Generalized Electromagnetic Fields Associated with the Hydrogen-Like Atom Problem

Abstract: It is known that the principle of minimal coupling in quantum mechanics determines a unique interaction form for a charged particle. By properly redefining the canonical commutation relation between (canonical) conjugate components of position and momentum of the particle, e.g. an electron, we restate the Dirac equation for the hydrogen-like atom problem incorporating a generalized minimal electromagnetic coupling. The corresponding interaction keeps the 1/|q| dependence in both the scalar potential \( V(|q|) \) and the vector potential \( A(q) \) (\( A(q) \sim 1/|q| \)). This problem turns out to be exactly solvable; moreover, the eigenstates and eigenvalues can be obtained in an elementary fashion. Some feasible models within this approach are discussed. Then we make a few remarks about the breaking of supersymmetry. Finally, we briefly comment on the possible Lie algebra (dynamical symmetry algebra) of these relativistic quantum systems.

Keywords: Canonical Quantum Variables; Dirac Equation; Hydrogen-Like Atom Problem.

1 Introduction

This article deals with a restatement of the Dirac equation in order to study certain hydrogen-like atom (H-LA) models in relativistic quantum mechanics. The H-LA is a quantum-mechanical system consisting of a nucleus with charge \( Z \) and one, say, electron of mass \( m_0 \) with charge \( e \) (\( e < 0 \)), which interact in accordance with Coulomb’s law. In the special case of \( Z = 1 \), when the nucleus is a proton, the H-LA is an ordinary hydrogen atom. Muonic-atoms (a muon in the Coulomb field of a nucleus) and positronium (a system consisting of an electron and a positron) may also be considered as instances of H-LAs. The H-LA problem is an exactly solvable special case of the general two-body problem in classical mechanics and in quantum mechanics. In the last decades, an enormous amount of work has been produced regarding H-LA systems: their various forms and applications to quantum and classical physics [1–8]. It is clear to everyone that the concept of hydrogen atom is a fundamental source to understand many concrete problems in physics [9–11]. Therefore, we limit ourselves by presenting certain mathematical relations between deep-seated symmetries in which specific kinds of H-LA systems are involved. Hence, at present, it is not our goal to give complete analytical solutions to the problems we address, as they are available (under various concepts) in the literature.

Below we quote some derivations from our own articles, not because we believe that they are the only important ones, but because they contain the analytical approach we will use from the outset. Thus, as a basic mathematical framework [12–15], let us first consider a general quantum system with (Hermitian) canonical variables \( Q_i \) and \( P_j \) satisfying the Heisenberg algebra

\[
[Q_i, P_j] = i\hbar \delta_{ij},
\]

where \( I = I_{n \times n} \otimes 1 \) represents an \( n \)-block identity matrix such that we may express these operators in the general form:

\[
Q_i = \eta \otimes q_i, \quad P_j = \eta \otimes p_j,
\]

with \( p_j = -i\hbar \partial_j \). Here, \( \eta \) is a constant \( n \times n \) self-adjoint matrix operator satisfying \( \eta^2 = I_{n \times n} \). From (2) we can define a label \( \Pi \) associated with each representation of the Heisenberg algebra (1):

\[
n \geq \Pi(Q_i, P_j) = |\text{tr}(\eta)| \geq 0.
\]

Representations satisfying \( \Pi = n \) correspond to the usual ones (\( \eta = I_{n \times n} \)), with \( Q_i, P_j \) reducible operators for \( n \geq 2 \). The Hilbert space is defined by \( \mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^n \) for each quantum system. It consists of \( n \)-component column vectors

\[
\Psi(q, t) = \begin{pmatrix} \psi_1(q, t) \\ \vdots \\ \psi_n(q, t) \end{pmatrix} \in \mathbb{C}^n,
\]

