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Bifurcation Dynamics and Pattern Formation of a Three-Species Food Chain System with Discrete Spatiotemporal Variables

Abstract: The bifurcation dynamics and pattern formation of a discrete-time three-species food chain system with Beddington–DeAngelis functional response are investigated. Via applying the centre manifold theorem and bifurcation theorems, the occurrence conditions for flip bifurcation and Neimark–Sacker bifurcation as well as Turing instability are determined. Numerical simulations verify the theoretical results and reveal many interesting dynamic behaviours. The flip bifurcation and the Neimark–Sacker bifurcation both induce routes to chaos, on which we find period-doubling cascades, invariant curves, chaotic attractors, sub–Neimark–Sacker bifurcation, sub–flip bifurcation, chaotic interior crisis, sub–period-doubling cascade, periodic windows, sub–periodic windows, and various periodic behaviours. Moreover, the food chain system exhibits various self-organized patterns, including regular and irregular patterns of stripes, labyrinth, and spiral waves, suggesting the populations can coexist in space as many spatiotemporal structures. These analysis and results provide a new perspective into the complex dynamics of discrete food chain systems.

Keywords: Bifurcation; Chaos; Discrete System; Food Chain; Turing Instability.

1 Introduction

The dynamics of food web has become an important branch of ecology and biology in recent decades. In nature, almost all organisms are embedded in food webs through energy flow and material flow. The analysis of predator–prey interactions by means of mathematical modelling is the key to the study of food web [1–3]. As the food chain of an ecosystem is generally more than two trophic levels, many important dynamic behaviours that may be critical to community function may arise under the interactions among three or more species [4]. A great deal of research works demonstrated that theoretical studies on food webs enable us to capture and profoundly comprehend the mechanisms of stability and diversity of ecosystems [4–6].

Early studies of food web dynamics mainly focused upon the stability or persistence of the systems [7, 8]. In 1991, Hastings and Powell [4] initially proposed a food chain model and demonstrated the occurrence of long-term chaotic dynamics, stirring up the study of chaotic dynamics of food webs. Chaotic dynamics explains the existence of disordered state in food webs and the dynamic transition between order and disorder in ecosystems. Moreover, Klebanoff and Hastings [9] further demonstrated that the emergence of chaos often takes place in a class of food web models, rather than only in a specific model. This is particularly important in ecology, where the description on dynamic behaviours of food webs may be greatly dependent upon the exact form of particular models.

The basic composition of a food web is the predator–prey interactions between two different trophic species. “Nutrition function” or “functional response,” describing the number of prey consumed by each predator per unit time, is a crucial element in modelling the predator–prey interactions and in determining the stability and the bifurcation dynamics of the food chain system [10]. Previous studies have shown that the dynamics of the food chain are greatly affected when the functional responses change between different trophic species. For example, based on the Hastings–Powell food chain model, Naji and Balasim...
[11] developed a three-species food chain model with the Beddington–DeAngelis functional response, obtained different persistence conditions of the food chain, and demonstrated different chaotic dynamics under realistic and biologically feasible parameter values.

In the studies of the dynamics of food webs, most of the dynamic models are time-continuous. In contrast to continuous models, time-discrete models provide another important way for revealing the nonlinear dynamic characteristics of food webs. Xu et al. [12] found the existence of periodic solutions in a discrete-time three-species Lotka–Volterra food-chain system. Dai et al. [13] proved the permanence of a Michaelis–Menten-type discrete three-species food chain model with delay under appropriate conditions. Fundinger et al. [14] revealed the presence of multiple attractors in a discrete three-species food web model. Matouk et al. [15] investigated the discretized version of fractional-order Hastings–Powell food chain model and obtained rich dynamic behaviours such as bifurcations and chaos [13]. Chen [16] and Li and Lu [17] provided the persistence of an n-dimensional discrete food web model and the existence of periodic solutions.

In existing approaches, there are only a few data on applying discrete models to study the food webs. In order to better understand the complex dynamic behaviours in discrete food chain systems, the present research launches an investigation on a discrete food chain, which is the simplest food web, with the consideration of time- and space-discrete variables. Via utilizing the centre manifold theorem and bifurcation theorems, the bifurcation dynamics of the food chain system is discovered. Moreover, the emergence of population dispersal allows a further penetration into the spatial distribution of different species. When Turing instability occurs, the discrete food chain system may present Turing pattern formation, i.e. the self-organization of regular and irregular patterns, revealing various coexistent ways for the species in the food chain. However, the pattern formation of a food web is still seldom studied by previous researchers.

The structure of this research is organized as follows. Section 2 presents the spatiotemporally discrete food chain model with Beddington–DeAngelis functional response and gives the stability analysis. Section 3 provides the bifurcation analyses and Turing instability analysis, determining the occurrence conditions for flip bifurcation, Neimark–Sacker bifurcation, and pattern formation. Section 4 verifies the theoretical results obtained in the previous section via numerical simulations and exhibits bifurcation diagrams and spatial patterns. The final section describes the concluding remarks.

2 Three-Species Food Chain System with Discrete Spatiotemporal Variables

2.1 Development of the Discrete Food Chain Model

The food chain system is composed of three species, denoted as X, Y, and Z. Z represents the top predator, which relies entirely on one food as a nutritional supplement, predating on the intermediate consumer Y; Y represents the intermediate consumer, and predates only on X; X indicates the basic prey, which is the lowest nutrient level species of the food chain. Based on previous research [11], spatial dispersal of the species is introduced into the food chain system, and therefore the spatiotemporal food chain model can be expressed as

\[
\frac{dX}{dT} = RX(1 - X/K) - F_1(X, Y)Y + \eta_1 \nabla^2 X,
\]

\[
\frac{dY}{dT} = E_1 F_1(X, Y)Y - F_2(Y, Z)Z - D_1 Y + \eta_2 \nabla^2 Y,
\]

\[
\frac{dZ}{dT} = E_2 F_2(Y, Z)Z - D_2 Z + \eta_3 \nabla^2 Z,
\]

where \( T \) indicates time; \( R \) and \( K \) are the intrinsic growth rate and the carrying capacity of the prey \( X \); \( E_1 \) and \( E_2 \) are the conversion rates of predators \( Y \) and \( Z \) on predating \( X \) and \( Y \), respectively; \( D_1 \) and \( D_2 \) are the mortality rates of the predators \( Y \) and \( Z \); \( \eta_i \) represents the diffusion coefficient of the corresponding species, and \( \nabla^2 \) represents the Laplacian operator of the two-dimensional space; \( F_1 \) and \( F_2 \) are the functional response. Here, we adopt the Beddington–DeAngelis type functional response in the food chain, i.e. for \( i = (1, 2) \),

\[
F_i(U, V) = A_i U / (B_i V + U + C_i),
\]

