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Unsteady Peristaltic Transport of a Particle-Fluid Suspension: Application to Oesophageal Swallowing

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Abstract: This model characterises the flow behaviour of suspended particles when swallowing through oesophagus. Transport of particle-fluid mixture induced by dilating peristaltic waves on a circular cylindrical tube was considered for investigation. Unsteady closed-form solutions for pressure gradient, velocity and stream function are obtained by applying regular perturbation technique up to the first order of wavenumber (the ratio of the tube radius to the wavelength). It is observed that the axial velocity of the fluid is greater than that of the solid particles almost everywhere. However, at the wall the axial velocity of the fluid is zero due to the no-slip condition imposed on it; but the suspended particulate material has non-zero positive axial velocity. Thus, the axial velocity of the suspended particles near the tube wall is more than that of the fluid velocity. It is further observed that the axial velocity is negative in the regions close to maximum occlusions giving way to instantaneous backward flow. It is also inferred that the maximum axial velocity of the particle-fluid suspension with non-zero wavenumber is more than that of the particle-free fluid with zero wavenumber. We examined the effect of volume fraction of suspended particulate matter on pressure gradient and velocity. An increase in the volume fraction diminished the pumping performance and also the axial and radial velocities. The results are also compared with those which were obtained for zero wavenumber. The outcome of the investigation endorses the doctors’ advice to patients suffering from achalasia, oesophageal stricture and oesophageal tumors to take liquid or food items with lesser solid contents. Streamline patterns are changed by increasing the flow rate while trapping occurs at high flow rates.

Keywords: Oesophageal Diseases; Oesophagus; Particle-Fluid Suspensions; Peristalsis; Two-Phase Flow.

1 Introduction

Peristalsis is a pumping mechanism by which a fluid can be transported through a tube/channel when contraction or expansion waves propagate along the tube wall. Most of the physiological fluids in humans or animals are propelled by continuous periodic muscular contractions and expansions of the ducts through which the fluids pass. Peristaltic pumping is involved in swallowing of food bolus through the oesophagus, embryo transport in the uterus, vasomotion of blood vessels, spermatic flows in the male reproductive tracts, transport of urine through the ureter and also in some engineering applications. The principle of peristalsis is used to design blood pump for heart lung machines, diabetic pumps and roller pumps. Numerous investigations including [1–8] over several years have thrown light on peristalsis, but it is still desirable to investigate it in new perspectives.

Study of the theory of particle-fluid mixture is immensely useful for understanding a number of physical phenomena including transportation of solid particles by liquids, mixing operations, particulate suspension theory of blood, flow of food suspension through oesophagus and intestines, urine flow through the ureters, transportation of liquid slurries in chemical and nuclear processing, etc. Several industrial food processes involve flow of food suspension in which the knowledge of flow properties is essential for assessing pumping requirements. Hung and Brown [9] investigated various geometric and dynamic effects on peristaltic transport of suspended solid particles in a fluid in a two-dimensional channel. Drew [10, 11] presented a two-phase flow model that accounts for a mixture of dispersed small particles in a fluid as the working medium. Srivastava and Srivastava [12] applied Drew’s model [10] to a particle-fluid mixture flowing in a channel and obtained perturbation solution which satisfies the momentum equations for the case in which amplitude ratio is small. The flow of diseased urine modelled as
particle-fluid suspension through the ureters was subsequently studied by Misra and Pandey [13] who concluded that the mean flow induced by peristaltic motion is proportional to the square of the amplitude ratio and depends on the mean pressure gradient. Ureters are muscular ducts that propel urine from the kidneys to the bladder by peristalsis. Jimenez-Lozano et al. [14] also presented a model for peristaltic flow in ureters due to a solitary wave with the objective of explaining the flow mechanics of a particle-fluid mixture. Mekheimer and Abdelmaboud [15] theoretically analysed the peristaltic flow through a uniform and non-uniform annulus filled with particle-fluid suspension by long wavelength approximation. Popularity of Drew’s model is revealed through a series of recent publications in biomechanics involving peristalsis [16–21] and rheological flow of blood [22, 23].

The digestive system is accountable for carrying out the food bolus from the mouth to the large intestine. In the oesophageal deglutition, food is ingested through the mouth and when swallowed passes into the pharynx which forces the food bolus rapidly into the oesophagus. This process ends with the transport of the bolus to the stomach by peristaltic contractions of the oesophageal wall. There are several food stuffs which are in the form of particulate suspensions in which the continuous phase is an aqueous solution. Examples are pasta products in sauces, yogurts with fruits, fruit preserves with seeds, vegetable soups, fruit in syrup, sugarcane juice with kiwifruit and many other homemade food items [24]. Generally, some of these food items have non-Newtonian behaviour but if the volume fraction of suspended particles is small then they behave as a Newtonian fluid. Moreover, any solid edible item is first masticated with teeth to break into very small particles, then it is mixed with saliva or some liquid such as water which can drag it into the abdomen through the oesophagus without harming it. What are the implications of the dragged solid particles from medical point of view is another aspect of the investigation. Hence, study of particle-fluid suspension is extremely important in relation to oesophageal swallowing. Suspensions are defined as heterogeneous or homogeneous material, in which rigid or deformable particles are suspended in a liquid. Oesophageal swallowing of food suspension is a two-phase flow model. This model is most appropriate if the dispersed particle phase behaves as a continuum. The continuum theory of mixtures is also applicable to hydrodynamics of some biological systems.