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\[
\Psi(q, t) = \begin{pmatrix} \psi_1(q, t) \\ \vdots \\ \psi_n(q, t) \end{pmatrix} \in \mathbb{C}^n,
\]
where each component $\psi_i$ is a complex valued function of the four-dimensional space-time coordinates $(q, t)$. The scalar product is given by

$$
(\Psi, \Phi) = \int_{\mathbb{R}^4} \sum_{i=1}^{n} \overline{\psi_i(q, t)} \phi_i(q, t) d^4q.
$$

The operator $Q$ consists of three self-adjoint operators $Q_i$ whose domains are defined by

$$
\int_{\mathbb{R}^4} (Q_i \Psi)^\dagger Q_i \Psi d^4q = \int_{\mathbb{R}^4} \sum_{i=1}^{n} |q_i \psi_i|^2 d^4q < \infty.
$$

The momentum operator $P_i = -i \hbar \eta \otimes \partial_i$ can be defined as the Fourier transformation of the position operator $Q_i$.

Minimal interactions can now be incorporated by means of the prescription

$$
P_\mu \rightarrow P_\mu = \frac{e}{c} A_\mu,
$$

with $A_\mu (\mu = 0, \ldots, 3)$, a gauge field; $e/c$, the electromagnetic (EM) coupling constant; and $P_0 = i \hbar I_{n \times n} \otimes \partial_0$. This is the basis for the so-called gauge principle by which the form of the interaction is determined on local gauge invariance. The covariant derivative $D_\mu = (i/\hbar) (P_\mu - (e/c) A_\mu)$ turns out to be of crucial importance for determining the (Abelian) EM field tensor of the theory:

$$
F_{\mu \nu} = (c/ie)[D_\mu, D_\nu].
$$

Namely, $D_\mu$ is the operator that generalizes EM-like interactions. Note that the equations written using the covariant derivative preserve their physical properties under gauge transformations.

## 2 A Restatement of the Dirac Equation and the H-LA Problem

To be specific, let us consider a spin-1/2 free particle (e.g. an electron) described by the Dirac wave equation [12–14]:

$$
H_0 \Psi(q, t) = \left( c \mathbf{a} \cdot \mathbf{p} + m_0 c^2 \beta \right) \Psi(q, t)
$$

$$
= \left( c \mathbf{\Sigma} \cdot \mathbf{p} + m_0 c^2 \beta \right) \Psi(q, t)
$$

$$
= i \hbar \frac{\partial}{\partial t} \Psi(q, t),
$$

with

$$
P_j = \gamma_5 p_j = -i\hbar \gamma_5 \partial_j,
$$

$$
\Sigma_j = \gamma_5 a_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix},
$$

where

$$
\beta = \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & I_{2 \times 2} \\ I_{2 \times 2} & 0 \end{pmatrix},
$$

which satisfy $[\gamma_5, a_j] = 0 = \{ \beta, \gamma_5 \}$. Hence, we formally define

$$
Q_0 = I_{4 \times 4} \otimes q_0, \quad Q_1 = \gamma_5 \otimes q_1,
$$

$$
P_0 = I_{4 \times 4} \otimes p_0, \quad P_j = \gamma_5 \otimes p_j,
$$

where $p_\mu = i\hbar \partial_\mu$ and $\eta = \gamma_5$. Therefore, $\Pi = 0$, i.e. $n(-4)$ is an even integer, and $Q_i, P_j$ are (formally) traceless self-adjoint operators. The operators $Q_\mu, P_\nu$ satisfy the canonical commutation relations

$$
[Q_\mu, Q_\nu] = 0 = [P_\mu, P_\nu], \quad [Q_\mu, P_\nu] = i\hbar g_{\mu \nu},
$$

in which $\mathbb{I} = I_{4 \times 4} \otimes 1$, and $g = \text{diag}(-1, 1, 1, 1)$ in the standard Dirac representation. We assume that $\Psi(q, t)$ is normalized, i.e. multiplied by a scalar constant such that