where \( A_i \), \( B_i \), and \( C_i \) are the parameters used in the functional response. \( A_i \) describes the effects of capture rate and handling time; \( B_i \) measures the magnitude of the mutual interference between individuals of the specialist predators; \( C_i \) is the half-saturation constant in the functional response. The Beddington–DeAngelis functional response is one of the classic functional responses widely used in predator–prey interactions [18, 19]. With the application of such functional response, a large variety of different types of long-term behaviours, including various bifurcations and stationary spatial patterns, are observed in predator–prey systems and food web systems [20].
For convenient analysis on the food chain, we need to reduce the number of parameters. Let $a_1 = A_1 E_1 K / D_1$, $b_1 = B_1 E_1$, $c_1 = C_1 / K$, $a_2 = A_2 E_2 K / D_1$, $b_2 = B_2 E_2$, $c_2 = C_2 / E_2 K$, $d = D_2 / D_1$, $t = T D_1$, $d_1 = \eta_1 / D_1$, $d_2 = \eta_2 / D_1$, $d_3 = \eta_3 / D_1$, then (1) can be simplified as

$$
\begin{align*}
\frac{dx}{dt} &= rx(1-x) - \frac{a_1 xy}{b_1 y + x + c_1} + d_1 \nabla^2 x, \\
\frac{dy}{dt} &= \frac{a_1 xy}{b_1 y + x + c_1} - a_2 y z - y + d_2 \nabla^2 y, \\
\frac{dz}{dt} &= \frac{a_2 y z}{b_2 z + y + c_2} - d_3 \nabla^2 z.
\end{align*}
(3)
$$

Based on the discretization of model system (3), a spatiotemporally discrete food chain model is further developed. In this research, the discrete food chain model is given by a coupled map lattice. For making discretization, we assume a time interval $\tau$ and a space interval $h$, which represent the time scale and spatial scale where the populations in the food chain interact with each other. A two-dimensional rectangular lattice that includes $n \times n$ grid cell is given, representing the space domain. Each grid cell means one spatial patch and is ascribed to three state variables, $x_{i,j,t}$, $y_{i,j,t}$, and $z_{i,j,t}$ $(i, j \in \{1, 2, 3, \ldots, n\}$ and $t \in N)$, which are defined in the discrete space and time, describing the corresponding nutrient level species densities in $(i, j)$ site and at the $t$th iteration. If we give an initial time $t_0$, then the time at the $t$th iteration is $t_0 + t \tau$. According to the framework of the coupled map lattice described in a previous approach [21], the dynamics of the spatiotemporally discrete predator–prey at each discrete step from $t$ to $t + 1$ iteration consists of two distinctly different stages, the dispersal stage and the “reaction” stage. The dispersal stage can be obtained by discretizing the spatial terms of (3), described by the following equations:

$$
\begin{align*}
x'_{i,j,t+1} &= x_{i,j,t} + \frac{\tau}{h^2} d_1 \nabla^2 x_{i,j,t}, \\
y'_{i,j,t+1} &= y_{i,j,t} + \frac{\tau}{h^2} d_2 \nabla^2 y_{i,j,t}, \\
z'_{i,j,t+1} &= z_{i,j,t} + \frac{\tau}{h^2} d_3 \nabla^2 z_{i,j,t},
\end{align*}
(4)
$$

where

$$
\begin{align*}
\nabla^2 x_{i,j,t} &= x_{i+1,j,t} + x_{i-1,j,t} + x_{i,j+1,t} + x_{i,j-1,t} - 4x_{i,j,t}, \\
\nabla^2 y_{i,j,t} &= y_{i+1,j,t} + y_{i-1,j,t} + y_{i,j+1,t} + y_{i,j-1,t} - 4y_{i,j,t}, \\
\nabla^2 z_{i,j,t} &= z_{i+1,j,t} + z_{i-1,j,t} + z_{i,j+1,t} + z_{i,j-1,t} - 4z_{i,j,t}.
\end{align*}
(5)
$$

Via discretizing the nonspatial part of (3), the reaction stage described in the spatiotemporal discrete food chain model can be expressed by the following equations:

$$
\begin{align*}
x_{i,j,t+1} &= f_1 (x'_{i,j,t}, y'_{i,j,t}, z'_{i,j,t}), \\
y_{i,j,t+1} &= g_1 (x'_{i,j,t}, y'_{i,j,t}, z'_{i,j,t}), \\
z_{i,j,t+1} &= h_1 (x'_{i,j,t}, y'_{i,j,t}, z'_{i,j,t}),
\end{align*}
(6)
$$

where $f_1$, $g_1$, and $h_1$ are the reaction functions determined by local interspecific and intraspecific interactions of the populations in the food chain, given by

$$
\begin{align*}
f_1(x, y, z) &= x + \tau \left( rx(1-x) - \frac{a_1 xy}{b_1 y + x + c_1} \right), \\
g_1(x, y, z) &= y + \tau \left( \frac{a_1 xy}{b_1 y + x + c_1} - \frac{a_2 y z}{b_2 z + y + c_2} - y \right), \\
h_1(x, y, z) &= z + \tau \left( \frac{a_2 y z}{b_2 z + y + c_2} - dz \right).
\end{align*}
(7)
$$

The coupled map lattice model described by (4–7) expresses a three-species food chain system with discrete spatiotemporal variables. All the values of the parameters in the spatiotemporally discrete food chain system are assumed to be positive. Simultaneously, the values of $x_{i,j,t}$, $y_{i,j,t}$, and $z_{i,j,t}$ are restricted to be nonnegative.

### 2.2 Linear Stability Analysis

When $\nabla^2 x_{i,j,t} = 0$, $\nabla^2 y_{i,j,t} = 0$, and $\nabla^2 z_{i,j,t} = 0$, the spatiotemporally discrete food chain system degenerates to a discrete-time system. To conveniently perform linear stability analysis, the discrete-time system is rewritten as a map:

$$
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\rightarrow
\begin{pmatrix}
x + \tau \left( rx(1-x) - \frac{a_1 xy}{b_1 y + x + c_1} \right) \\
y + \tau \left( \frac{a_1 xy}{b_1 y + x + c_1} - \frac{a_2 y z}{b_2 z + y + c_2} - y \right) \\
z + \tau \left( \frac{a_2 y z}{b_2 z + y + c_2} - dz \right)
\end{pmatrix}.
(8)
$$

According to the definition of mapping fixed points [22], four nonnegative fixed points of discrete-time food chain system are obtained, i.e.

$$
\begin{align*}
E_0: (0, 0, 0), \\
E_1: (1, 0, 0), \\
E_2: (x_2, y_2, 0), \\
E^* : (x^*, y^*, z^*),
\end{align*}
(9)
$$
in which
\begin{align*}
x_2 &= \frac{1}{2b_1r}(1 + a_1 + b_1r + \Delta), \\
y_2 &= \frac{1}{2b_1}(1 - a_1 + b_1r + \Delta) \\
\Delta &= \sqrt{4b_1c_1r + (-1 - a_1 - b_1r)^2}, \\
y^* &= \frac{r(1 - x^*)(c_1 + x^*)}{a_1 - b_1r + b_1rx^*}, \quad z^* = \frac{(a_2 - d)y^* - c_2d}{b_2d}, \quad (10)
\end{align*}
and \(x^*\) is one real positive root of the following cubic equation:
\begin{align*}
b_1r^2x^3 + \left( a_1r - 2b_1r^2 - e_1r \right)x^2 \\
+ \left( b_1r^2 - a_1r + e_1r(1 - c_1) - e_2b_1r \right)x \\
+ c_1e_1r - e_2a_1 + e_2b_1r = 0, \quad (11)
\end{align*}
where \(e_1 = (a_2 + b_2 - d)/b_2, \quad e_2 = c_2d/b_2.