Many solutions of peristaltic flows with various approximations are present in the literature. The authors [1–3] solved problems by using long wavelengths at low Reynolds number approximation in infinite and finite length tubes in which the wavenumber and Reynolds number tend to zero. Pandey et al. [25] analysed the variation of pressure from the cervical to the distal end of the oesophagus during swallowing with these approximations. The novelty of the investigation lies in the fact that it theoretically discovered the reason for the pressure rise in the distal part of the oesophagus reported by Kahrilas et al. [26] by using the anatomical measurements reported by Xia et al. [27]. The conclusion they drew was that the wave-amplitude increases progressively as the bolus is swallowed. With this idea Pandey and Tiwari [28] investigated oesophageal swallowing for the Casson fluid.

Achalasia is a dysfunction that causes inadequate lower sphincter relaxation of the oesophagus. As a consequence of it, oesophageal clearance is hindered. A possible treatment for patients to overcome this is by application of drugs or operation [29]. This analysis is also planned to look for finding alternative or supportive ways for a remedial measure.

In a two-phase flow like this, a particulate matter suspension, which cannot move itself, is dragged by the fluid mixed with this. As the two phases will have different velocities, there are several queries to investigate such as which one leads, which one lags behind, what is the mutual relation in the middle of the tube, whether it is different near the tubular boundary, etc.

In light of the literature presented above and the discussion that followed, we aim to model the swallowing of particulate suspension in a Newtonian fluid through oesophagus in order to investigate the impact of the presence and concentration of suspended particles in the food.

The entire analysis is in dimensionless quantities. We use a regular perturbation method in terms of the wavenumber for oesophageal swallowing in which the wavenumber is small but not zero. Flow variables are presented in power series of the wavenumber to obtain closed-form solution up to the first order. As the wavenumber is very small and the higher order equations are very much involved, no further analyses will be carried out. The velocities of the two phases will be deduced separately. Pressure equation will also be formulated. The interrelation of the two phases as well as the influence of the particle volume fraction is to be investigated.

2 Mathematical Formulation

We consider the oesophagus as a circular cylindrical tube of finite length. The geometrical form of the peristaltic wave in oesophageal wall was given by Li and Brasseur [2] and modified by Misra and Pandey [3]. The observation
of high pressure zone in the lower part of oesophagus by Kahrilas et al. [26] motivated Pandey et al. [25], in view of the experimental reports of Xia et al. [27], to reformulate the wall equation with dilating wave amplitude as

\[ H'(x', t') = a - \Psi_0 e^{k' x'} \cos^2 \frac{\pi}{\lambda}(x' - ct'), \]

where \( H', x', t', a, \Psi_0, k', \lambda \) and \( c \) are the radial distance of wall, axial coordinate, time, radius of the tube, amplitude of the wave, wave velocity, respectively (Fig. 1).

The governing equations are taken in the form of two sets of continuity and momentum equations, one each meant for the fluid phase and the particulate phase based on the two-phase model of Drew [10]. The governing equations for the fluid and particle phases in the axisymmetric cylindrical coordinates are as follows:

**Fluid phase:**

\[
\frac{1}{r} \frac{\partial r \left((1 - C) r' v'_f \right)}{\partial r'} + \frac{\partial (1 - C) u'_f}{\partial x'} = 0, \tag{2}
\]

\[
(1 - C) \rho_f \left( \frac{\partial v'_f}{\partial r'} + v'_f \frac{\partial v'_f}{\partial r'} + u'_f \frac{\partial v'_f}{\partial x'} \right)
= -(1 - C) \frac{\partial p'}{\partial r'} + (1 - C) \mu_s(C)
\left\{ \frac{\partial}{\partial r'} \left( \frac{1}{r} \frac{\partial r (r' v'_f)}{\partial r'} \right) + \frac{\partial^2 v'_f}{\partial x'^2} \right\}
+ CS(v'_p - v'_f),
\]

\[
(1 - C) \rho_f \left( \frac{\partial u'_f}{\partial r'} + v'_f \frac{\partial u'_f}{\partial r'} + u'_f \frac{\partial u'_f}{\partial x'} \right)
= -(1 - C) \frac{\partial p'}{\partial x'} + (1 - C) \mu_s(C)
\left\{ \frac{1}{r} \frac{\partial}{\partial r'} \left( r' \frac{\partial u'_f}{\partial r'} \right) + \frac{\partial^2 u'_f}{\partial x'^2} \right\}
+ CS(u'_p - u'_f), \tag{3}
\]

**Particle phase:**

\[
\frac{1}{r} \frac{\partial (C r' v'_p)}{\partial r'} + \frac{\partial (C u'_p)}{\partial x'} = 0, \tag{5}
\]

\[
C \rho_p \left( \frac{\partial v'_p}{\partial r'} + v'_p \frac{\partial v'_p}{\partial r'} + u'_p \frac{\partial v'_p}{\partial x'} \right)
= -C \frac{\partial p'}{\partial r'} + CS(v'_f - v'_p), \tag{6}
\]

\[
C \rho_p \left( \frac{\partial u'_p}{\partial r'} + v'_p \frac{\partial u'_p}{\partial r'} + u'_p \frac{\partial u'_p}{\partial x'} \right)
= -C \frac{\partial p'}{\partial x'} + CS(u'_f - u'_p), \tag{7}
\]

where \( u'_f, v'_f, u'_p, v'_p, \rho_f, \rho_p, C, (1 - C) \rho_f, C \rho_p, p, S \) and \( \mu_s(C) \), respectively, represent axial velocity of the fluid phase, radial velocity of the fluid phase, actual density of the fluid, actual density of the particle material, volume fraction of the particles in the mixture, fluid phase density, particle phase density, pressure, drag coefficient of interaction for the force exerted by one phase on the other and the effective viscosity of suspension. For the present problem, the Stokes drag coefficient for a small particle at low Reynolds number, \( S = 9 \mu_0/4 \rho_p \), and the Einstein's formula, \( \mu_s = \mu_0 \mu_r \), shall be used, where \( \mu_0 \) is the fluid viscosity, \( \rho_p \) is the particle radius and \( \mu_r(C) = 1 + 5C/2 \) [10].

