Rustam Ali* and Prasanta Chatterjee

Three-Soliton Interaction and Soliton Turbulence in Superthermal Dusty Plasmas

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Abstract: Propagation and interaction of three solitons are studied within the framework of the Korteweg-de Vries (KdV) equation. The KdV equation is derived from an unmagnetised, collision-less dusty plasma containing cold inertial ions, stationary dusts with negative charge, and non-inertial kappa-distributed electrons, using the reductive perturbation technique (RPT). Adopting Hirota’s bilinear method, the three-soliton solution of the KdV equation is obtained and, as an elementary act of soliton turbulence, a study on the soliton interaction is presented. The concavity of the resulting pulse is studied at the strongest interaction point of three solitons. At the time of soliton interaction, the first- and second-order moments as well as the skewness and kurtosis of the wave field are calculated. The skewness and kurtosis decrease as a result of soliton interaction, whereas the first- and second-order moments remain invariant. Also, it is observed that the spectral index \( \kappa \) and the unperturbed dust-to-ion ratio \( \mu \) have great influence on the skewness and kurtosis of the wave field.

Keywords: Korteweg-de Vries (KdV) Equation; Soliton; Soliton Interaction; Turbulence; Unmagnetised Plasma.

1 Introduction

During last few decades, the study of linear and nonlinear wave propagation in dusty plasma has attracted the attention of many researchers because of its applicability in different plasma devices, laboratory experiments, and space plasmas [1–7]. In addition to the usual electrons, ions, and neutral particles, a dusty plasma contains charged and massive dust grains. Because of the presence of the large number of charged particles, a dusty plasma supports different kinds of eigen modes such as the dust acoustic mode [8], dust drift mode [9], Shukla-Varma mode [10], dust lattice mode [11], dust cyclotron mode [12], dust ion acoustic (DIA) mode [13], and dust Berstain-Green-Kruskal mode [14]. Recently, researchers have shown great interest in the study of linear and nonlinear DIA wave propagation [15–21]. The existence of low-frequency DIA waves in a dusty plasma was theoretically predicted in [15] and later confirmed by Barkan et al. [22] experimentally. Recently, Pakzad et al. [23] showed that the DIA wave pattern is affected by the non-extensive parameter and also by the relative density of the plasma constituents. Very recently, Chatterjee et al. [24] studied the DIA wave in the framework of the damped forced Korteweg-de Vries (KdV) equation. They showed that the structure of the DIA solitary wave is heavily affected by the strength and frequency of the external periodic force.

It is known that the plasmas in space or in the laboratory may contain a large number of high-energy particles, which can be modeled by a Lorentzian distribution or a kappa distribution [25]. The presence of a substantially large number of superthermal particles can significantly change the rate of resonant energy transfer between the particles and plasma waves [26–28]. A three-dimensional generalised Lorentzian or kappa distribution function [29] can be written as follows:

\[
f_\kappa (v) = \frac{\Gamma (\kappa + 1)}{(\pi \kappa \theta^2)^{3/2} \Gamma (\kappa - 1/2)} \left( 1 + \frac{v^2}{\kappa \theta^2} \right)^{-(\kappa + 1)}, \tag{1}
\]

where \( \theta = ((\kappa - 3/2)(2k_B T_e)/m)^{1/2} \) is the effective thermal speed, modified by spectral index \( \kappa \), with \( T_e \) as the characteristic kinetic temperature and \( k_B \) is the Boltzmann constant, and \( \Gamma \) is the gamma function arising from the normalisation of \( f_\kappa (v) \) such that \( \int f_\kappa (v) dv = 1 \). It is clear that for the physically realistic thermal speed, one requires \( \kappa > 3/2 \). Low values of \( \kappa \) represent a hard spectrum with a strong non-Maxwellian (power law-like) tail having a power-law form at high speeds, while in the limit \( \kappa \to \infty \), the kappa distribution function reduces to the well-known Maxwell-Boltzmann distribution [27]. By integrating the kappa distribution over the velocity space, one can

*Corresponding author: Rustam Ali, Department of Mathematics, Sikkim Manipal Institute of Technology, Sikkim Manipal University, Majitar, Rangpo, East-Sikkim 737136, India; and Department of Mathematics, Siksha Bhavana, Visva Bharati University, Santiniketan 731235, India, E-mail: rustamali24@gmail.com, rustam.ali@smiit.smu.edu.in
Prasanta Chatterjee: Department of Mathematics, Siksha Bhavana, Visva Bharati University, Santiniketan 731235, India, E-mail: prasantachatterjee1@rediffmail.com
obtain the normalised form of hot electron number density as [30]

\[ n_e = \left( 1 - \frac{\phi}{\kappa - 3/2} \right)^{-\kappa + 1/2}. \] (2)