As per the method of solving cubic equation, the sufficient conditions for positive fixed point of map (8) can be determined. If either of the following conditions holds,
\begin{align*}
\text{Case 1:} \quad & b_1r^3e_1 + b_2^2r^5e_1 \left( 2b_1r^2 + re_1 - a_1r \right) > 0, \\
& 0 < c_1 < \min \left( c_1^*, \frac{e_2(a_1 - b_1r)}{e_1r} \right), \quad (12) \\
\text{Case 2:} \quad & b_1r^3e_1 + b_2^2r^5e_1 \left( 2b_1r^2 + re_1 - a_1r \right) < 0, \\
& 0 < c_1^* < c_1, \quad (13)
\end{align*}
map (8) has only one positive fixed point. In the above conditions, the value of \(c_1^*\) is calculated as
\begin{align*}
c_1^* &= \frac{1}{9b_1re_1(b_1r + e_1 - a_1)} \left\{ 2a_1^3 - 3a_1^2(b_1r + 2e_1) \\
+ a_1 \left[ 6b_1re_1 + 6e_1^2 - 3b_1^2r^2 \right] + b_1r^2(2r - 9e_2) \\
+ 3b_1^2re_1(r - 3e_2) - 3b_1re_1^2 - 2e_1^3 \right\}. \quad (14)
\end{align*}

In order to investigate the local dynamics of the food chain around the fixed points, the stability of various fixed points is analysed via the method of Jacobian matrix. The Jacobian matrix associated to map (8) at any point can be described by
\begin{align*}
J(x, y, z) &= \begin{pmatrix}
a_{11} & a_{12} & 0 \\
a_{21} & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{pmatrix}, \quad (15)
\end{align*}
where
\begin{align*}
a_{11} &= 1 + \tau \left( r(1 - 2x) - \frac{(b_1y + c_1)a_1y}{(b_1y + x + c_1)^2} \right), \\
a_{12} &= -\tau \left( x + c_1 \right) a_1x \frac{1}{(b_1y + x + c_1)^2}, \\
a_{21} &= \tau \left( b_1y + c_1a_1y \right) \frac{1}{(b_1y + x + c_1)^2}, \\
a_{22} &= 1 + \tau \left( \frac{(x + c_1)a_1x}{(b_1y + x + c_1)^2} - \frac{(b_2z + c_2)a_2z}{(b_2z + y + c_2)^2} - 1 \right), \\
a_{23} &= -\tau \left( \frac{(y + c_2)a_2y}{(b_2z + y + c_2)^2} \right), \\
a_{32} &= \tau \left( \frac{(b_2y + c_2)a_2z}{(b_2z + y + c_2)^2} \right), \\
a_{33} &= 1 + \tau \left( \frac{(y + c_2)a_2y}{(b_2z + y + c_2)^2} - d \right). \quad (16)
\end{align*}

We substitute the value of each fixed point into matrix (15) and calculate the three eigenvalues, \(\lambda_1, \lambda_2, \) and \(\lambda_3\). If one of the three eigenvalues satisfies \(|\lambda_i| > 1 \quad (i = 1, 2, \) or 3\), then the corresponding fixed point is unstable; if the three eigenvalues all satisfy \(|\lambda_i| < 1 \quad (i = 1, 2, \) and 3\), then the corresponding fixed point is stable.

For the fixed point \(E_0\), it is obvious that \(\lambda_1 = 1 + \tau r, \lambda_2 = 1 - \tau, \lambda_3 = 1 - dr\). As all parameters of the system are positive and therefore \(\lambda_1 > 1\), we know that \(E_0\) is unstable for any parametric condition. For the fixed point \(E_1\), we have
\begin{align*}
J(E_1) &= \begin{pmatrix}
1 - \tau r & -\tau \frac{a_1}{1 + c_1} & 0 \\
0 & 1 + \tau \left( \frac{a_1}{1 + c_1} - 1 \right) & 0 \\
0 & 0 & 1 - dr
\end{pmatrix}, \quad (17)
\end{align*}
which has three eigenvalues as \(\lambda_1 = 1 - \tau r, \lambda_2 = 1 + \tau (a_1/(1 + c_1) - 1)\), and \(\lambda_3 = 1 - dr\). Therefore according to above stability criterion, the conditions for that \(E_1\) is stable are determined as \(a_1 - c_1 - 1 < 0 \) and \(0 < \tau < \min \left( \frac{2}{3}, \frac{2}{3}, \frac{2(c_1 + 1)}{1 + c_1 - a_1} \right)\).

Similarly, the three eigenvalues corresponding to the fixed point \(E_2\) are \(\lambda_{1,2} = \frac{1}{2}(a_{11} + a_{22} \pm \sqrt{a_{11}^2 + 4a_{12}a_{21} - 2a_{11}a_{22} + a_{22}^2})\) and \(\lambda_3 = a_{33}\), where \(x, y, \) and \(z\) in \(a_{ij}\) take the values in those of \(E_2\). Thus, when \(a_{11}a_{22} - a_{12}a_{21} < 1, a_{12}a_{21} - a_{21}a_{12} < 1 < a_{11} + a_{22} < 1 + a_{11}a_{22} - a_{12}a_{21} \), and \(-1 < a_{33} < 1, E_2\) is stable. Likewise, substituting the value of \(E^*\) into \(J(x, y, z)\) yields the following characteristic equation
\begin{align*}
\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0. \quad (18)
\end{align*}

According to research [23], when parameters satisfy \(1 + C_1 + C_2 + C_3 > 0, 3 + C_1 - C_2 - 3C_3 > 0, 1 - C_2 + \)}
$C_3(C_1 - C_2) > 0$, and $1 - C_1 + C_2 - C_3 > 0$, $E^*$ is stable, where $C_1 = -(a_{11} + a_{22} + a_{33})$, $C_2 = a_{11}a_{22} + a_{11}a_{33} + a_{12}a_{21} - a_{23}a_{32}$, $C_3 = a_{11}a_{23}a_{31} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}$, and $x$, $y$, and $z$ in $a_{ij}$ take the values in those of $E^*$.

Based on the linear stability analysis, the following results can be obtained for the discrete-time food chain:

1. As $E_0$ is unstable regardless of parameter variation, the food chain system always retains the existence of population(s);
2. Around the stable fixed point $E_1$, the discrete-time food chain degenerates to the classic discrete-time Logistic system;
3. Around the stable fixed point $E_2$, the discrete-time food chain degenerates to the discrete-time Beddington–DeAngelis predator–prey system; and
4. When $E^*$ is stable, the discrete-time food chain reaches a state where the three species coexist and keeps at stationary level of population densities.