The following dimensionless parameters are now introduced into the analysis:

\[
\begin{align*}
&x = \frac{x'}{\lambda}, \quad r = \frac{r'}{a}, \quad t = \frac{ct'}{\lambda}, \\
&h = \frac{H'}{a}, \quad k = k' \lambda, \quad u_f = \frac{u'_f}{c}, \\
&u_p = \frac{u'_p}{c}, \quad v_f = \frac{v'_f}{c} \delta, \quad \delta = \frac{a}{\lambda}, \\
&v_p = \frac{v'_p}{c} \delta, \quad Q = \frac{Q'}{\pi a^2 c}, \quad \rho = \frac{\rho_p}{\rho_f}, \\
&\theta = \frac{\psi}{a}, \quad p = \frac{p' a^2}{\mu_s c}, \quad Re_0 = \frac{\rho_c a c}{\mu_0}, \\
&Re = \delta Re_0, \quad M = g \left( \frac{a}{r_p} \right)^2 \end{align*}
\]

where \( \delta, Re_0, Re \) and \( M \) are respectively, the wavenumber, the Reynolds number, the modified Reynolds number and the drag parameter. Introduced dimensionless quantities
reduce the wall equation (1) and governing equations (2)–(7) to
\[
h(x, t) = 1 - \delta e^{kx} \cos^2 \pi (x - t).
\]
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( (1 - C) r v_f \right) + \frac{\partial}{\partial x} (1 - C) u_f = 0, \tag{10}
\]
\[
\delta^3 (1 - C) \Re_0 \left( \frac{\partial v_f}{\partial t} + v_f \frac{\partial v_f}{\partial r} + u_f \frac{\partial v_f}{\partial x} \right) = -\mu_f (1 - C) \frac{\partial p}{\partial r} + \mu_f (1 - C) \frac{\partial p}{\partial x} + C (u_f - v_f), \tag{11}
\]
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \rho \delta C v_p \right) + \frac{\partial}{\partial x} (\rho \delta C u_p) = 0, \tag{13}
\]
\[
\rho \delta^3 \Re_0 \left( \frac{\partial v_p}{\partial t} + v_p \frac{\partial v_p}{\partial r} + u_p \frac{\partial v_p}{\partial x} \right) = -\mu_c \rho \frac{\partial p}{\partial r} + \delta^2 C (v_f - v_p), \tag{14}
\]
\[
\rho \delta^3 \Re_0 \left( \frac{\partial u_p}{\partial t} + v_p \frac{\partial u_p}{\partial r} + u_p \frac{\partial u_p}{\partial x} \right) = -\mu_c \rho \frac{\partial p}{\partial x} + C (u_f - u_p). \tag{15}
\]

Boundary conditions are the essential requirements for obtaining solution of a system of differential equations. However, in a practical problem such as one undertaken here, physics of fluid and solid particles have to be properly taken into consideration. For instance, no solid particle can stick to a solid boundary else the definition of rigidity will be violated. In fact, it is dragged by the fluid which can stick to the boundary. Hence, we cannot impose no-slip condition on solid particles at the boundary of the tubular wall. The dimensionless boundary conditions, to be imposed on the fluid molecules and the solid particles for the sake of solution may be put as follows:
\[
\left. \begin{array}{l}
    u_f |_{r=h} = 0, \\
    \left. \frac{\partial u_f}{\partial r} \right|_{r=0} = 0, \\
    v_f |_{r=0} = 0, \\
    \left. \frac{\partial v_f}{\partial r} \right|_{r=h} = \frac{\partial h}{\partial t}, \\
    \frac{\partial u_p}{\partial r} |_{r=0} = 0, \\
    v_p |_{r=0} = 0.
\end{array} \right\} \tag{16}
\]

### 3 Perturbation Solution

To solve the problem, a regular perturbation expansion in terms of wavenumber $\delta (\ll 1)$, is used and assumed that particle volume fraction, $C$, is low and is of the form $C = \delta C^{(1)}$. We consider the solutions for the fluid and particle velocities and the pressure of the form
\[
\left. \begin{array}{l}
    u_f(r, x, t) = u_f^{(0)} + \delta u_f^{(1)} + O(\delta^2), \\
    v_f(r, x, t) = v_f^{(0)} + \delta v_f^{(1)} + O(\delta^2), \\
    u_p(r, x, t) = u_p^{(0)} + \delta u_p^{(1)} + O(\delta^2), \\
    v_p(r, x, t) = v_p^{(0)} + \delta v_p^{(1)} + O(\delta^2), \\
    p(r, x, t) = p^{(0)} + \delta p^{(1)} + O(\delta^2).
\end{array} \right\} \tag{17-21}
\]

Substituting these expansions in (10)–(15), and comparing the coefficients of like powers of $\delta$, we get a set of linear differential equations as given below.

The zeroth-order system, i.e. coefficients of $\delta^0$ equated on the two sides, is
\[
\left. \begin{array}{l}
    \frac{1}{r} \frac{\partial}{\partial r} \left( r v_f^{(0)} \right) + \frac{\partial}{\partial x} (v_f^{(0)}) = 0, \\
    \frac{\partial}{\partial r} \left( r u_p^{(0)} \right) + \frac{\partial}{\partial x} (u_p^{(0)}) = 0, \\
    \frac{\partial p^{(0)}}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_f^{(0)}}{\partial r} \right),
\end{array} \right\} \tag{22-24}
\]
subject to the boundary conditions
\[
\left. \begin{array}{l}
    \frac{\partial u_f^{(0)}}{\partial r} |_{r=0} = 0, \\
    u_f^{(0)} |_{r=h} = 0, \\
    v_f^{(0)} |_{r=0} = 0, \\
    v_f^{(0)} |_{r=h} = \frac{\partial h}{\partial t}.
\end{array} \right\} \tag{25}
\]
The first-order system, i.e. coefficients of δ equated on the two sides, is