$$
||\Psi(q, t)|| = \int_{\mathbb{R}^3} |\Psi(q, t)|^2 |q|^3 q d^4q = 1,
$$

where "$||\ldots||$" denotes the scalar product in $\mathbb{R}^4 = L^2(\mathbb{R}^3) \otimes \mathbb{C}$, just as is usually defined in the Dirac theory. In relativistic quantum mechanics, we specify a Hermitian minimal substitution of the EM field as

$$
P_0 \rightarrow P_0 - \frac{e}{c} V(q), \quad \mathbf{P} \rightarrow \mathbf{P} - \frac{e}{c} \mathbf{A}(q),
$$

where $q = |q|$. A general interaction that keeps the $1/q$ dependence in the gauge field components is defined by

$$
V(q) = -\frac{Ze}{q}, \quad \mathbf{A}(q) = -\varsigma(Ze)i\beta \gamma_5 \frac{\mathbf{q}}{q}.
$$

Here, $\varsigma = \varsigma(Ze)$ is a real constant whose nature and explicit dependence on $Ze$ will be considered below. We observe that the vector field $\mathbf{A}$ satisfies hermiticity:
\[ A^\dagger(q) = A(q). \] Thus, the wave equation for the interacting particle becomes

\[ H\Psi(q, t) = \left( c\Sigma \cdot \left( p + \frac{e\Sigma}{c} b \right) + m_0 c^2 \beta - \frac{Ze^2}{q} \right) \Psi(q, t) \]

\[ = \frac{i h}{\partial t} \Psi(q, t), \quad (17) \]

with \( H^\dagger = H. \) Note that in this equation

\[ i\beta \gamma^3 \Sigma \cdot \dot{q} = i\beta \alpha \cdot \dot{q} = i \begin{pmatrix} 0 & \sigma \cdot \dot{q} \\ -\sigma \cdot \dot{q} & 0 \end{pmatrix}. \quad (18) \]

Before solving the eigenvalue problem for the bound states associated with (17), it is straightforward to prove that the operators

\[ K = \beta(\Sigma \cdot L + h), \quad J = L + \frac{h}{2} \Sigma, \quad (19) \]

with \( L = q \times p, \) the orbital angular momentum operator, are constants of motion:

\[ [H, K] = 0, \quad [H, J] = 0. \quad (20) \]

In this regard, the operator \( \sigma \cdot q \) has a similar mathematical behavior to that of \( \sigma \cdot p. \) In fact,

\[ [K, \alpha \cdot p] = 0 = \begin{bmatrix} K, \beta \frac{\alpha \cdot q}{q^2} \end{bmatrix}, \]

\[ [J, \alpha \cdot p] = 0 = \begin{bmatrix} J, \beta \frac{\alpha \cdot q}{q^2} \end{bmatrix}. \quad (21) \]

Following a standard procedure [9–11], the stationary states of energy \( E \) can be written as

\[ \Psi_E(q, t) = \begin{pmatrix} \psi_0(q, t) \\ \psi_1(q, t) \end{pmatrix} \]

\[ = \frac{1}{\sqrt{q}} \begin{pmatrix} G(q) \gamma_{jj_1l_1}(q) \\ iF(q) \gamma_{jj_1l_1}(q) \end{pmatrix} \exp\left(-\frac{i}{\hbar} E t\right), \quad (22) \]

where \( \gamma_{jj_1l_1} \) is the normalized total angular momentum function, with

\[ L^2 \gamma_{jj_1l_1} = \hbar^2(l + 1) \gamma_{jj_1l_1}, \]

\[ J^2 \gamma_{jj_1l_1} = \hbar^2(j + 1) \gamma_{jj_1l_1}, \]

\[ K \gamma_{jj_1l_1} = \hbar \kappa \gamma_{jj_1l_1}. \quad (23) \]