# 3 Bifurcation Analyses

## 3.1 Flip Bifurcation

Parameter $r$ is chosen as the bifurcation parameter. When flip bifurcation occurs around the fixed point $E^*$, the stability of $E^*$ will change with the variation of bifurcation parameter. Simultaneously, the discrete system may generate a new period-2 orbit. According to the flip bifurcation theorem, one of the eigenvalues of the fixed point at the bifurcation point equals $-1$. Let $\lambda_1 = -1$ and substituting it into (18), then we have

$$C_1(r) - C_2(r) - C_3(r) = 1. \quad (19)$$

We separate the bifurcation parameter $r$ in the Jacobian matrix (15), i.e.

$$J(x, y, z) = \begin{pmatrix} 1 + \tau m_{11} & \tau m_{12} & 0 \\ \tau m_{12} & 1 + \tau m_{22} & \tau m_{23} \\ 0 & \tau m_{32} & 1 + \tau m_{33} \end{pmatrix}, \quad (20)$$

in which

$$m_{11} = r(1 - 2x^* - \frac{(b_1y^* + c_1)a_1y^*}{b_1y^* + x^* + c_1})^2,$$

$$m_{12} = -\frac{a_1x^*(x^* + c_1)}{(b_1y^* + x^* + c_1)^2}, \quad m_{21} = \frac{a_1y^*(b_1y^* + c_1)}{(b_1y^* + x^* + c_1)^2},$$

$$m_{22} = \frac{a_1x^*(x^* + c_1)}{(b_1y^* + x^* + c_1)^2} - \frac{a_2z^*(b_2z^* + c_2)}{(b_2z^* + y^* + c_2)^2} - 1,$$

$$m_{23} = -\frac{a_2y^*(y^* + c_2)}{(b_2z^* + y^* + c_2)^2}, \quad m_{32} = \frac{a_2z^*(b_2z^* + c_2)}{(b_2z^* + y^* + c_2)^2}, \quad m_{33} = \frac{a_2y^*(y^* + c_2)}{(b_2z^* + y^* + c_2)^2} - d. \quad (21)$$

Combining (19) and (20), we get

$$A_1r^3 + A_2r^2 + A_3r - 8 = 0, \quad (22)$$

where

$$A_1 = m_{11}m_{32}m_{33} - m_{12}m_{21}m_{33} = 8m_{32}m_{12},$$

$$A_2 = 2(m_{12}m_{21} + m_{23}m_{31} - m_{22}m_{33}) - m_{11}m_{22} - m_{11}m_{33},$$

$$A_3 = -4(m_{11} + m_{22} + m_{33}). \quad (23)$$

When the conditions

$$27 \left( \frac{2A_2^3}{27A_1^4} - \frac{8}{A_1} - A_2A_3 \right)^2 + 4 \left( \frac{A_3}{A_1} - \frac{A_3}{3A_1} \right)^3 > 0, \quad (24)$$

or

$$27 \left( \frac{2A_2^3}{27A_1^4} - \frac{8}{A_1} - A_2A_3 \right)^2 + 4 \left( \frac{A_3}{A_1} - \frac{A_3}{3A_1} \right)^3 \leq 0 \quad (25)$$

are established, (22) has at least one positive real root. Via solving (22), the flip bifurcation point is determined as the minimum positive real root of (22). In this research, we denote this flip bifurcation point as $r^*$. With the condition of $r = r^*$, the three eigenvalues of $J(E^*)$ are given as follows:

(a) if $(1 - C_1(r^*))^2 - 4C_3(r^*) \geq 0$ and $C_3 \neq -C_1, C_1 - 2$, we have

$$\lambda_1 = -1,$$

$$\lambda_{2,3} = \frac{1 - C_1(r^*) \pm \sqrt{(1 - C_1(r^*))^2 - 4C_3(r^*)}}{2}; \quad (26)$$

(b) if $(1 - C_1(r^*))^2 - 4C_3(r^*) < 0$ and $C_3 \neq 1$, we have

$$\lambda_1 = -1,$$

$$\lambda_{2,3} = \frac{1 - C_1(r^*) \pm \sqrt{4C_3(r^*) - (1 - C_1(r^*))^2}}{2}i. \quad (27)$$
It would be convenient to analyse the system dynamics if we transform the fixed point \( E^* \) of map (8) to the origin. Let
\[
u = x - x^*, \quad \nu = y - y^*, \quad \nu = z - z^*, \quad \bar{\tau} = \tau - \tau^*,
\]
and considering \( \bar{\tau} \) as a new dependent variable. Applying the Taylor expansion around \((x^*, y^*, z^*, \tau^*)\), then map (8) is transformed into
\[
\begin{pmatrix}
u \\
\nu \\
\nu
\end{pmatrix}
\rightarrow J(E^*)
\begin{pmatrix}
u \\
\nu \\
\nu
\end{pmatrix}
+ \begin{pmatrix}
f(u, v, w, \bar{\tau}) \\
g(u, v, w, \bar{\tau}) \\
h(u, v, w, \bar{\tau})
\end{pmatrix},
\]
where
\[
f(u, v, w, \bar{\tau}) = \sum_{2 \leq h + j + 3 \leq 3}
\begin{pmatrix}
\tilde{f}_{j_1 j_2 j_3}
\end{pmatrix}
\begin{pmatrix}
\tilde{u}^j \tilde{v}^j \tilde{w}^h \tilde{\bar{\tau}}^j + O(\tilde{\zeta}^4),
\end{pmatrix}
\]
\[
g(u, v, w, \bar{\tau}) = \sum_{2 \leq h + j + 3 \leq 3}
\begin{pmatrix}
\tilde{g}_{j_1 j_2 j_3}
\end{pmatrix}
\begin{pmatrix}
\tilde{u}^j \tilde{v}^j \tilde{w}^h \tilde{\bar{\tau}}^j + O(\tilde{\zeta}^4),
\end{pmatrix}
\]
\[
h(u, v, w, \bar{\tau}) = \sum_{2 \leq h + j + 3 \leq 3}
\begin{pmatrix}
\tilde{h}_{j_1 j_2 j_3}
\end{pmatrix}
\begin{pmatrix}
\tilde{u}^j \tilde{v}^j \tilde{w}^h \tilde{\bar{\tau}}^j + O(\tilde{\zeta}^4),
\end{pmatrix}
\]

The coefficient expressions in the Taylor series are
\[
\begin{pmatrix}
\tilde{f}_{j_1 j_2 j_3} \\
\tilde{g}_{j_1 j_2 j_3} \\
\tilde{h}_{j_1 j_2 j_3}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{(j_1 + j_2 + j_3)! \partial u^{j_1} \partial v^{j_2} \partial w^{j_3} \partial \bar{\tau}^{j_4}}
\frac{1}{(j_1 + j_2 + j_3)! \partial u^{j_1} \partial v^{j_2} \partial w^{j_3} \partial \bar{\tau}^{j_4}}
\frac{1}{(j_1 + j_2 + j_3)! \partial u^{j_1} \partial v^{j_2} \partial w^{j_3} \partial \bar{\tau}^{j_4}}
\end{pmatrix},
\]
where \( \zeta = |u| + |v| + |w| + |\bar{\tau}| \) and \( O(\tilde{\zeta}^4) \) are a function with order at least four in the variables \( u, v, w, \bar{\tau} \). In all coefficients of (31), the value of \( \tau \) is given as \( \tau = \tau^* \).