In weakly dispersive media, the soliton is an essential part of the nonlinear wave field, and their deterministic dynamics in the framework of the KdV equation is well studied [31–33]. Several authors have also studied the soliton and multi-soliton solutions of the KdV and KdV-like equations and their mutual interactions [34–38]. When a large number of interacting waves propagate in different directions with different velocities, their interactions lead to a fast change in the wave pattern and the characteristic of the wave field is described within the framework of statistical theory. Such a theory is termed ‘the theory of wave turbulence’ [39, 40]. Zakharov [41, 42] has discussed the characteristics of wave turbulence in an integrable system. Soliton turbulence is specified by the kinetic equations describing the parameters of the associated scattering problem and is a specific part of wave turbulence. In 1971, Zakharov [41] for the first time described the fundamental role of pair-wise soliton collisions within the framework of the KdV equation, which was later confirmed [43–45]. Soliton turbulence in an integrable system is slightly degenerate because of the conservation of solitons in the interaction process. In turbulence theory, it is very important to know the wave field distribution and the moments (mean, variance, skewness, and kurtosis) of the random wave field, which are obtained from measurements [46–49]. Pelinovsky et al. [50, 51] studied the two-soliton interaction as an elementary act of soliton turbulence in the framework of KdV and modified KdV equations. They showed that the nonlinear interaction of two solitons leads to a decrease of third- and fourth-order moments of the wave field, whereas the first and second moments remain constant. Recently, some works have been reported on soliton turbulence [52–56]. Very recently, Shurgalina [57, 58] studied different features of soliton turbulence in the framework of the Gardner equation with negative and positive cubic nonlinearity.

In this article, our goal is to study the properties of skewness and kurtosis of the random wave field due to three-soliton interaction and the effect of different plasma parameters on the skewness and kurtosis of the wave field at the time of soliton interaction. The rest of the article is organised as follows. The model equations are presented in Section 2. In Section 3, the derivation of the KdV equation and the three-soliton solution are given. Sections 4 and 5 present the effect of soliton interaction on the statistical characteristic of the wave field and the effect of different plasma parameters on soliton turbulence, respectively. Section 6 presents the conclusions.

2 Model Equations

In this work, an unmagnetised dusty plasma with cold inertial ions, stationary dust with negative charge, and inertia-less, \( \kappa \)-distributed electrons is considered. The normalised basic equations governing the DIA waves are given by

\[ \frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0, \] (3)

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial x}, \] (4)

\[ \frac{\partial^2 \phi}{\partial x^2} = (1 - \mu) \left( 1 - \frac{\phi}{\kappa - 3/2} \right)^{-\kappa + 1/2} - n + \mu, \] (5)

where \( n \) is the ion number density normalised to \( n_0 \); \( n_e \) is the number density of the electrons; and \( u \) is the ion velocity normalised to the ion fluid speed \( c_s = \sqrt{k_B T_e/m_i} \), with \( m_i \) as the mass of ions, \( T_e \) as the temperature of electrons, \( k_B \) as the Boltzmann constant. The electrostatic wave potential \( \phi \) is normalised to \( k_B T_e/e \). The space and time variables are normalised to the electron Debye radius \( \lambda_D = \sqrt{k_B T_e/4\pi n_e e^2} \) and the inverse of the cold electron plasma frequency \( \omega_{pe}^{-1} = \sqrt{m_i/4\pi n_e e^2} \), respectively. Here, \( \mu = Z e n_d/e n_0 \), with \( n_{d0} \) being the number density of dust particles and \( Z_d \) the dust charge number.

3 Derivation of the KdV Equation and Its Three-Soliton Solution

To derive the KdV equation, we apply the reductive perturbation technique (RPT). According to RPT, the independent variables are

\[ \xi = e^{1/2}(x - vt), \] (6)

\[ \tau = e^{3/2} t. \] (7)

The dependent variables are expanded as

\[ n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \cdots, \] (8)

\[ u = 0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \] (9)

\[ \phi = 0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \cdots. \] (10)
Putting (2) and (6)–(10) in (3)–(5) and comparing coefficients of lowest powers of $\varepsilon$, we obtain the linear propagation speed in the low-frequency limit as

$$v^2 = \frac{1}{a(1-\mu)},$$

with $a = \frac{\kappa-\frac{1}{2}}{k-\frac{3}{2}}$.