\[ \frac{1}{r} \frac{d}{dr} \left( rv^{(1)} \right) + \frac{du^{(1)}}{dx} = 0, \]  

\[ \frac{dp^{(1)}}{dr} = 0, \]  

\[ \text{Re}_0 \left( \frac{dv^{(0)}}{dt} + v^{(0)} \frac{du^{(0)}}{dx} + u^{(0)} \frac{dp^{(0)}}{dx} \right) \]  

\[ = - \frac{\partial p^{(1)}}{\partial x} - \frac{3C^{(1)}}{2} \frac{\partial p^{(0)}}{\partial x} + \frac{3C^{(1)}}{2} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u^{(0)}}{\partial r} \right) \]

\[ + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{du^{(1)}}{dr} \right) + MC^{(1)} (u^{(0)} - u^{(0)}), \]  

\[ C^{(1)} \left( \frac{1}{r} \frac{dv^{(0)}}{dr} + \frac{du^{(0)}}{dx} \right) = 0, \]  

\[ C^{(1)} \frac{dp^{(0)}}{dr} = 0, \]  

\[ C^{(1)} \frac{dp^{(0)}}{dx} = MC^{(1)} (u^{(0)} - u^{(0)}), \]  

subject to the boundary conditions

\[ v^{(0)} \bigg|_{r=h} = 0, \quad \left. \frac{du^{(1)}}{dr} \right|_{r=0} = 0, \quad \left. u^{(1)} \right|_{r=h} = 0, \]  

\[ \left. \frac{du^{(0)}}{dr} \right|_{r=0} = 0, \quad \left. v^{(1)} \right|_{r=0} = 0. \]  

We need to express the analytical results in terms of time-averaged volume flow rate which is defined as \( \overline{Q}(x) = \int_0^1 Q(x, t) dt \), with the volume flow rate as a sum of that for the two phases, i.e.

\[ Q(x, t) = Q_f(x, t) + Q_p(x, t), \]  

where \( Q_f(x, t) = 2 \int_0^h (1 - C) u_f dr \) and \( Q_p(x, t) = 2 \int_0^h C u_p dr \) are the instantaneous volume flow rates for the fluid and particle phases, respectively. It is a tedious job due to several involved expressions. Hence in order to avoid complications, we use the transformations from the unsteady laboratory frame to steady wave frame for this purpose only. All the other analyses will be later, from Section 3.1 onwards, carried out once again in the unsteady laboratory frame.

The wave frame parameters, given on the left side of the equality sign, are related to the corresponding parameters in the laboratory frame, given on the right side, in the non-dimensional form by

\[ X = x - t, \quad R = r, \quad U_i(R, X) = u_i(r, x, t) - 1, \]

\[ V_i(R, X) = v_i(x, r, t), \quad q = Q(x, t) - h^2, \]  

where \( (R, X), (V_i, U_i) \) and \( q \) are, respectively, the coordinate system, the velocity field \( i = f, p \) and the flow rate in the wave frame.

In view of (34), \( \overline{Q}(x) = q + \int_0^1 h^2 dt \), and hence

\[ q = Q(x, t) - h^2 = \overline{Q}(x) - 1 + \delta e^{kx} - \frac{3}{8} \delta^2 e^{2kx}. \]  

The regular perturbation expansions for \( Q \) and \( \overline{Q} \) are, respectively, \( Q = Q^{(0)} + \delta Q^{(1)} + O(\delta^2) \) and \( \overline{Q} = \overline{Q}^{(0)} + \delta \overline{Q}^{(1)} + O(\delta^2) \).

### 3.1 Solution of the Zeroth-Order System

Integrating (24) with respect to \( r \), in view of (23), and using the first boundary condition of (25), we get

\[ \frac{\partial u^{(0)}}{\partial r} = \frac{\partial p^{(0)}}{\partial x} r \frac{\partial}{\partial x}, \]

which, on integrating once more with respect to \( r \) and using the second boundary condition of (25), gives

\[ u^{(0)} = \frac{1}{4} \frac{\partial p^{(0)}}{\partial x} (r^2 - h^2). \]  

Solving the continuity equation (22) together with (36) and using the third boundary condition of (25), it yields

\[ v^{(0)} = \frac{r}{4} \left( \frac{h \frac{\partial h}{\partial x} \frac{\partial p^{(0)}}{\partial x} - \frac{\partial^2 p^{(0)}}{\partial x^2} \left( \frac{r^2}{4} - \frac{h^2}{2} \right)}{h^2} \right). \]  

Now applying the fourth boundary condition of (25) in (37) and simplifying, we get

\[ \frac{h^3}{16} \frac{\partial^2 p^{(0)}}{\partial x^2} + \frac{h^2}{4} \frac{\partial h}{\partial x} \frac{\partial p^{(0)}}{\partial x} = \frac{\partial h}{\partial t}, \]

from which, the zeroth-order pressure gradient is derived as

\[ \frac{\partial p^{(0)}}{\partial x} = g(t) + \frac{16}{h^4} \int_0^x h(s, t) \frac{\partial h(s, t)}{\partial t} ds, \]  

where \( g(t) \) is an arbitrary function of \( t \).
Therefore, the zeroth-order pressure at an arbitrary point along the length of the oesophagus is given by

\[ p^{(0)}(x, t) = p^{(0)}(0, t) + \int_0^x g(t) + 16 \int_0^{x_1} h(s, t) \frac{\partial h(s, t)}{\partial x} ds \, dx_1. \]  \hspace{1cm} (39)