Here, \( l_a = j \pm 1/2, l_b = j \pm 1/2 \) when \( \kappa = \pm(j + 1/2). \) Thus, the Dirac equation (17) is equivalent to the set of first-order differential equations

\[ \frac{dF}{dp} - \frac{(\kappa + \zeta)}{\rho} \frac{F}{G} = \left( \sqrt{\frac{A}{B}} - \frac{Ze}{\rho} \right) G, \]

\[ \frac{dG}{dp} + \frac{(\kappa + \zeta)}{\rho} \frac{G}{F} = \left( \sqrt{\frac{B}{A}} + \frac{Ze}{\rho} \right) F, \quad (24) \]

where

\[ A = \frac{m_0 c^2 - E}{\hbar c}, \quad B = \frac{m_0 c^2 + E}{\hbar c}, \quad \rho = \sqrt{ABq}, \]

\[ Ze^2 \hbar c \cdot \zeta = \frac{e^2}{\hbar c}, \quad (25) \]

with \( \alpha = e^2/\hbar c \approx 1/137.04 (\alpha \ll 1), \) the fine-structure constant. We now look for solutions in the form of series:

\[ F(\rho) = \exp(-\rho)\rho^s N_{v=0} a_v \rho^v, \]

\[ G(\rho) = \exp(-\rho)\rho^s N_{v=0} b_v \rho^v. \quad (26) \]

From (24) and (26), we obtain the recursion relations:

\[ (s + v - \kappa - \zeta) a_v + Zab_v - a_{v-1} = \sqrt{\frac{A}{B}} b_{v-1}, \]

\[ (s + v + \kappa + \zeta) b_v - Za a_v - b_{v-1} = \sqrt{\frac{B}{A}} a_{v-1}, \quad (27) \]

For \( v = 0, \) we get

\[ \begin{pmatrix} s - \kappa - \zeta & Ze \\ -Ze & s + \kappa + \zeta \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = 0. \quad (28) \]

Given the fact that \( a_0, b_0 \neq 0, \) from (28) we obtain

\[ s = \pm \sqrt{(\kappa + \zeta)^2 - (Za)^2} > -\frac{1}{2}, \quad (29) \]

as \( F \) and \( G \) must behave better than \( \rho^{-1/2} \) at the origin. Note that, for restricted values of \( \zeta \) and \( \kappa, \) \( s \) can assume negative values, in contrast to the usual H-LA problem. In this section, we will study only instances where \( s \geq 0. \) For \( v = N + 1 \) and \( a_{N+1} = b_{N+1} = 0, \) in order to terminate the series, we obtain \( a_N = -b_N \sqrt{AB}. \) Thus, from (27), we get

\[ \sqrt{m_0 c^2 - E^2} \left( N + \sqrt{(\kappa + \zeta)^2 - (Za)^2} \right) = Z a, \quad (30) \]
so that the energy eigenvalues become

\[
E_{n,k}(\zeta) = + m_0 c^2 \left( 1 + \frac{(Z\alpha)^2}{n^2 + (k + \zeta)^2 - (Z\alpha)^2} \right)^{-1/2},
\]

(31)

where \( n = N + |k| = N + j + 1/2 \) is the principal quantum number of the nonrelativistic theory of the hydrogen atom. We now argue about the nature of the real constant \( \zeta \) (proportional to the EM gauge coupling constant) that multiplies the vector field \( A(q) \) in (15) and (17). It is clear that there is freedom to choose a fixed value of \( \zeta \) in order to describe the corresponding quantum system. However, we conjecture that a reasonable election for this coupling constant essentially correlates to the factor that multiplies the \( 1/q \) function in \((e/c) V(q)\): \( \sim Z\alpha \). Namely, \( \zeta = e\zeta / hc \) has to be a function of \( Z\alpha \): \( \zeta = \zeta(Z\alpha) \), provided that \( d(\zeta(Z\alpha))/dZ > 0 \) as increasing \( Z \), the absolute value of the binding energy, also increases. This assumption should be correct because we are considering a possibly realizable physical system where the EM gauge potential is an individual four-vector: \( A^\mu = (V, A) \). To illustrate this point, we choose \( \zeta \) in terms of single powers in \( Z\alpha \):

\[
\zeta_i(Z\alpha) = (Z\alpha)^i \chi_i,
\]

where there is no summation on the index \( i \). Here, \( \chi_i \) is real parameter, with \( i = 1, 2, 3 \ldots \) whose values depend on the particular problem we are dealing with.