Taking the coefficients of the next higher order of $\varepsilon$, we obtain the following KdV equation:

$$\frac{\partial \phi_1}{\partial \tau} + A \phi_1 \frac{\partial \phi_1}{\partial \xi} + B \frac{\partial^3 \phi_1}{\partial \xi^3} = 0,$$

where $A = \left(\frac{3-2b(1-\mu)\nu}{2\nu}\right)$ and $B = \frac{\nu^3}{\nu^3}$, with $b = \frac{k-1/2}{(\kappa-1/2)(\kappa+1/2)}$.

The coefficient $A$ of the nonlinear term $\phi_1^2$ of the KdV equation (12) becomes zero for $\kappa = \frac{1}{4-3\nu}$. So, assuming $\kappa \neq \frac{1}{4-3\nu}$ ($\mu < 1$) and making the transformations $\xi = B^{1/3} \xi$, $\phi_1 = 6A^{-1}B^{1/3} \phi_1$, and $\tau = \bar{\tau}$ to the KdV equation (12), we obtain the following standard KdV equation:

$$\frac{\partial \phi_1}{\partial \bar{\tau}} + 6 \phi_1 \frac{\partial \phi_1}{\partial \xi} + \frac{\partial^3 \phi_1}{\partial \xi^3} = 0.$$  

Using Hirota’s bilinear method, the three-soliton solution of (13) is obtained as [59]

$$\phi_1 = 2 \frac{\partial^2}{\partial \xi^2} (\ln f),$$

where $f(\xi, \tau) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A_{12}^2 e^{\theta_1+\theta_2} + A_{23}^2 e^{\theta_2+\theta_3} + A_{13}^2 e^{\theta_1+\theta_3} + A^2 e^{\theta_1+\theta_2+\theta_3}$, with

$$\theta_i = -2 \left( \frac{\eta_i}{B^{1/3}} \xi - 4 \eta_i^3 \tau - \alpha_i \right), \quad i = 1, 2, 3.$$  

From (16), we have

$$\phi_1 = \frac{12B^{1/3} N}{A} \frac{\partial^2}{\partial \xi^2} (\ln f),$$

where

$$D = \left(1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A_{12}^2 e^{\theta_1+\theta_2} + A_{23}^2 e^{\theta_2+\theta_3} + A_{13}^2 e^{\theta_1+\theta_3} + A^2 e^{\theta_1+\theta_2+\theta_3} \right)^2$$

and

$$N = 4 \eta_1^2 e^\theta_1 + 4 \eta_1^2 e^{\theta_1+\theta_2} + 4 \eta_1^2 e^{\theta_1+\theta_3} + 4 \eta_1^2 e^{\theta_1+\theta_2+\theta_3} + 4 \eta_1^2 A^2 e^{\theta_1+\theta_2+\theta_3} + 4 \eta_1^2 A^2 e^{\theta_1+\theta_2+\theta_3} - 8 \eta_1 \eta_2 e^{\theta_1+\theta_2} + 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} - 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} + 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} + 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} + 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} + 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} + 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} + 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} + 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} + 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} + 8 \eta_1 \eta_2 A^2 e^{\theta_1+\theta_2+\theta_3} + \cdots$$
After a rigorous and very complicated mathematical calculation, \( N \) becomes

\[
N = 4\eta_1^2 A^2 A_{23}^2 e^{\theta_i + \theta_j} + 4\eta_2^2 A^2 A_{13}^2 e^{\theta_i + \theta_j}
+ 4\eta_3^2 A^2 A_{12}^2 e^{\theta_i + \theta_j}.
\]

From (18), we have

\[
\phi_1 = \frac{12B^{1/3}}{A}
\left(4\eta_1^2 A^2 A_{23}^2 e^{\theta_i + \theta_j}
+ 4\eta_2^2 A^2 A_{13}^2 e^{\theta_i + \theta_j}
+ 4\eta_3^2 A^2 A_{12}^2 e^{\theta_i + \theta_j}
\right)
\left(1 + e^{\theta_i} + e^{\theta_j} + A_{12}^2 e^{\theta_i + \theta_j}
+ A_{13}^2 e^{\theta_i + \theta_j}ight)^2
\]

\[
= \frac{12B^{1/3}}{A}
\left(4\eta_1^2 A^2 A_{23}^2 e^{\theta_i}
+ 4\eta_2^2 A^2 A_{13}^2 e^{\theta_i}
+ 4\eta_3^2 A^2 A_{12}^2 e^{\theta_i}
\right)
\left(e^{-\theta_i - \theta_j} + e^{-\theta_i - \theta_j} + e^{-\theta_i - \theta_j} + e^{-\theta_i - \theta_j}
+ A_{12}^2 e^{\theta_i}
+ A_{13}^2 e^{\theta_i} + A_{13}^2 e^{\theta_i} + A^2
\right)^2.
\]