The arbitrary function \( g(t) \) may be evaluated by putting \( x = l \) in (39) as

\[ g(t) = \left\{ \begin{array}{l}
p^{(0)}(l, t) - p^{(0)}(0, t) \\
+ 16 \int_0^l h(s, t) \frac{\partial h(s, t)}{\partial x} ds x_1 \end{array} \right\} \]

Further, the zeroth-order flow rate, in view of (33), may be given by

\[ Q^{(0)} = Q_f^{(0)} + Q_p^{(0)} = 2 \int_0^h u_f^{(0)} r dr + 0 = -\frac{1}{8} \frac{\partial p^{(0)}}{\partial x} h^4. \]

Note that, in view of (35), we have \( Q^{(0)} = Q^{(0)} - 1 + \frac{\partial}{\partial x} e^{kx} - \frac{3}{2} \frac{\partial}{\partial x} e^{2kx} + h^2 \). Therefore,

\[ \frac{\partial p^{(0)}}{\partial x} = -8 \left\{ Q^{(0)} - 1 + \frac{\partial}{\partial x} e^{kx} - \frac{3}{2} \frac{\partial}{\partial x} e^{2kx} + h^2 \right\} \]

\[ = P_0 \text{ (say)}. \]  \hspace{1cm} (40)

Hence, from (36), (37) and (41), the zeroth-order axial and radial velocities of the fluid, in terms of zeroth-order time-averaged flow rate, are given by

\[ u_f^{(0)} = \frac{P_0}{q} \left( r^2 - h^2 \right), \]  \hspace{1cm} (42)

\[ v_f^{(0)} = \frac{r}{q} \left\{ \frac{h}{\frac{\partial}{\partial x}} P_0 - \frac{\partial P_0}{\partial x} \left( r^2 - h^2 \right) \right\}. \]  \hspace{1cm} (43)

### 3.2 Solution of the First-Order System

From (31) and (42), the zeroth-order axial velocity of the solid particles is given by

\[ u_p^{(0)} = \frac{P_0}{q} \left( r^2 - h^2 - \frac{q}{M} \right). \]  \hspace{1cm} (44)

Integrating continuity equation (29) together with (44) with respect to \( r \) and using the first boundary condition of (32), the zeroth-order radial velocity of the particle is obtained as

\[ v_p^{(0)} = \frac{r}{q} \left\{ P_0 \frac{\partial h}{\partial x} - \frac{\partial P_0}{\partial x} \left( r^2 - h^2 - \frac{2}{M} \right) \right\}. \]  \hspace{1cm} (45)

Equations (26)–(28) of the first-order system are solved similarly as described in the solution of zeroth-order system.

The first-order solutions for the axial and radial velocities of the fluid (the details given in the Appendix A) are given by

\[ u_f^{(1)} = N_1 \left( r^6 - h^6 \right) + N_2 \left( r^8 - h^8 \right) \]

\[ + \left( N_3 + \frac{P_{1h} + C_{1h} P_0}{4} \right) \left( r^6 - 3r^4 h^2 \right) \]

\[ + \frac{1}{4} \left( \frac{\partial N_1}{\partial x} + \frac{1}{4} \frac{\partial P_1}{\partial x} + \frac{C_{1h}}{4} \frac{\partial P_0}{\partial x} \right) \left( r^4 - 2r^2 h^2 \right) \]

\[ + \frac{1}{4} \left( 12N_1 \frac{\partial h}{\partial x} + 8N_3 \frac{\partial h}{\partial x} + 3N_4 \frac{\partial h}{\partial x} + P_{1h} P_0 h C_{1h} \right) \frac{\partial h}{\partial x} \]  \hspace{1cm} (46)

The expressions for \( P_1, N_1, N_2 \) and \( N_3 \) are given by

\[ P_1 = -6N_1 h^6 + \frac{16}{3} N_3 h^2 + 4N_3 \]

\[ - \frac{P_0 C_{1h}}{h^2} \left( \frac{8}{M} + \frac{h^2}{2} \right) + \frac{8Q^{(1)}}{h^2}. \]  \hspace{1cm} (48)

where

\[ N_1 = \frac{Re_0}{1152} P_0 \frac{\partial P_0}{\partial x}, \]  \hspace{1cm} (49)

\[ N_2 = \frac{Re_0}{16} \left( \frac{1}{3} \frac{\partial P_0}{\partial t} - \frac{3P_0 \frac{\partial P_0}{\partial x} h^2}{8} \right). \]  \hspace{1cm} (50)

\[ N_3 = \frac{Re_0}{12} \left( \frac{h^2}{2} \frac{\partial P_0}{\partial t} - \frac{P_0 h \frac{\partial h}{\partial x}}{8} + \frac{3P_0 \frac{\partial P_0}{\partial x} h^4}{32} \right), \]  \hspace{1cm} (51)

and \( Q^{(1)} = Q^{(1)} - \frac{2}{3} N_1 h^8 - \frac{2}{3} N_2 h^6 - \left( \frac{N_1}{2} + \frac{1}{8} \frac{\partial P_1}{\partial x} + \frac{P_{1h} C_{1h}}{8} \right) h^4 - \frac{P_0 C_{1h}}{M} h^2 \) (see Appendix B).
The zeroth- and first-order solutions of those given in (17)–(20), together constitute the required results for the fluid and particle velocities. Therefore, the axial and radial velocities of the fluid in the fixed frame are

\[ u_f = \frac{P_0}{4} \left( r^2 - h^2 \right) + \delta \left\{ N_1 \left( r^6 - h^6 \right) + N_2 \left( r^4 - h^4 \right) \right\} + \left( N_3 + \frac{P_1 + C^{(1)}P_0}{4} \right) \left( r^2 - h^2 \right) \right\} . \]  

