $u(x, t) = e^{u(t) + ikx}$ is the solution for the linearization of (11) around the fixed point $\hat{u}$, then the associated characteristic equation is

$$1 = 2 \frac{\lambda_e A_0}{\sigma_e} \int_{-\infty}^{\infty} e^{-\frac{s^2 + p^2}{\sigma_e^2}} e^{-iky} \, dy \int_{-\infty}^{\infty} e^{-\frac{s^2 + p^2}{\sigma_i^2}} e^{-\nu_e dy} \, ds - \frac{\lambda_i B_0}{\sigma_i} \int_{-\infty}^{\infty} e^{-\frac{s^2 + p^2}{\sigma_i^2}} e^{-iky} \, dy \int_{-\infty}^{\infty} e^{-\frac{s^2 + \nu_e^2}{\sigma_i^2}} e^{-\nu_e dy} \, ds,$$

that is

$$1 = A_0 \left( 1 - \text{erf} \left( \frac{\mu \sigma_e}{2 \lambda_e} \right) \right) e^{-\frac{\nu_e^2}{\sigma_e}} - B_0 \left( 1 - \text{erf} \left( \frac{\mu \sigma_i}{2 \lambda_i} \right) \right) e^{-\frac{\nu_e^2}{\sigma_i}},$$

where $A_0 = \frac{a_e \nu_e}{2}$, $B_0 = \frac{a_i \nu_i}{2}$.

For $\mu = i\omega$, (13) can be written as

$$1 = A_0 \left( 1 - \text{erf} \left( \frac{i \omega \sigma_e}{2 \lambda_e} \right) \right) e^{-\frac{\nu_e^2}{\sigma_e}} - B_0 \left( 1 - \text{erf} \left( \frac{i \omega \sigma_i}{2 \lambda_i} \right) \right) e^{-\frac{\nu_e^2}{\sigma_i}}.$$ (14)

Separating the real and imaginary parts of the above equation, we obtain

$$1 = A_0 e^{-\frac{\nu_e^2}{\sigma_e}} \text{erfi} \left( \frac{\omega \sigma_e}{2 \lambda_e} \right) - B_0 e^{-\frac{\nu_e^2}{\sigma_i}} \text{erfi} \left( \frac{\omega \sigma_i}{2 \lambda_i} \right) - \frac{\nu_e^2}{\sigma_i}.$$ (15)

Next, we take the kernel with the form of Mexican hat (i.e., lateral inhibitory) as an example to simulate by XPPAUT. Suppose $\sigma_e = 0.01$, $\sigma_i = 0.1$, $a_e = 1.5$, $a_l = 1$, $\theta_e = 0.25$, $\lambda_e = 0.1$, $\lambda_l = 0.5$, $\nu_e = 5$, $\nu_i = 65$ in system (11). The relationship between $\nu_e$ and $k$ is shown in Figure 5a; this is the minimum $k^* = 46.855$, and the bifurcation point is $\nu_e^* = 2.846$. We take $\nu_e = 3$ to obtain the vertical stripes caused by Turing bifurcation.

Taking $\sigma_e = 0.02$, $\sigma_i = 0.05$, $a_e = 1.5$, $a_l = 1$, $\theta_e = 0.25$, $\lambda_e = 0.1$, $\lambda_l = 0.5$, $\nu_e = 5$, $\nu_i = 65$ in system (11), the relationship between $\nu_e$ and $k$ is shown in Figure 5a, i.e., $k^* = 6.141$, $\nu_e^* = 2.546$, $\nu_e^* = 4.223$. $\beta_1 = -1354.7$, $\gamma_1 = -2709.4$ in Theorem 2 imply stable travelling waves (see Figure 6b).
Figure 5: (a) Plot of $\nu_e$ corresponding to $k$, indicating the wave number, i.e. the number of the vertical stripes, is the minimum $k^* = 46.855$, and the bifurcation point is $\nu_e^* = 2.846$. (b) Vertical stripes of (11) with $\nu_e = 3$ caused by Turing bifurcation.

Figure 6: (a) Plot of $\nu_e$ corresponding to $k$, indicating the wave number, i.e. the number of the travelling waves, is the minimum $k^* = 6.141$, and the bifurcation point is $\nu_e^* = 4.223$. (b) Travelling waves of (11) with $\nu_e = 5$ caused by Turing bifurcation.

Figure 7: (a) Plot of $\nu_e$ corresponding to $k$, indicating the wave number, i.e. the number of the checkerboard spots, is the minimum $k^* = 33.103$, and the bifurcation point is $\nu_e^* = 3.294$. (b) Checkerboard spots of (11) with $\nu_e = 4$ caused by Turing bifurcation.
Similarly, taking $\sigma_e = 0.02, \sigma_i = 0.1, \alpha_e = 1.5, \alpha_i = 1, \theta_e = \theta_i = 0.25, \lambda_e = 0.1, \lambda_i = 0.05, v_e = 4, v_i = 20$, we can obtain the plot of $\nu_e$ corresponding to $k$ (Fig. 7) and the pattern formation of (11). Here, $k^* = 33.103, \omega^* = 3.167, \nu_k^* = 3.294, \beta_1 = -1.2926, \gamma_1 = -0.6463$ imply stable spots.