Now, we want to calculate the binding energy for some instances of \( \zeta \): \( E_{\text{bind}}^{(G)} = E_{n,k}(\zeta) - m_0 c^2 \). The parameter \( \zeta_i(Z\alpha) \) will represent a (hypothetical) relativistic quantum system. If \( \chi_i = 1 \) \((i = 1, 2, 3)\) and \( Z\alpha \ll 1 \), the corresponding energies become

\[
E_{\text{bind}}^{(G)} = - m_0 c^2 \left( \frac{1}{n^2} + \frac{(Z\alpha)^2}{n^2 + (k + \zeta)^2 - (Z\alpha)^2} - \frac{1}{|k|} \right),
\]

(33a)

\[
E_{\text{bind}}^{(G)} = - m_0 c^2 \left( \frac{1}{n^2} + \frac{(Z\alpha)^2}{n^2} \left( \frac{1}{|k|} - \frac{3}{4n} \right) \right) + O \left( (Z\alpha)^4 \right),
\]

(33b)

\[
E_{\text{bind}}^{(G)} = - m_0 c^2 \left( \frac{1}{n^2} + \frac{(Z\alpha)^2}{n^2} \left( \frac{1}{|k|} - \frac{3}{4n} \right) \right) + O \left( (Z\alpha)^4 \right),
\]

It is clear that generally there are numerous possible expressions for \( \zeta(Z\alpha) \). However, at present, we restrict ourselves to the simplest forms of binding energies.

The first term of each one of these equations is similar to the energy spectrum of the bound states in the nonrelativistic approximation. The second term in (33a) is a fine structure energy correction of the order of \( (Z\alpha)^3 \), which is absent in the usual Dirac spectrum. The corresponding third term in this equation is modified by a piece proportional to \( 1/|k| \), being also a fine structure correction. In \( E_{\text{bind}}^{(G)} \), there is also a modification of the spectrum in the second term. Finally, (33c) contains a small contribution of the order of \( (Z\alpha)^3 \) with respect to (33d). In this case, for the hydrogen system (\( Z = 1 \)) with \( n = 2 \), we have

\[
\Delta E_{n=2, \pm 1}^{(G)} = \pm \frac{m_0 c^2 a^5}{16} = \pm 0.666 \times 10^{-6} \text{ eV}
\]

\[
= \pm 2\pi h \times 160.0 \text{ MHz}.
\]

(34)

Furthermore, \( \Delta E_{n=2, -2}^{(G)} = - m_0 c^2 a^5 / 16 \). It is interesting to compare the magnitude of this quantity with the well-known result for the (nonrelativistic) Lamb shift [16]:

\[
\Delta E_{n=2, l=0}^{\text{(Lamb)}} = \frac{m_0 c^2 a^5}{6\pi} \ln \left( \frac{1}{a} \right)
\]

\[
= 2.760 \times 10^{-6} \text{ eV}
\]

\[
= 2\pi h \times 667.4 \text{ MHz}.
\]

(35)

Although these quantities are of the same order of magnitude, they seem to be neither physically nor analytically related as the Lamb shift arises mostly from the interaction of a bound electron with the zero-point fluctuations of the free EM field. Hence, possible corrections to the usual Dirac formula, which could be compared with experimental observations, should be found in \( E_{\text{bind}}^{(G\geq 4)} \) beyond the order of \( (Z\alpha)^5 \).
Note that the corresponding total spread in energy of the component levels is

\[ E^{(\text{Dirac})}_{n,j=n-1/2} - E^{(\text{Dirac})}_{n,j=1/2} = \frac{1}{2} (Z \alpha)^4 \frac{m_0 c^2 n - 1}{n}, \]  

(36c)

which form the fine structure of the states for a given \( n > 1 [17, 18] \). Applications of (33a–c) to H-LA systems remain to be properly studied.