Using the transformation defined in (34), the stream function, \( \psi_f(r, x, t) \), in the fixed frame may be obtained by the solution of the differential equation, \( d\psi_f = 2r(u_f - 1)dr - 2rv_fdx \). This is an exact differential equation; therefore stream function may be obtained by evaluating

\[ 2 \left\{ \frac{P_0}{4} \left( r^2 - h^2 \right) + \delta \left\{ N_1 \left( r^6 - h^6 \right) + N_2 \left( r^4 - h^4 \right) \right\} + \left( N_3 + \frac{P_1 + C^{(1)}P_0}{4} \right) \left( r^2 - h^2 \right) \right\} - 1 \right\} dr. \]

Thus we have

\[ \psi_f(r, x, t) = r^2 \left\{ \frac{P_0}{8} \left( r^2 - 2h^2 \right) + \delta \left\{ \frac{N_1}{4} \left( r^6 - 4h^6 \right) \right\} + \frac{N_2}{3} \left( r^4 - 3h^4 \right) \right\} + \frac{1}{8} \left( 4N_3 + P_1 + C^{(1)}P_0 \right) \left( r^2 - 2h^2 \right) - 1 \right\}. \]  

For wavenumber, \( \delta \rightarrow 0 \), (52)–(55) reduce to the corresponding equations derived by Shapiro et al. [1].

### 3.3 Pressure Gradient

The perturbation expansion for pressure gradient is

\[ \frac{\partial p}{\partial x} = \frac{\partial p^{(0)}}{\partial x} + \delta \frac{\partial p^{(1)}}{\partial x} + O(\delta^2). \]

Therefore, the solution for pressure gradient in terms of time-averaged volume flow rate, in view of (41) and (48), yields

\[ \frac{\partial p}{\partial x} = -8 \left\{ \left( \frac{Q^{(0)}}{h^4} - 1 + \frac{\Theta e^{kx}}{h^4} - \frac{3}{5} \frac{\Theta^2 e^{2kx}}{h^4} + \frac{h^2}{h^4} \right) \right\} - 8 \left\{ \frac{Q^{(0)}}{h^4} - 1 + \frac{\Theta e^{kx}}{h^4} - \frac{3}{5} \frac{\Theta^2 e^{2kx}}{h^4} + \frac{h^2}{h^4} \right\} \right\}. \]  

### 3.4 Stream Function

The flow patterns of fluid are also given by counters of the constant stream function, \( \Psi_f \), in the moving frame defined as

\[ d\Psi_f = 2RU_f dR - 2RV_f dX. \]  

In order to have an estimate of applicability of the analytical work presented here, we consider the length and the diameter of the oesophagus as 25–30 cm [30] and 1.8–2.1 cm [31] respectively. Suspended particles have a uniform radii of 0.04 cm. We also consider that oesophagus can contain two boluses at a time while swallowing.

With these dimensions of the oesophagus, the following parameters are evaluated as \( \delta = 0.06 \) and \( M = 1139 \). The analytical results are expressed in the fixed frame up to the first order of the time-averaged volume flow rate \( Q = Q^{(0)} + \delta Q^{(1)} \). Then computer codes are developed by substituting \( Q^{(0)} = Q - \delta Q^{(1)} \) in the solutions (52)–(55) for numerical evaluations of the analytical results. Diagrams are drawn for the pressure gradient, the axial and radial velocities and the streamlines of the flow shown in Figures 2–8 with various parameters as assumed below.

The axial velocity of the fluid together with that of the solid particles along the axis of the oesophagus is presented in Figure 2 at the fixed radial distance \( r = 0.3 \) and the temporal values (a) \( t = 0.0 \) (b) \( t = 0.4 \). This figure is based on (44) and (52). Other parameters are taken as \( \delta = 0.06, k = 0.02, C = 0.12, \Theta = 0.7, Re_0 = 5, \Theta = 1.5, Q^{(1)} = 15, M = 1139 \). At \( t = 0.0 \) (Fig. 2a), we
observe that the axial velocity of the fluid is greater than that of the solid particles almost everywhere. But analysing (44) and (52), we infer that at the wall the axial velocity of the fluid is zero but the suspended particulate material has non-zero positive axial velocity. This means that the axial velocity of the suspended particles near the tube wall is more than that of the fluid velocity. It has been illustrated in Figure 3 by plotting graphs of the axial velocities close to the tube wall. In Figure 2b, we also observe that the axial velocity is negative in the regions close to
maximum occlusions giving way to instantaneous backward flow. Backward flow is found in a small region with maximum occlusion. Therefore, the net flow will be positive. Further, it is observed that the magnitude of the velocity at the second occlusion point is more than at the first occlusion point which is due to dilating wave amplitude.

The impact of dilating wave amplitude on the axial velocity is depicted in Figure 4 in which the various parameters are taken as $k = 0.0, 0.05, 0.1$, $x = 0.3$, $t = 0.9$, $\delta = 0.06$, $\mathcal{C} = 0.12$, $\theta = 0.7$, $Re_0 = 5$, $\mathcal{Q} = 1.5$, $\mathcal{Q}^{(1)} = 20$, $M = 1139$. Solid, dashed and the solid line with marker correspond, respectively, to $k = 0.0$, $k = 0.05$ and $k = 0.1$.

Figure 4: Radial profiles of the axial velocity, respectively, of the (a) fluid and (b) solid particles versus the radial distance at the fixed axial position $x = 0.3$ and time $t = 0.9$ showing the impact of amplitude dilation parameter $k$. Other parameters are taken as $\delta = 0.06$, $\mathcal{C} = 0.12$, $\theta = 0.7$, $Re_0 = 5$, $\mathcal{Q} = 1.5$, $\mathcal{Q}^{(1)} = 20$, $M = 1139$. Solid, dashed and the solid line with marker correspond, respectively, to $k = 0.0$, $k = 0.05$ and $k = 0.1$.