Additionally, system (11) can show other different spatiotemporal patterns. Initially, we always set $\sigma_e = 0.02, \lambda_e = 0.1, \alpha_e = 1.5, \alpha_i = 1, \theta_e = \theta_i = 0.25$. Figure 8 demonstrates that other parameters varying with the system (11) undergo different dynamics, including horizontal stripes and networks.

4 Conclusion

In this article, we focused on a neural network with nonlocal delay in a real line. We discussed the dynamics near the positive equilibrium of the system. This study revealed the relationship between the linear part of the system and stable equilibrium that provides the sufficient condition for stability. In addition, the eigenvalue analysis implies bifurcation behavior, taking several typical interaction kernels as examples in detail. The investigation of bifurcation shows the pattern formation of a neural model. Whether a pattern is stable or not is further determined by the normal form from multiple scale reduction.

As an application of the theory, a seashell model with spatiotemporal weighting was studied. We found that the stable equilibrium depends on both excitatory and inhibitory neurons, and especially, the shape and range of the active function in the network determine the dynamics. Additionally, the bifurcation set was plotted, showing the effect of excitatory and inhibitory synaptic scales on pattern formation. With the guidance of our theory, proper parameters were chosen to confirm the existence of various patterns, including vertical and horizontal stripes, networks, and checkerboard spots corresponding to their wave numbers.

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Appendix

According to the description above, assume there exists $(k_0, \omega_0)$ such that

$$\int_0^\infty \int_{-\infty}^\infty K_0(s, y)e^{-ik_0y}e^{-i\omega_0s}dy = 1,$$

system (1) undergoes Turing-Hopf bifurcation; as a result, the eigenfunctions of $L$ have the form $ze^{i\omega_0(t+ik_0x)}$.
\( e^{i\omega_0 t - ik_0 x} + \bar{z}e^{-i\omega_0 t + ik_0 x}, \, we^{i\omega_0 t + ik_0 x} \) where \( z, w \) are time-dependent functions; therefore, the solution of (1) can be written as

\[
u(x, t, \tau) = e[z(\tau)e^{i\omega_0 t + ik_0 x} + w(\tau)e^{-i\omega_0 t - ik_0 x} + c.c.] + e^2\nu_2 + e^3\nu_3 + \cdots, \quad (A.1)
\]

where \( \tau = e^2t \) and c.c. is the complex conjugate of the preceding terms.

Neglecting the spatial variable, we provide the following observation to make sure that the time scale is nonzero.

\[
h(\tau) = \int_{-\infty}^{\infty} q(s)z(\tau)e^{i\omega_0(t-s)}ds
\]

\[
= \int_{-\infty}^{\infty} q(s)[z(\tau) - e^{-2\gamma}s]ds
\]

\[
= \gamma_0 z(\tau) - e^{-2\gamma}z(\tau)(\tau), \quad (A.2)
\]

where \( \gamma_0 = \int_0^{\infty} q(s)ds \neq 0, \gamma_* = \int_0^{\infty} sq(s)ds \neq 0 \).

Assume parameter \( p = \exp_1 + e^2p_2 + \cdots \). In order to seek the solutions of (1), substitute (A.1) into (5), and compare the coefficients of \( e^3 \), neglecting the high-order terms of the parameters

\[
Lu_2 = u_2 - \int_{-\infty}^{\infty} K_0(s, y)u_2(x - y, t - s)dyds
\]

\[
= p_1 \int_{-\infty}^{\infty} K_1(s, y)[z(\tau)e^{i\omega_0(t-s - ik_0(x-y))}
\]

\[
+ w(\tau)e^{i\omega_0(t-s - ik_0(x-y))} + c.c.]dyds
\]

\[
+ z^2e^{2i\omega_0 t}e^{-2ik_0 x}K_2(2\omega_0, 2k_0)
\]

\[
+ 2z\bar{z}K_2(0, 0) + \bar{z}^2e^{-2i\omega_0 t}e^{-2ik_0 x}K_2(2\omega_0, 2k_0)
\]

\[
+ 2\omega\bar{\omega}K_2(0, 0) + \omega^2e^{-2i\omega_0 t}e^{2ik_0 x}K_2(2\omega_0, 2k_0) + \cdots, \quad (A.3)
\]

where \( \hat{K}_1(m_1, m_2) = \int_0^{\infty} \int_{-\infty}^{\infty} K(s, y)e^{-im_1s}e^{-im_2y}dyds \), \( i = 0, 1, 2, \cdots \), \( m_1 = j\omega_0, m_2 = jk_0, j = 2v, v \in \mathbb{Z} \).