As described in literature [24], an effective method for bifurcation analysis is applying the centre manifold theorem, which enables us to restrict our attention to the flow within the centre manifold. Applying the following invertible transformation
\[
\begin{pmatrix}
u \\
\nu \\
\nu
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 \\
\theta_1 & \theta_2 & \theta_3 \\
\theta_4 & \theta_5 & \theta_6
\end{pmatrix}
\begin{pmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{w}
\end{pmatrix}
\]

For \( i = 1, 2, 3 \), \( j = 4, 5, 6 \), we have
\[
\theta_i = \frac{\lambda_i - a_{11}}{a_{12}},
\]
\[
\theta_j = \frac{\lambda_j - a_{11}(a_{22} - \lambda_3) - a_{22}\lambda_3 - a_{12}a_{21}}{a_{12}a_{23}},
\]
then we obtain the normal form of map (29) as
\[
\begin{pmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{w}
\end{pmatrix}
\rightarrow \begin{pmatrix}
-1 & 0 & 0 \\
0 & \lambda_2 & \lambda_3 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{w}
\end{pmatrix}
+ \begin{pmatrix}
\tilde{f}(\tilde{u}, \tilde{v}, \tilde{w}, \bar{\tau}) \\
\tilde{g}(\tilde{u}, \tilde{v}, \tilde{w}, \bar{\tau}) \\
\tilde{h}(\tilde{u}, \tilde{v}, \tilde{w}, \bar{\tau})
\end{pmatrix},
\]
in which
\[
\begin{pmatrix}
\tilde{f}(\tilde{u}, \tilde{v}, \tilde{w}, \bar{\tau}) \\
\tilde{g}(\tilde{u}, \tilde{v}, \tilde{w}, \bar{\tau}) \\
\tilde{h}(\tilde{u}, \tilde{v}, \tilde{w}, \bar{\tau})
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 \\
\theta_1 & \theta_2 & \theta_3 \\
\theta_4 & \theta_5 & \theta_6
\end{pmatrix}^{-1}
\begin{pmatrix}
f(u, v, w, \bar{\tau}) \\
g(u, v, w, \bar{\tau}) \\
h(u, v, w, \bar{\tau})
\end{pmatrix},
\]
and
\[
\begin{pmatrix}
\tilde{f}(\tilde{u}, \tilde{v}, \tilde{w}, \bar{\tau}) \\
\tilde{g}(\tilde{u}, \tilde{v}, \tilde{w}, \bar{\tau}) \\
\tilde{h}(\tilde{u}, \tilde{v}, \tilde{w}, \bar{\tau})
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 \\
\theta_1 & \theta_2 & \theta_3 \\
\theta_4 & \theta_5 & \theta_6
\end{pmatrix}^{-1}
\begin{pmatrix}
f(u, v, w, \bar{\tau}) \\
g(u, v, w, \bar{\tau}) \\
h(u, v, w, \bar{\tau})
\end{pmatrix}.
\]
It should be noticed that the three eigenvalues of fixed points \( E^* \) considered in (34–36) are all real numbers. Actually, \( \lambda_2 \) and \( \lambda_3 \) may be conjugate complex numbers. In such case, the submatrix \( \begin{pmatrix}
\lambda_2 & 0 \\
0 & \lambda_3
\end{pmatrix} \) in map (34) is changed to be
\[
\begin{pmatrix}
\frac{\lambda_2 + \lambda_3}{2} & -\frac{\lambda_2 - \lambda_3}{2i} \\
\frac{\lambda_2 - \lambda_3}{2i} & \frac{\lambda_2 + \lambda_3}{2}
\end{pmatrix},
\]
where \( i = \sqrt{-1} \). However, the following analysis for both cases is similar, and therefore, for convenience, we just present the analysis for the case of map (34) in this research.

The centre manifold \( W^c(0, 0, 0, 0) \) of (34) at the fixed point \( (0, 0, 0, 0) \) is then determined. Applying the centre manifold theorem, a centre manifold \( W^c(0, 0, 0, 0) \) exists and can be approximately represented by the following:
\[
W^c(0, 0, 0, 0) = \{ (\tilde{u}, \tilde{v}, \tilde{w}, \bar{\tau}) \in R^4 | \tilde{v} = \gamma_1(\tilde{u}, \bar{\tau}), \tilde{w} = \gamma_2(\tilde{u}, \bar{\tau}), \gamma_1(0, 0) = 0, \gamma_2(0, 0) = 0, D\gamma_1(0, 0) = 0, D\gamma_2(0, 0) = 0 \},
\]
where \( \gamma_1(\tilde{u}, \bar{\tau}) \) and \( \gamma_2(\tilde{u}, \bar{\tau}) \) are assumed to be
\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix}
= \begin{pmatrix}
\sum_{1 \leq j + k \leq 2}
\gamma_{1j}^{jk} |\tilde{u}|^j |\bar{\tau}|^k + O\left((|\tilde{u}| + |\bar{\tau}|)^3\right) \\
\sum_{1 \leq j + k \leq 2}
\gamma_{2j}^{jk} |\tilde{u}|^j |\bar{\tau}|^k + O\left((|\tilde{u}| + |\bar{\tau}|)^3\right)
\end{pmatrix}.
\]
Via calculation, it can be determined that the values of the coefficients in (38) are
\[
\begin{align*}
\gamma_{10}^{(0)} &= \gamma_{1}^{(0)} = \gamma_{2}^{(0)} = 0, \\
\gamma_{20}^{(0)} &= \mathcal{G}^{2000}/(1 - \lambda_2), \\
\gamma_{11}^{(0)} &= -\mathcal{G}_{1001}^{1}(1 + \lambda_2), \\
\gamma_{21}^{(0)} &= \mathcal{G}_{1001}^{2}/(1 + \lambda_3), \\
\gamma_{12}^{(0)} &= -\mathcal{H}_{1001}^{1}(1 + \lambda_3).
\end{align*}
\]

Accordingly, we consider the dynamics of map (38) restricted to the centre manifold \( WC(0,0,0,0) \). This part of dynamics can be described by

\[
F: \tilde{u} \rightarrow -\tilde{u} + \omega_1 \tilde{u}^2 + \omega_2 \tilde{u} \tilde{v} + \omega_3 \tilde{u}^2 \tilde{v} + \omega_4 \tilde{u} \tilde{v}^2 + \omega_5 \tilde{u}^3 + O \left( \| \tilde{u} \| \right)^4,
\]

where

\[
\begin{align*}
\omega_1 &= \mathcal{F}^{2000}, \\
\omega_2 &= \mathcal{F}^{1001}, \\
\omega_3 &= \mathcal{F}^{2001} + \mathcal{F}^{1001}^{11} + \mathcal{F}^{1001}^{20} + \mathcal{F}^{1001}^{12}, \\
\omega_4 &= \mathcal{F}^{1002} + \mathcal{F}^{1001}^{11} + \mathcal{F}^{1001}^{12}, \\
\omega_5 &= \mathcal{F}^{2000} + \mathcal{F}^{1001}^{20} + \mathcal{F}^{1001}^{12}.
\end{align*}
\]

According to the flip bifurcation theorem described in Guckenheimer and Holmes [24], the occurrence of flip bifurcation for map (40) requires that two discriminatory quantities \( \eta_1 \) and \( \eta_2 \) are nonzero, i.e.

\[
\begin{align*}
\eta_1 &= \left( \frac{\partial^2 F}{\partial \tilde{u}^2} + \frac{1}{2} \frac{\partial^3 F}{\partial \tilde{u} \partial \tilde{v}^2} \right) \neq 0, & (\tilde{u} = 0, \tilde{v} = 0), \\
\eta_2 &= \left( \frac{1}{6} \frac{\partial^3 F}{\partial \tilde{u}^3} + \left( \frac{1}{2} \frac{\partial^2 F}{\partial \tilde{u}^2} \right)^2 \right) \neq 0, & (\tilde{u} = 0, \tilde{v} = 0).
\end{align*}
\]

Hence, if the above conditions are established, the discrete food chain system undergoes a flip bifurcation at \( E^* \). Moreover, if \( \eta_2 > 0 \), the period-2 points bifurcating from \( E^* \) are stable; if \( \eta_2 < 0 \), the bifurcating period-2 points are unstable.