Assuming \( \tau \gg 1 \), (19) can be asymptotically (up to exponentially small terms) approximated as

\[
\phi_1 = \frac{12B^{1/3}}{A}
\left(4\eta_1^2 A^2 A_{23}^2 e^{\theta_i}
+ 4\eta_2^2 A^2 A_{13}^2 e^{\theta_i}
+ 4\eta_3^2 A^2 A_{12}^2 e^{\theta_i}
\right)
\left(1 + A_{13}^2 e^{\theta_i} + A^2
\right)^2
\]

\[
= \frac{48A^2 B^{1/3}}{A}
\left(\eta_1^2 A_{23}^2 e^{\theta_i}/(A_{23}^2 e^{\theta_i} + A^2)^2ight)
\]
Because of the complete integrability of the KdV equation (12), interaction of the solitons is elastic, and after interaction they regain the properties of the soliton [41, 43–45, 60]. Three solitons with different amplitudes propagate in time, and since the speed of the solitons is proportional to their amplitudes, the nonlinear interaction of the three solitons takes place in a certain time and space. Choosing

\[
\{ \alpha_1 = -\frac{\eta_1}{2B^{1/3}} \Delta_1, \quad \alpha_2 = -\frac{\eta_2}{2B^{1/3}} \Delta_2, \quad \alpha_3 = -\frac{\eta_3}{2B^{1/3}} \Delta_3 \},
\]

we observe that the interaction of the soliton elements occurs at the origin, i.e. at the point \( \xi = 0, \tau = 0 \) [61]. For above choice of \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) and assuming \( \eta_1 > \eta_2 > \eta_3 \), the amplitude of the resulting peak is obtained as

\[
\phi^* = \frac{12B^{1/3}}{A} (\eta_1^2 - \eta_2^2 + \eta_3^2).
\] (22)

The pulse shape at the instant of soliton interaction is determined by the equation

\[
\frac{\partial^2 \phi_1}{\partial \xi^2} (0, 0) = -\frac{24}{AB^{1/3}} (\eta_1^2 - \eta_2^2 + \eta_3^2)^2 - 2(\eta_1^2 - \eta_2^2)(\eta_2^2 - \eta_3^2)).
\] (23)

Equation (23) indicates the concavity of wave profile at the strongest interaction point. The negative value of (23) indicates that the wave profile is concave downward. Therefore, the wave profile may maintain a single peak or a triple peak status at the strongest interaction point. The positive value of (23) implies that the wave profile is concave upward and it will maintain the two-peak status at \( \tau = 0 \). Equation (23) is positive or negative according to whether \((1 - R_1 + R_2)^2 - 2(1 - R_1)(R_1 - R_2)\) is negative or positive, where \( R_1 = \frac{\eta_1}{\eta_3} < 1 \) and \( R_2 = \frac{\eta_2}{\eta_3} < 1 \) with \( R_1 > R_2 \). Figure 1 shows the region of the positive and negative values of \( \frac{\partial \phi_1}{\partial \xi^2} (0, 0) \).

Figure 2 shows the interaction process of the three-soliton solution of (12) for \( R_1 = 0.38 \) and \( R_2 = 0.08 \) with \( \kappa = 2.1, \mu = 0.1, \eta_1 = 0.4 \). It is observed that for significant difference between the soliton amplitude ratios \( R_1 \) and \( R_2 \) and maintaining \( \frac{\partial \phi_1}{\partial \xi^2} (0, 0) < 0 \), the three solitons interact simultaneously and merge to a single peak at \( \xi = 0, \tau = 0 \).

Figure 3 presents the interaction process of three solitons for \( R_1 = 0.38 \) and \( R_2 = 0.35 \) with \( \kappa = 2.1, \mu = 0.1, \eta_1 = 0.4 \). It is observed that when \( R_1 \) and \( R_2 \) do not differ significantly, and maintaining the negative sign of \( \frac{\partial \phi_1}{\partial \xi^2} (0, 0) \), then the two fastest solitons interact first and overtake the slowest soliton.