### 3 The Breaking of Supersymmetry

It is known that within the framework of supersymmetric quantum mechanics one can obtain the complete energy spectrum and eigenfunctions of the Dirac equation for an attractive Coulomb potential [19, 20], where \( G \) is the “large” component in the nonrelativistic limit. Certainly, the radial functions \( G \) and \( F \) must be multiplied by the appropriate two-component angular eigenfunctions to make up the full four-component solutions of the Dirac equation. Specifically, we consider the case, not merely of academic interest, where the EM four-vector potential component \( A^\mu \) is treated in an even-handed way:

\[ A^\mu = (V(q), A(q)) = -\frac{Ze}{q} (1, +i\beta y\gamma^5 \mathbf{q}). \]

(37)

If \( A(q) = 0 \), we can compare the two ladders of levels for a fixed value of \( \kappa \), a pair of degenerate levels corresponds to the same value of \( j \) but opposite values of \( \kappa \) except for the lowest state of the pair of ladders when only the negative value of \( \kappa \) corresponds to an eigenstate [11, 19]. However, for \( A^\mu \) given in (37), i.e. \( \zeta(Z \alpha) = Z \alpha \), the resulting energy eigenvalues become

\[ E_{n,\kappa} = m_0 c^2 \left( 1 + \frac{(Z \alpha)^2}{\left( n - |\kappa| + \sqrt{(\kappa + Z \alpha)^2 - (Z \alpha)^2}\right)^2} \right)^{-1/2}, \]

(38)

so that a relative plus or minus sign between \( \kappa \) and \( Z \alpha \) appears in \((Z \alpha + Z \alpha)^2\). This means that \( E_{n,\kappa} \approx E_{n,-\kappa} \) \((\kappa \neq n)\) for \( Z \alpha \ll 1 \). We may say that in general the presence of \( \zeta = \zeta(Z \alpha) \) in (31) is related to the breaking of supersymmetry that connects the “fermionic” and “bosonic” components of the spectrum (except for the missing groundstate). The states are classified in complete accordance to the levels of the hydrogen atom \((Z = 1)\), as summarized in detail in Table 1. Following the analysis in [9], a more general situation (for \( Z \geq 1 \)) is illustrated.

### Table 1: As with the spectrum of a Dirac electron in a Coulomb potential [9], the classification of bound states of the particle for \( Z = 1 \) according to (9) under minimal replacement \( P^\mu \rightarrow P^\mu - (e/c)A^\mu \) with \( A^\mu \) given in (37).

<table>
<thead>
<tr>
<th>State</th>
<th>( n )</th>
<th>( l )</th>
<th>( j )</th>
<th>( N )</th>
<th>( \kappa )</th>
<th>( E_{\text{bind}}[\text{eV}] )</th>
<th>( E_{\text{bind}}(\text{Dirac})[\text{eV}] )</th>
</tr>
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<tbody>
<tr>
<td>1s_{\uparrow}</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>-1</td>
<td>-13.806785</td>
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<td>1/2</td>
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<tr>
<td>2p_{\uparrow}</td>
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<td>1/2</td>
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<tr>
<td>3p_{\uparrow}</td>
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<td>1/2</td>
<td>2</td>
<td>-1</td>
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<td>-1.511779</td>
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<tr>
<td>3p_{\uparrow}</td>
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<td>1/2</td>
<td>2</td>
<td>1</td>
<td>-1.504451</td>
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</tr>
<tr>
<td>3d_{\uparrow}</td>
<td>3</td>
<td>2</td>
<td>3/2</td>
<td>1</td>
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<td>4p_{\uparrow}</td>
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<td>1/2</td>
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<td>3/2</td>
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</tr>
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</table>