### 3.2 Neimark–Sacker Bifurcation

According to the Neimark–Sacker bifurcation theorem, when map (8) undergoes Neimark–Sacker bifurcation at the fixed point \( E^* \), the pair of conjugate complex eigenvalues has module as 1, and the absolute value of the real eigenvalue is not equal to 1. This requires equation (18) to be transformed into

\[
\left( \lambda^2 + p_1 \lambda + 1 \right) \lambda - p_2 = 0,
\]

and \( p_1^2 - 4 < 0 \) as well as \( p_2 = \pm 1 \). Combining with the relationship between (19) and (43), we have

\[
\left\{ \begin{array}{l}
p_1 - p_2 = C_1, \\
1 - p_1 p_2 = C_2, \\
p_2 = -C_3.
\end{array} \right.
\]

Based on the relationship of (44), the occurrence of the Neimark–Sacker bifurcation needs to satisfy

\[
1 + C_3(r)(C_1(r) - C_3(r)) = C_2(r).
\]

Under given parametric conditions, a critical value \( r_0 \) can be calculated by (45), i.e. \( r_0 \) is the minimum positive real root of (45).

With the satisfaction of \( r = r_0 \), we translate the fixed point \( E^* \) of map (8) to the origin with the translation

\[
\begin{align*}
\tilde{u} &= x - x^*, & \tilde{v} &= y - y^*, & \tilde{w} &= z - z^*.
\end{align*}
\]

By applying Taylor expansion, map (8) is now transformed to the following new map:

\[
\begin{align*}
&u \\
v &\quad - f(E) \left( \begin{array}{c}
u \\
w \end{array} \right) + \left( \begin{array}{c} f(u, v, w, 0) \\
g(u, v, w, 0) \\
h(u, v, w, 0) \end{array} \right),
\end{align*}
\]

where

\[
\begin{align*}
f(u, v, w, 0) &= \sum_{2 \leq h + j + h \leq 3} F_{ijh} u^h v^j w^h + O(\zeta^4), \\
g(u, v, w, 0) &= \sum_{2 \leq h + j + h \leq 3} G_{ijh} u^h v^j w^h + O(\zeta^4), \\
h(u, v, w, 0) &= \sum_{2 \leq h + j + h \leq 3} H_{ijh} u^h v^j w^h + O(\zeta^4).
\end{align*}
\]

Obviously, the expansion in (48) is the same as the (30) expression described above, but with \( r = r_0 \). As the translation does not alter the stability of the fixed point, the three eigenvalues of the Jacobian matrix associated with map (47) at the origin can be written as (assume the real eigenvalue is \( \lambda_3 \))

\[
\lambda_{1,2}(r_0) = \frac{-p_1(r_0)}{2} \pm \frac{i}{2} \sqrt{4 - p_1^2(r_0)} = \alpha \pm i\beta,
\]

\[
\lambda_3(r_0) = p_2(r_0).
\]

Simultaneously, the occurrence of the Neimark–Sacker bifurcation also needs the satisfaction of the following two conditions:

\[
\varphi = \left. \frac{d\lambda_{1,2}(r)}{dr} \right|_{r=r_0} = \frac{C_1(r_0)C_3(r_0) + C_1(r_0)C_3(r_0) - 2C_2(r_0)C_3(r_0) - C_2(r_0)}{C_1(r_0)C_3(r_0) - 2C_2(r_0)} = 0,
\]
\( (\lambda_1, \lambda_2(\tau_0))^\pi \neq 1, \quad \pi = 1, 2, 3, 4. \) \hfill (51)

The condition (51) can be further simplified as
\[ C_\lambda(\tau_0) - C_\lambda(\tau_0) \neq 0, 1. \] \hfill (52)

Under the satisfaction of the above conditions, we now investigate the norm form of map (47). Via the transformation
\[
\begin{pmatrix}
  u \\
  v \\
  w
\end{pmatrix} \rightarrow \begin{pmatrix}
  1 & 0 & 1 \\
  \delta_1 & \delta_2 & \delta_3 \\
  \delta_4 & \delta_5 & \delta_6
\end{pmatrix} \begin{pmatrix}
  \tilde{u} \\
  \tilde{v} \\
  \tilde{w}
\end{pmatrix},
\]
where
\[
\delta_1 = \frac{\alpha - a_{11}}{a_{12}}, \quad \delta_2 = -\frac{\beta}{a_{12}}, \quad \delta_3 = \frac{\lambda_3 - a_{11}}{a_{12}},
\delta_4 = \frac{(\alpha - a_{11})(\alpha - a_{22}) - \beta^2 - a_{12}a_{21}}{a_{12}a_{23}},
\delta_5 = -\frac{\beta}{a_{12}a_{23}}(2\alpha - a_{11} - a_{22}),
\delta_6 = \frac{(\lambda_1 - a_{11})(\lambda_3 - a_{22}) - a_{12}a_{21}}{a_{12}a_{23}},
\]
map (47) is changed to
\[
\begin{pmatrix}
  \tilde{u} \\
  \tilde{v} \\
  \tilde{w}
\end{pmatrix} \rightarrow \begin{pmatrix}
  \alpha - \beta & 0 & 0 \\
  \beta & \alpha & 0 \\
  0 & 0 & \lambda_3
\end{pmatrix} \begin{pmatrix}
  \tilde{u} \\
  \tilde{v} \\
  \tilde{w}
\end{pmatrix} + \begin{pmatrix}
  \bar{f}(\tilde{u}, \tilde{v}, \tilde{w}) \\
  \bar{g}(\tilde{u}, \tilde{v}, \tilde{w}) \\
  \bar{h}(\tilde{u}, \tilde{v}, \tilde{w})
\end{pmatrix},
\]
where
\[
\begin{pmatrix}
  \bar{f}(\tilde{u}, \tilde{v}, \tilde{w}) \\
  \bar{g}(\tilde{u}, \tilde{v}, \tilde{w}) \\
  \bar{h}(\tilde{u}, \tilde{v}, \tilde{w})
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & 1 \\
  \delta_1 & \delta_2 & \delta_3 \\
  \delta_4 & \delta_5 & \delta_6
\end{pmatrix}^{-1} \begin{pmatrix}
  f(u, v, w, 0) \\
  g(u, v, w, 0) \\
  h(u, v, w, 0)
\end{pmatrix},
\]
and
\[
\bar{f}(\tilde{u}, \tilde{v}, \tilde{w}) = \sum_{2 \leq h + j + k \leq 3} \bar{f}_{hjkh} \tilde{u}^h\tilde{v}^j\tilde{w}^k + O(\chi^4),
\]
\[
\bar{g}(\tilde{u}, \tilde{v}, \tilde{w}) = \sum_{2 \leq h + j + k \leq 3} \bar{g}_{hjkh} \tilde{u}^h\tilde{v}^j\tilde{w}^k + O(\chi^4),
\]
\[
\bar{h}(\tilde{u}, \tilde{v}, \tilde{w}) = \sum_{2 \leq h + j + k \leq 3} \bar{h}_{hjkh} \tilde{u}^h\tilde{v}^j\tilde{w}^k + O(\chi^4).
\]