The nonhomogeneous equation (A.3) has a solution if and only if the so-called solvability condition is satisfied [36]. That is, if

\[
\int_{-\infty}^{\infty} K_1(s, y)e^{i\omega_0 s}e^{ik_0 y}dyds \equiv \text{Re} \beta_1 = 0,
\]

then (A.3) can be solved. At the same time, we can obtain \( p_1 = 0 \).

Based on the discussion above, it is reasonable to assume

\[
u_2 = a_2z^2e^{2i\omega_0 t}e^{2ik_0 x} + a_0zz^\bar{z}
\]

\[
+ b_2w^2e^{2i\omega_0 t}e^{-2ik_0 x} + b_0w\bar{w}
\]

\[
+ c_1zw e^{2i\omega_0 t} + c_2z\bar{w}e^{2ik_0 x} + \cdots
\]

and comparing the coefficients of \( z^2e^{2i\omega_0 t}e^{2ik_0 x} \),

\[
L(a_2z^2e^{2i\omega_0 t}e^{2ik_0 x}) = a_2z^2e^{2i\omega_0 t}e^{2ik_0 x}
\]

\[
- a_2z^2e^{2i\omega_0 t}e^{2ik_0 x}K_2(2\omega_0, 2k_0)
\]

\[
= z^2e^{2i\omega_0 t}e^{2ik_0 x}K_2(2\omega_0, 2k_0), \quad (A.4)
\]

\[
a_2 = \frac{\hat{K}_2(2\omega_0, 2k_0)}{1 - K_0(2\omega_0, 2k_0)}.
\]

Similarly

\[
b_2 = a_2, a_0 = b_0 = \frac{2\hat{K}_2(0, 0)}{1 - K_0(0, 0)},
\]

\[
c_1 = \frac{2\hat{K}_2(2\omega_0, 0)}{1 - K_0(0, 0)}, \quad c_2 = \frac{2\hat{K}_2(0, 2k_0)}{1 - K_0(0, 2k_0)},
\]

where \( \hat{K}_0(m_1, m_2) \neq 1, m_1 = j\omega_0, m_2 = jk_0, j = 2v, v \in \mathbb{Z} \).

We have the truncation of \( u \) up to the second order:

\[
u(x, t, \tau) = e[z(\tau)e^{i\omega_0 t + ik_0 x} + w(\tau)e^{i\omega_0 t - ik_0 x} + c.c.] + e^2[a_2z^2e^{2i\omega_0 t}e^{2ik_0 x} + a_0zz^\bar{z}] + \cdots \quad (A.5)
\]

where \( a_0 \) and \( a_2 \) are defined above. Besides, owing to

\[
z^{(2 \tau - t)} = z(e^2t - e^2s)
\]

\[
= z(\tau - e^2s)
\]

\[
= z(\tau) - e^2sz_s + \cdots,
\]

\[
w^{(2 \tau - t)} = w(\tau) - e^2sw_s + \cdots, \quad (A.6)
\]

and continuing to compare the coefficients of \( e^3 \), one can obtain

\[
Lu_3 = u_3 - \int_{-\infty}^{\infty} K_0(s, y)u_3(x - y, t - s)dyds
\]

\[
= \int_{-\infty}^{\infty} K_0(s, y)[-sz\tau e^{i\omega_0(t-s) + ik_0(x-y)}
\]

\[
- sw\tau e^{i\omega_0(t-s) - ik_0(x-y)} + c.c.]dyds
\]
We can also suppose \( u_3 = k_1z^2e^{i\omega_0(t-\tau)x} + k_2ze^{i\omega_0(\tau-x)} \). Actually, to obtain \( z_T \) and \( w_T \), we only need to consider the terms with \( e^{i\omega_0(t-\tau)x} \) and \( e^{i\omega_0\tau-x} \), which should make the left-hand side of (1) zero. Comparison of the coefficients of \( e^{i\omega_0(t-\tau)x} \) and \( e^{i\omega_0\tau-x} \) leads to the normal form of the Turing-Hopf bifurcation

\[
\begin{align*}
\dot{z}_T &= z(\alpha p_2 + \beta z\bar{z} + \gamma w\bar{w}) , \\
\dot{w}_T &= w(\alpha p_2 + \beta w\bar{w} + \gamma z\bar{z}) ,
\end{align*}
\]

where \( \alpha = \alpha_1 + i\alpha_2 = \frac{\bar{K}(\omega_0, k_0)}{M} \), \( \beta = \beta_1 + i\beta_2 = \frac{2\bar{K}(\omega_0, k_0)\alpha_2}{M} + 3\bar{K} \), \( \gamma = \gamma_1 + i\gamma_2 = \frac{2\bar{K}(\omega_0, k_0)(c_1 + c_2 + c_3) + 3\bar{K}(\omega_0, k_0)}{M} \), \( M = \int_0^\infty \int_{-\infty}^\infty K_0(s, y) e^{-i\omega_0 s - i\omega_0 y} dy ds \).