The three-soliton interaction process is presented in Figure 4 for \( R_1 = 0.85 \) and \( R_2 = 0.17 \) with \( \kappa = 2.1, \mu = 0.1, \eta_1 = 0.4 \), for which (23) is positive. Here, the two slowest solitons interact first and both are overtaken by the fastest soliton.

The interaction of a large number of propagating waves in conservative systems leads to a fast changing of the wave pattern. In such a scenario, the statistical theory is suitable for describing the wave field. Such a theory is called weak wave turbulence. Soliton turbulence is usually energetically harder than the ordinary weakly turbulent plasma description. The conservation laws are very important in the study of turbulence. For a scalar partial differential equation with two independent variables \( x, t \) and a single dependent variable \( u \), the conservation law can be written as

\[
\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0,
\] (24)

where \( T \) and \( X \) are the ‘conserved density’ and the ‘flux’, respectively, and both are polynomials of the solution \( u \) and its derivatives with respect to the space variable \( x [62] \). If both \( T \) and \( \frac{\partial X}{\partial x} \) are integrable over the interval \( (−\infty, \infty) \), then on the assumption that \( X \to 0 \) as \( |X| \to \infty \), (24) can be integrated to give

\[
\frac{d}{dt} \left( \int_{−\infty}^{\infty} Tdx \right) = 0,
\]

\[
\Rightarrow \int_{−\infty}^{\infty} Tdx = \text{constant}.
\] (25)

Equation (25) is invariant with time and is termed the ‘invariant of motion’ or ‘constant of motion’ [63, 64].
It is known that the KdV equation forms a completely integrable Hamiltonian system and, hence, possesses an infinite numbers of conserved quantities [31–33]. The first four conservation laws of the KdV equation (12) are as follows:

\[ I_1 = \int_{-\infty}^{\infty} \phi_1(\xi, \tau) d\xi, \quad (26) \]

\[ I_2 = \int_{-\infty}^{\infty} \phi_2(\xi, \tau) d\xi, \quad (27) \]

\[ I_3 = \int_{-\infty}^{\infty} \left[ \phi_3^4(\xi, \tau) - \frac{3B}{A} \left( \frac{\partial \phi_1}{\partial \xi} \right)^2 \right] d\xi, \quad (28) \]

\[ I_4 = \int_{-\infty}^{\infty} \left[ \phi_4^4(\xi, \tau) - \frac{12B}{A} \phi_1 \left( \frac{\partial \phi_1}{\partial \xi} \right)^2 + \frac{36B^2}{5A^2} \left( \frac{\partial^2 \phi_1}{\partial \xi^2} \right)^2 \right] d\xi. \quad (29) \]
The first three integrals (26)–(28) correspond to the conservation of mass, momentum, and energy, respectively. These integrals are conserved in the process of the wave field evolution, and using the non-interacting solitons (21), the analytical calculation of \( I_1, I_2, I_3, \) and \( I_4 \) are as follows:

\[
I_1 = \frac{4\sqrt{3B}}{\sqrt{A}}(A_1^{1/2} + A_2^{1/2} + A_3^{1/2}), \quad (30)
\]

\[
I_2 = \frac{8\sqrt{3B}}{3\sqrt{A}}(A_1^{3/2} + A_2^{3/2} + A_3^{3/2}), \quad (31)
\]

\[
I_3 = \frac{8\sqrt{3B}}{5\sqrt{A}}(A_1^{5/2} + A_2^{5/2} + A_3^{5/2}), \quad (32)
\]

\[
I_4 = \frac{32\sqrt{3B}}{35\sqrt{A}}(A_1^{7/2} + A_2^{7/2} + A_3^{7/2}), \quad (33)
\]

From (30)–(33), it is observed that values of the integrals \( I_1, I_2, I_3, \) and \( I_4 \) increase as the amplitudes of the interacting solitons increase.