All the states are nondegenerate due to the presence of the magnetic field \( B \). See end of the text and Figure 5 for explanation of this fine structure.
in the figures. For states with \( \kappa = -1 \), energy values can be calculated up to \( 1 - 2Z\alpha > 0 \), i.e., up to \( Z \approx 69 \); for the ground state \(( n = 1, \kappa = -1)\), we have

\[
E_{1, -1} = m_0c^2\left(1 + \frac{(Z\alpha)^2}{\sqrt{1 - 2Z\alpha}}\right)^{-1/2}. \tag{39}
\]

With increasing \( Z \), the absolute value of the binding energy also increases, as it must do (Fig. 1). The slope \( dE_{n, -n}/dZ \to -\infty \) as \( Z \to n/2 \), with \( E_{n, -n}(n/2) = 0 \). But, at these points, at least the single particle interpretation collapses because quantum electrodynamical effects should be considered. For \(( n, -(n-1)) \) states, with \( n > 1 \), the energy becomes imaginary when \( \kappa(\kappa + 2Z\alpha) < 0 \) as shown in Figure 2. Nevertheless, for \(( n, n - 1) \) states, the particle is much less tightly bound to the point nucleus as is outlined in Figure 3. To be brief, we will not consider the (alternative) system where \( A^\rho = -(Ze/q)(1, -i\beta\gamma_5\mathbf{q}) \), which can be solved following similar steps as above.

Finally, we briefly discuss the case where \( s \) in (29) assumes negative values:

\[
s = -\sqrt{(\kappa + Z\alpha)^2 - (Z\alpha)^2} > -1/2. \tag{40}
\]

Namely, \( Z\alpha > (1/4 - \kappa^2)/2\kappa \), which is valid only if \( \kappa < 0 \). Notice that this is a consequence due to the presence of the magnetic field \( B \) given in (44) for \( \zeta(Z\alpha) = Z\alpha \). The situation is depicted in Figure 4. There is a “bifurcation” point of energy \( E = 4m_0c^2|\kappa|/(1 + 4|\kappa|^2) \) at

**Figure 1:** The energy \( m_0c^2 \) of a spin-1/2 particle in the \(( n, -n) \) states as a function of \( Z\alpha \). With increasing \( Z \), the absolute value of the binding energy sharply increases.

**Figure 2:** Solution of (17) for the EM four-potential (37). For the \( n \) level, there are no solutions when \( n + 1/2 > Z\alpha > n/2 \). This should change for \( 2\kappa \geq 1 \) if the finite size of the central nucleus is taken into account.

**Figure 3:** The energy \( m_0c^2 \) in the \(( n, n - 1) \) states \( n \geq 2 \) as a function of \( Z\alpha \). With increasing \( Z \), the absolute value of the binding energy also increases. However, the particle is much less bound to the nucleus. In fact, \( E_{n, n-1}(Z\alpha) \to 0 \) as \( Z \to \infty \).

**Figure 4:** Graph of the energy eigenvalues as a function of \( Z\alpha \) for negative values of \( s \), where only \( \kappa < 0 \) is allowed. For \( Z\alpha > |\kappa|/2 \), the solution \( E_{n, \kappa}(Z\alpha) \) is imaginary. See text at the end Section 3 for related explanation.
Za = \left( |\kappa|^2 - 1/4 \right)/2|\kappa|. Surprisingly, for \( n \geq |\kappa| + 1 \), the higher the principal quantum number \( n \), the smaller the slope of \( E_{n,k}(Za) \). This effect could be interpreted as \( B \) (indirectly) “counteracting” the electric field of the nuclei.