The centre manifold theorem is also applied for studying the dynamics of map (55). Let \( W^c(0, 0, 0) \) denote the centre manifold of map (55) at the origin and assume it be approximately described by
\[
W^c(0, 0, 0) = \{(\tilde{u}, \tilde{v}, \tilde{w}) \in \mathbb{R}^3 | \tilde{w} = \gamma_3(\tilde{u}, \tilde{v}),
\gamma_3(0, 0) = 0, D\gamma_3(0, 0) = 0\},
\]
in which
\[
\gamma_3(\tilde{u}, \tilde{v}) = \sum_{1 \leq j + k \leq 2} \gamma_3^j \tilde{u}^j\tilde{v}^k + O((|\tilde{u}| + |\tilde{v}|)^3).
\]

Applying map (55) on both sides of (59) and keeping balance for the one-order and two-order terms of the variables, then we can directly obtain \( \gamma_3^{01} = 0, \gamma_3^{10} \), and
\[
\gamma_3^{20} = \frac{\bar{h}_{200}(\alpha^2 - \lambda_3) - \bar{h}_{020}\beta^2}{(1 - \lambda_3)(\alpha^4 + \beta^4 + \lambda_3^2 + 2(\beta^2 - \alpha^2)\lambda_3^2)},
\gamma_3^{02} = \frac{\bar{h}_{020}(\alpha^2 + \lambda_3^2 + \beta^2\lambda_3 - 2\alpha^2\lambda_3) + \bar{h}_{020}\beta^2(1 + \lambda_3) + \bar{h}_{110}\alpha\beta(\lambda_3 - 1)}{(1 - \lambda_3)(\alpha^4 + \beta^4 + \lambda_3^2 + 2(\beta^2 - \alpha^2)\lambda_3^2)},
\gamma_3^{11} = \frac{\bar{h}_{110}(\alpha^2 - \lambda_3) - \beta^4 + 2\alpha\beta(\bar{h}_{020}\beta^2 + \bar{h}_{200}(\lambda_3 - \alpha^2))}{(1 - \lambda_3)(\alpha^4 + \beta^4 + \lambda_3^2 + 2(\beta^2 - \alpha^2)\lambda_3^2)}.
\]

Considering the dynamics of map (55) restricted to the centre manifold \( W^c(0, 0, 0) \) and substituting (59) into map (55) yield
\[
\begin{pmatrix}
  \tilde{u} \\
  \tilde{v}
\end{pmatrix} \rightarrow \begin{pmatrix}
  \alpha - \beta & 0 \\
  \beta & \alpha
\end{pmatrix} \begin{pmatrix}
  \tilde{u} \\
  \tilde{v}
\end{pmatrix} + \begin{pmatrix}
  \bar{F} \\
  \bar{G}
\end{pmatrix},
\]
where
\[
\begin{pmatrix}
  \bar{F} \\
  \bar{G}
\end{pmatrix} = \begin{pmatrix}
  \sum_{2 \leq j + k \leq 3} \xi_{j1}^k \tilde{u}^j\tilde{v}^k + O((|\tilde{u}| + |\tilde{v}|)^4) \\
  \sum_{2 \leq j + k \leq 3} \xi_{j2}^k \tilde{u}^j\tilde{v}^k + O((|\tilde{u}| + |\tilde{v}|)^4)
\end{pmatrix}.
\]
By calculation, the values of the coefficients in map (62) are
\[ \begin{align*}
\xi_1^{20} &= \tilde{f}_{200}, \quad \xi_1^{02} = \tilde{f}_{020}, \quad \xi_1^{11} = \tilde{f}_{110}, \\
\xi_1^{21} &= \tilde{f}_{210} + \tilde{f}_{011}y^2 + \tilde{f}_{101}y^3, \\
\xi_2^{02} &= \tilde{h}_{200} + \tilde{h}_{011}y^2 + \tilde{h}_{110}y^3, \\
\xi_3^{00} &= \tilde{f}_{200} + \tilde{f}_{011}y^2 + \tilde{f}_{101}y^3, \\
\xi_2^{02} &= \tilde{h}_{200} + \tilde{h}_{011}y^2 + \tilde{h}_{110}y^3, \\
\xi_1^{21} &= \tilde{f}_{210} + \tilde{f}_{011}y^2 + \tilde{f}_{101}y^3, \\
\xi_1^{20} &= \tilde{f}_{200} + \tilde{f}_{011}y^2 + \tilde{f}_{101}y^3, \\
\xi_1^{30} &= \tilde{f}_{200} + \tilde{f}_{011}y^2 + \tilde{f}_{101}y^3, \\
\xi_1^{03} &= \tilde{f}_{003} + \tilde{f}_{011}y^2, \\
\xi_1^{02} &= \tilde{f}_{003} + \tilde{f}_{011}y^2, \\
\xi_1^{01} &= \tilde{f}_{003} + \tilde{f}_{011}y^2.
\end{align*} \]

(63)

If map (62) experiences Neimark–Sacker bifurcation at \((0, 0)\), the following condition must hold according to the Neimark–Sacker bifurcation theorem, i.e.
\[ a = -\text{Re} \left( \frac{1 - 2\lambda_2}{1 - \lambda_1} \rho_{21} \rho_{20} \right) - \frac{1}{2} |\rho_{11}|^2 - |\rho_{02}|^2 + \text{Re}(\lambda_2 \rho_{21}) \neq 0, \]

(64)

where
\[ \rho_{20} = \frac{1}{8} \left( (\tilde{F}'_{u} - \tilde{F}'_{v} + 2\tilde{G}'_{u}) + i (\tilde{G}'_{u} - \tilde{G}'_{v} + 2\tilde{F}'_{u}) \right), \]
\[ \rho_{11} = \frac{1}{4} \left( (\tilde{F}'_{u} + \tilde{F}'_{v}) + i (\tilde{G}'_{u} + \tilde{G}'_{v}) \right), \]
\[ \rho_{02} = \frac{1}{8} \left( (\tilde{F}'_{u} - \tilde{F}'_{v} - 2\tilde{G}'_{u}) + i (\tilde{G}'_{u} - \tilde{G}'_{v} + 2\tilde{F}'_{u}) \right), \]
\[ \rho_{21} = \frac{1}{16} \left( (\tilde{F}'_{u} + \tilde{F}'_{v} + \tilde{G}'_{u} + \tilde{G}'_{v}) \right. \\
\left. + i (\tilde{G}'_{u} + \tilde{G}'_{v} - \tilde{F}'_{u} - \tilde{F}'_{v}) \right) \]
\[ = \frac{1}{8} \left( 3\xi_1^{20} + \xi_1^{12} + \xi_1^{21} + 3\xi_1^{03} \right), \]
\[ + i \left( 3\xi_2^{20} + \xi_2^{12} + \xi_2^{21} + 3\xi_1^{03} \right). \]

(65)

The above analysis demonstrates that, when conditions (43), (44), (45), (50), (52), and (64) are satisfied, the discrete-time food chain system undergoes Neimark–Sacker bifurcation at the fixed point \(E^*\). Moreover, if \(a < 0\) and \(\varphi > 0\), an attracting invariant circle bifurcates from \(E^*\) for \(\tau > \tau_0\); if \(a > 0\) and \(\varphi > 0\), a repelling invariant circle bifurcates for \(\tau < \tau_0\).