It is observed that greater the dilation parameter, the higher is the axial velocity at the fixed axial positions for both the fluid and particles velocities.

Figure 5a illustrates the impact of the particulate suspension on the axial velocity of the fluid along the radius at a specific axial location $x = 0.3$ and at an instant $t = 0.9$ comprising three distinct curves for different volume fractions $\mathcal{C} = 0.0, 0.12, 0.24$. We take other parameters as $k = 0.02$, $\theta = 0.8$, $Re_0 = 5$, $\mathcal{Q}^{(1)} = 20$. Examining the behaviour of the present figure, we observe that the axial velocity of the fluid decreases with increasing volume fraction. The curves are all analogous in the sense that they decrease from their individual maximum at the axis to the minimum near the wall surface as expected. We draw another graph for the axial fluid velocity with the same parameters as in Figure 5a except for $\delta = 0$, i.e. particle-free fluid which is shown in Figure 5b. Comparing Figure 5a and 5b, we find that the maximum axial velocity of particle-fluid suspension with non-zero wavenumber is more than that of particle-free fluid with zero wavenumber.

The characteristics of the radial velocity of the fluid varying radially at the fixed axial position $x = 0.3$ and different volume fractions $(\mathcal{C} = 0.0, 0.12, 0.24)$ are exhibited in Figure 6. Analysing all the curves of the depicting figure, we see that all the curves diminish from zero on the axis as these move away from it and finally move towards the wall to attain some finite value.
Figure 5: The impact of $\delta$ on the relation between the axial velocity of the fluid and the radial distance for different volume fractions at the fixed axial position $x = 0.3$ for $k = 0.02$, $\theta = 0.8$, $Re_0 = 5$, $\overline{Q} = 1.5$, $\overline{Q}^{(1)} = 20$, $t = 0.9$, $M = 1139$, (a) $\delta = 0.06$, (b) $\delta = 0.0$. Solid line, dashed line and the centre lines, respectively, correspond to $C = 0.0$, $C = 0.12$ and $C = 0.24$.

Figure 6: Radial velocity profile of the fluid along the radial distance for different volume fraction of particles at $x = 0.2$ for $t = 0.3$, $k = 0.02$, $\theta = 0.6$, $Re_0 = 5$, $\delta = 0.06$, $\overline{Q} = 1.5$, $\overline{Q}^{(1)} = 20$, $M = 1139$. Solid line, dashed line and the solid line with marker correspond, respectively, to $C = 0.0$, $C = 0.12$ and $C = 0.24$.

on the wall surface. These natures of curves reflect the presence of wall motion in the transverse direction. It is interesting to see that all the curves become concave near the wall which means that the radial velocity changes its nature near the wall. A rise in the volume fraction of suspended particles diminishes the magnitude of the radial velocity of the fluid.

Figure 7a displays the pressure gradient of the fluid-particle mixture versus the time-averaged volume flow rate $\overline{Q}$ for different volume fractions ($C = 0.0$, 0.12, 0.24) at
the fixed axial position and the fixed time. We randomly choose the axial position $x = 0.8$ and the time $t = 0.4$.

For the qualitative interpretation of the analytical result we take other parameters as $k = 0.02$, $\emptyset = 0.6$, $Re_0 = 5$, $Q^{(1)} = 15$. A close observation reveals that as the volume fraction increases from 0.0 to 0.24, the pressure gradient too declines. This means that the dispersed small particles in the fluid medium affect the pumping characteristics. It is also observed that below some fixed flow rate, the pressure gradient is adverse, i.e. $\frac{\partial p}{\partial x} > 0$; but above that the pressure gradient is favourable, i.e. $\frac{\partial p}{\partial x} < 0$. It is inferred that a positive pressure gradient hinders the flow while a negative one enhances it. In Figure 7b, $\frac{\partial p}{\partial x}$ vs. $\overline{Q}$ is also plotted under the same condition as in Figure 7a but at the wavenumber, $\delta = 0$. Comparing Figure 7b with 7a, we note that the pressure gradient varies linearly with the volume flow rate in Figure 7b but not in Figure 7a. The reason behind this is that the expressions defined by $N_1$, $N_2$ and $N_3$ contain quadratic terms in $\overline{Q}$ but when $\delta = 0$, these terms are absent in the pressure gradient and only the linear term of $\overline{Q}$ is left.

There are several diseases like achalasia, oesophageal stricture and oesophageal tumors in which swallowing is very difficult. Pressure gradient profile suggests that patients suffering from these diseases may be advised to consume food items with less particulate suspensions. Larger pressure gradient for low volume fraction is advised for comfortable swallowing.

The fluid motion is also described with the help of streamlines and stream functions. A streamline is an imaginary curve in the flow field of the fluid such that the tangent at each of the points of the curve gives the direction of the local velocity at that point at an instant. Streamlines in the fixed frame with $k = 0.02$, $\emptyset = 0.6$, $Re_0 = 5$, $Q^{(1)} = 15$, $x = 0.8$, $t = 0.4$, (a) $\delta = 0.06$, (b) $\delta = 0.0$. Solid line, dashed line and the solid line with marker correspond to $C = 0.0$, $C = 0.12$ and $C = 0.24$, respectively.