It is well known that the dynamics of multiple solitons is affected significantly by the mutual interactions of the solitons [41, 42]. To understand the effect of three-soliton interaction on the statistical moments of the random wave field, the following integrals are considered:

\[
\mu_n = \int_{-\infty}^{\infty} \phi_1^2 d\xi, \quad n = 1, 2, 3, \ldots \quad (34)
\]

The integrals in (34) are related to the statistical moments of the wave field. The first integral moment (\( \mu_1 \)) and the second integral moment (\( \mu_2 \)) represent the mean and variance of the random wave field, respectively. The first and second integral moments \( \mu_1 \) and \( \mu_2 \) are the same as Kruskal’s integral \( I_1 \) and \( I_2 \), and hence they are conserved for the three-soliton solution (16), which agrees with (30) and (31) (see Fig. 5). Thus, it can be said that the mean and variance of the wave field do not get affected by the nonlinear interactions of the three solitons. The third and fourth integral moments are defined by

\[
M_3(\tau) = \frac{\mu_3}{\mu_2^{3/2}}, \quad (35)
\]

\[
M_4(\tau) = \frac{\mu_4}{\mu_2^{2}}, \quad (36)
\]

Figure 4: Interaction process of three-soliton solution of (12) for \( \kappa = 2.1, \mu = 0.1, R_1 = 0.85, R_2 = 0.17 \) with \( \eta_1 = 0.4 \).

Figure 5: Time dependence of the integrals \( I_1 \) and \( I_2 \) in the three-soliton interaction with \( \kappa = 2.1, \mu = 0.1, R_1 = 0.38, R_2 = 0.35 \) with \( \eta_1 = 0.4 \).
respectively. The third and fourth moments \( M_3(\tau) \) and \( M_4(\tau) \) characterise the skewness and kurtosis (throughout this article, the word ‘kurtosis’ refers to the normalised fourth moment and not its difference from the Gaussian value 3 – ‘excess kurtosis’) of the random wave field. The statistical measure of the vertical asymmetry of the wave field is given by the skewness \( M_3(\tau) \), and kurtosis provides information on the probability of occurrence of extreme waves [65]. The structures of \( \mu_3 \) and \( \mu_4 \) are different from those of Kruskal’s integrals \( I_1 \) and \( I_6 \). Therefore, the skewness \( M_3(\tau) \) and kurtosis \( M_4(\tau) \) will not be conserved in the dominant interaction region. Figure 6 presents the kurtosis and skewness are always positive, as all the three solitons are due to the three-soliton interaction. Here, the skewness and kurtosis decrease in the dominant interaction region. Figure 6 also shows that the skewness \( M_3(\tau) \) and kurtosis \( M_4(\tau) \) deviates by about 8.03 % from \( M_0^3 \) calculated using (16) and 7.25 % from \( M_0^4 \) due to the three-soliton interaction. Figure 6 shows that the skewness and kurtosis are always positive, as all the three solitons are positive.

5 Effect of the Plasma Parameters on Soliton Turbulence

It has been observed that the soliton amplitude increases as the parameters \( \kappa \) and \( \mu \) increase. To show the effect of the plasma parameters \( \kappa \) and \( \mu \) on soliton turbulence, the maximum deviation \( M_i^\kappa = M_i^{\max} - M_i^{\min} \) \((i = 3, 4)\) and the relative deviation \( M_i^{\kappa \ast} = \frac{M_i^{\max} - M_i^{\min}}{M_i^0} \) \((i = 3, 4)\) in third and fourth moments as a function of \( \kappa \) and \( \mu \) are calculated (Figs. 7–10). From Figure 7, it is observed that an increase in the value of \( \kappa \) decreases the maximum deviation in third and fourth moments. Also, the maximum deviation in third and fourth moments due to the interaction of solitons decreases as the parameter \( \mu \) increases (Figure 8). Therefore, the spectral index \( \kappa \) and the parameter \( \mu \) (unperturbed dust-to-ion ratio) have strong influence on the soliton turbulence for DIA waves in a dusty plasma system.
helpful to understand the nonlinear features of the three-soliton solutions in Earth’s mesosphere, cometary tails, and Jupiter’s magnetosphere.

6 Conclusions

In this work, we studied the propagation and interaction of three solitons within the framework of the KdV equation in an unmagnetised, collision-less dusty plasma consisting of $\kappa$ distributed electrons. The KdV equation was derived using RPT, and by applying Hirota’s bilinear method the three-soliton solution of the KdV equation was obtained. The cancavity of the resulting peak was studied and, depending on amplitude ratios of the solitons, three different types of soliton interaction were shown. The skewness and kurtosis of the random wave field, which are crucial in turbulence theory, were calculated. It was observed that the skewness and kurtosis decrease in the soliton interaction region. Also, it was observed that the plasma parameters $\kappa$ and $\mu$ have great influence on the skewness and kurtosis of the random wave field for DIA waves. The results may be

References