### 4 Concluding Remarks

We have found that the systems governed by the wave equation (17) are exactly solvable in relativistic quantum mechanics. We also showed that these systems have simple symmetry properties: as they conserve total angular momentum, they must be O(3) invariant. The nature of this symmetry is not geometrical but dynamical. However, the complete symmetry Lie algebra of each system is yet to be discovered. As a hint to solve this question, we note that the Dirac hydrogen atom with either spin or pseudospin symmetry has O(4) symmetry [21] (which, in the present general problem \( \zeta(Za) \neq 0 \), should be approximate for \( Za \ll 1 \)). A well-known case of “hidden” SO(4,2) symmetry is the so-called dynamical symmetry of the hydrogen atom. The O(4) part of this symmetry is more familiar and has been known for a long time under the name of Runge–Lenz (RL) vector [22, 23]. Note that, in this regard, we have found that the energy eigenfunctions are formally similar to those of the usual relativistic H-LA problem. However, it is known that RL symmetry is spoiled for relativistic mechanics. The generalization of the RL vector to include the spin degrees of freedom was introduced by Johnson and Lippmann (Johnson–Lippmann operator) [24]. Hidden symmetry operator for the Dirac equation in a Coulomb field shows that this operator may be reduced to the one by Johnson and Lippmann to include the spin degrees of freedom. The \( l \)-degeneracy of the hydrogen atom spectrum in nonrelativistic quantum mechanics does not exist for the Dirac equation because the RL vector is no longer conserved in this case [24, 25].

It is worth mentioning here that the Dirac oscillator [26–30] can also be reobtained through a minimal interaction of the form (15) in the Dirac equation. This is done by making the self-adjoint substitution \( P \rightarrow P - (e/c)A(q) \), with \( A(q) = (m_0 \omega c/e)i\beta\gamma_5 q \), where \( \omega \) is the frequency of the oscillator. This gauge field gives rise to a harmonic oscillator with a strong spin-orbit coupling, which introduces an infinite degeneracy. This oscillator has a hidden supersymmetry, responsible for the special properties of its spectrum [31]. It is interesting to note that the vector field \( A_j(q) \) in (15) is a Hermitian operator. This feature is lacking in Moshinsky’s approach [26]. On the other hand, we have observed that the interaction of a charge particle with the four-potential (37) leads to departures from the Coulomb potential spectrum. When the field deviates from a Coulomb field, the transformed states for the eigenfunction components no longer belong to a supersymmetric pair, and the spectrum loses the ladder structure characteristic of supersymmetric pairing.

Finally, we observe that \( A_j(q) \) is both a radial vector and singular at \( q = 0 \). Then one can naively define the associated magnetic field as \( B(q) = \nabla \times A(q) \), which vanishes in \( \mathbb{R}^3 \). However, from (8), a natural definition of magnetic field in this theory is given by

\[
B_k(q) = F_{ij}(q) = (c/ie)[D_i, D_j] \\
= \beta(\nabla \times A(q) + A(q) \times \nabla)_k \neq 0,
\]

where \( A_j(q) = -i\beta\gamma_5 A_j(q) \). In (41), the central field \( A(q) \sim q/q \), then \( \nabla \times A(q) = 0 \). Furthermore,

\[
\nabla \cdot (A(q) \times \nabla) \sim \frac{i}{\hbar} \nabla \cdot \left( \frac{L}{q^2} \right) = 0,
\]

so that

\[
\nabla \cdot B(q) = 0,
\]

as one would expect. It is noteworthy that

\[
B(q) = \frac{2e'(Za)}{\hbar} \beta \frac{L}{q^2}
\]

directly induces the splitting of atomic energy levels relative to the Dirac spectrum (a fact we leave without proof here), as exemplified in Table 1 for \( \zeta(Za) \neq 0 \). Hence, this splitting corresponds to a modification of fine structure in H-LAs when, in general, \( \zeta(Za) \neq 0 \) (Fig. 5).

The analysis of subsequent terms in the power series expansion of \( E_{\text{bind}}^{(k)} \) in \( Za \) remains to be done. It would take us too far afield to make here a discussion on such

![Figure 5: Fine structure energy corrections (not to scale) for the \( n = 1, 2, 3 \) states of hydrogen in the presence of field (37) according to values of \( E_{\text{bind}} \) (Tab. 1).](image-url)
a problem; nevertheless, it provides a direction for future research.

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**References**