Figure 7: The effect of $\delta$ on the relation between the pressure gradient and the time-averaged volume flow rate for different volume fraction of particles with $k = 0.02$, $\emptyset = 0.6$, $Re_0 = 5$, $Q^{(1)} = 15$, $M = 1139$, $x = 0.8$, $t = 0.4$. (a) $\delta = 0.06$, (b) $\delta = 0.0$. Solid line, dashed line and the solid line with marker correspond to $C = 0.0$, $C = 0.12$ and $C = 0.24$, respectively.
\( \delta = 0.06, \overline{Q}^{(1)} = 20, M = 1139, t = 1.0 \) at different time-averaged volume flow rates (\( \overline{Q} = 1.2, 3.2, 5.0, 5.1 \)) are shown in Figure 8. A close look of this figure reveals that the fluid streamlines are generally similar to the shape of the wall for small \( \overline{Q} \) (Figure 8a). Another observation is that when \( \overline{Q} \) is increased, the streamlines change its shape and above a certain \( \overline{Q} \), the central streamline splits to engulf a ring-shaped bolus of the fluid as a closed

\[ \text{Figure 8: Streamlines in the fixed frame with } k = 0.02, C = 0.12, \theta = 0.6, \text{Re}_0 = 5, \delta = 0.06, \overline{Q}^{(1)} = 20, M = 1139, t = 1.0 \text{ at different time-averaged volume flow rates (a) } \overline{Q} = 1.2, \text{ (b) } \overline{Q} = 3.2, \text{ (c) } \overline{Q} = 5.0 \text{ and (d) } \overline{Q} = 5.1. \]
streamline as depicted in Figure 8b–d. This trapped bolus is now pushed ahead along with the peristaltic wave. This leads us to infer that trapping takes place at high flow rates. This phenomenon, termed as trapping, was first discovered by Shapiro et al. [1].

5 Conclusions

Peristaltic transport of particle-fluid suspension through oesophagus is investigated theoretically by regular perturbation technique. The impact of volume fraction of particles on the pressure gradient and the velocity is examined and streamline patterns are obtained. The presence of particles affects the pumping performance and velocity.

It is observed that the axial velocity of the fluid is greater than that of the solid particles almost everywhere. However, at the wall the axial velocity of the fluid is zero due to the no-slip condition imposed on it; but the suspended particulate material has non-zero positive axial velocity. Thus, the axial velocity of the suspended particles near the tube wall is more than that of the fluid velocity. It is further observed that the axial velocity is negative in the regions close to maximum occlusions giving way to instantaneous backward flow. Backward flow is created in a small region with maximum occlusion. Hence, the net flow will be positive. Further, the magnitude of the velocity at the second occlusion point is more than that of the fluid velocity.

An increment in the volume fraction of the suspended particles diminishes the pressure gradient and hence also the axial and radial velocities. The research endorses the advice of the doctors to the patients suffering from achalasia, oesophageal stricture and oesophageal tumors to consume liquid or food items with lesser solid contents.

Streamline patterns are changed by increasing flow rate while trapping occurs at high flow rates.

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Appendix A

Using (42) and (44) in (28), we have

\[
\begin{align*}
\text{Re}_0 \left[ \frac{1}{4} \frac{\partial P_0}{\partial t} \left( r^2 - h^2 \right) - \frac{h P_0}{2} \frac{\partial h}{\partial t} \right] & \\
+ \left\{ \frac{\text{rh} P_0}{4} \frac{\partial h}{\partial x} - \frac{r}{16} \frac{\partial P_0}{\partial x} \left( r^2 - 2 h^2 \right) \right\} \frac{P_0}{2} & \\
+ \frac{P_0}{4} \left( r^2 - h^2 \right) \left\{ \frac{1}{4} \frac{\partial P_0}{\partial x} \left( r^2 - h^2 \right) - \frac{h P_0}{2} \frac{\partial h}{\partial x} \right\} & \\
= - \frac{\partial p^{(1)}}{\partial x} + \frac{1}{r} \frac{\partial \left( \frac{\partial u_i^{(1)}}{\partial r} \right)}{\partial r} & \\
+ MC \left\{ \frac{P_0}{4} \left( r^2 - h^2 - \frac{4}{M} \right) - \frac{P_0}{4} \left( r^2 - h^2 \right) \right\}.
\end{align*}
\]  
(A1)
Integrating (A1) with respect to \( r \) and using the second boundary condition of (32), we get

\[
\frac{du_f^{(1)}}{dr} = \frac{r}{2} \frac{dp_f^{(1)}}{dx} + \frac{rP_0C^{(1)}}{2} \left( 1 + \frac{r}{16} \frac{dp}{dt} \right) \left( r^2 - 2h^2 r \right) - \frac{rhP_0}{4} \frac{dh}{dt} \]

\[+ \frac{r^2 P_0^2}{16} \frac{dh}{dx} + \frac{P_0}{384} \frac{dp}{dx} \left( 2r^5 - 6h^3 r^3 + 12r h^4 \right) \]

(A2)

Integrating (A2) with respect to \( r \) and using the third boundary condition of (32), we obtain \( u_f^{(1)} \) given in (46). Further, using (46) in (26) and integrating it with respect to \( r \) under the fifth boundary condition of (32), we get \( v_f^{(1)} \) given by (47).

**Appendix B**

In view of (33), the first-order volume flow rate in the fixed frame is \( Q^{(1)} = Q_f^{(1)} + Q_p^{(1)} \), where \( Q_f^{(1)} = 2 \int_0^h u_f^{(1)} rd r - 2C^{(1)} \int_0^h u_f^{(0)} rd r \) and \( Q_p^{(1)} = 2C^{(1)} \int_0^h u_p^{(0)} rd r \). Therefore, using (42), (44) and (46), we derive the first-order volume flow rate as

\[
Q^{(1)} = -\frac{3}{4} N_1 h^8 - \frac{2}{3} N_2 h^6 \]

\[+ \left( \frac{N_3}{2} + \frac{1}{8} \frac{dp^{(1)}}{dx} + \frac{P_0C^{(1)}}{8} \right) h^6 - \frac{P_0C^{(1)}}{M} h^2. \]

(B1)

In view of (35), it is also noted that the first-order time averaged volume flow rate, \( \bar{Q}^{(1)} = Q^{(1)} \).

**References**