### 3.3 Turing Instability

When spatially heterogeneous perturbations occur at the homogeneous states, the spatiotemporally discrete food chain system may converge to heterogeneous states [21]. In such case, the system experiences Turing bifurcation, also known as Turing instability. To determine the occurrence conditions for the Turing bifurcation, Turing instability analysis is performed. First, spatially heterogeneous perturbations are introduced to perturb the stable homogeneous stationary state \(E^*\), given by

\[ \tilde{x}_{(i,j,t)} = x_{(i,j,t)} - x^*, \quad \tilde{y}_{(i,j,t)} = y_{(i,j,t)} - y^*, \]
\[ \tilde{z}_{(i,j,t)} = z_{(i,j,t)} - z^*. \]

(66)

Based on the approach of Bai and Zhang [25], the eigenvalues of operator \(\nabla^2_a\) are determined as

\[ \lambda_{kl} = 4 \left( \sin^2 \left( \frac{(k-1)\pi}{n} \right) + \sin^2 \left( \frac{(l-1)\pi}{n} \right) \right), \]
\[ k, l \in \{1, 2, 3, \ldots, n\}. \]

(67)

Substituting (66) into the spatiotemporally discrete food chain system leads to

\[ \tilde{x}_{(i,j,t+1)} = a_{11} \left[ \tilde{x}_{(i,j,t)} + \frac{\tau}{\delta^2} d_1 \nabla^2_a \tilde{x}_{(i,j,t)} \right] \\
+ a_{12} \left[ \tilde{y}_{(i,j,t)} + \frac{\tau}{\delta^2} d_2 \nabla^2_a \tilde{y}_{(i,j,t)} \right] \\
+ O \left( \left( \tilde{x}_{(i,j,t)} + \tilde{y}_{(i,j,t)} + \tilde{z}_{(i,j,t)} \right)^2 \right), \]
\[ \tilde{y}_{(i,j,t+1)} = a_{21} \left[ \tilde{x}_{(i,j,t)} + \frac{\tau}{\delta^2} d_1 \nabla^2_a \tilde{x}_{(i,j,t)} \right] \\
+ a_{22} \left[ \tilde{y}_{(i,j,t)} + \frac{\tau}{\delta^2} d_2 \nabla^2_a \tilde{y}_{(i,j,t)} \right] \\
+ a_{23} \left[ \tilde{z}_{(i,j,t)} + \frac{\tau}{\delta^2} d_3 \nabla^2_a \tilde{z}_{(i,j,t)} \right] \\
+ O \left( \left( \tilde{x}_{(i,j,t)} + \tilde{y}_{(i,j,t)} + \tilde{z}_{(i,j,t)} \right)^2 \right), \]
\[ \tilde{z}_{(i,j,t+1)} = a_{32} \left[ \tilde{y}_{(i,j,t)} + \frac{\tau}{\delta^2} d_2 \nabla^2_a \tilde{y}_{(i,j,t)} \right] \\
+ a_{33} \left[ \tilde{z}_{(i,j,t)} + \frac{\tau}{\delta^2} d_3 \nabla^2_a \tilde{z}_{(i,j,t)} \right] \\
+ O \left( \left( \tilde{x}_{(i,j,t)} + \tilde{y}_{(i,j,t)} + \tilde{z}_{(i,j,t)} \right)^2 \right). \]

(68)

When the disturbance is extremely weak, the high-order terms in (66) can be negligible. Using \(\lambda_{kl}\) multiply the equation (68), we have

\[ X_{(i,j,t+1)}^{(k,l)} = a_{11} X_{(i,j,t)}^{(k,l)} + a_{12} X_{(i,j,t)}^{(k,l)} \]
\[ + \frac{\tau}{\delta^2} a_{11} d_1 X_{(i,j,t)}^{(k,l)}, \]
\[ + \frac{\tau}{\delta^2} a_{12} d_2 X_{(i,j,t)}^{(k,l)}, \]
\[
X_{kl}^{ij}(t+1) = a_{21}X_{kl}^{ij}(t) + a_{22}X_{kl}^{ij}(t) + \frac{r}{\delta}a_{21}d_{1}X_{kl}^{ij}\nabla_{i}x_{i}(t), \\
+ \frac{r}{\delta}a_{22}d_{2}X_{kl}^{ij}\nabla_{j}y_{j}(t), \\
+ \frac{r}{\delta}a_{23}d_{3}X_{kl}^{ij}\nabla_{z}(z_{i},j), \\
X_{kl}^{ij}(t+1) = a_{32}X_{kl}^{ij}(t) + a_{33}X_{kl}^{ij}(t) + \frac{r}{\delta}a_{32}d_{2}X_{kl}^{ij}\nabla_{i}y_{i}(t), \\
+ \frac{r}{\delta}a_{33}d_{3}X_{kl}^{ij}\nabla_{z}(z_{i},j). \\
\] (69)

Cumulative summation of the above formula (69), we can get

\[
\sum X_{kl}^{ij}(t+1) = a_{11}X_{kl}^{ij}(t) + a_{12}X_{kl}^{ij}(t) + \frac{r}{\delta}a_{11}d_{1}X_{kl}^{ij}\nabla_{i}x_{i}(t), \\
+ \frac{r}{\delta}a_{12}d_{2}X_{kl}^{ij}\nabla_{j}y_{j}(t), \\
+ \frac{r}{\delta}a_{13}d_{3}X_{kl}^{ij}\nabla_{z}(z_{i},j), \\
\sum X_{kl}^{ij}(t+1) = a_{21}X_{kl}^{ij}(t) + a_{22}X_{kl}^{ij}(t) + \frac{r}{\delta}a_{21}d_{1}X_{kl}^{ij}\nabla_{i}x_{i}(t), \\
+ \frac{r}{\delta}a_{22}d_{2}X_{kl}^{ij}\nabla_{j}y_{j}(t), \\
+ \frac{r}{\delta}a_{23}d_{3}X_{kl}^{ij}\nabla_{z}(z_{i},j), \\
\sum X_{kl}^{ij}(t+1) = a_{32}X_{kl}^{ij}(t) + a_{33}X_{kl}^{ij}(t) + \frac{r}{\delta}a_{32}d_{2}X_{kl}^{ij}\nabla_{i}y_{i}(t), \\
+ \frac{r}{\delta}a_{33}d_{3}X_{kl}^{ij}\nabla_{z}(z_{i},j). \\
\] (70)

Let

\[
\bar{x}_{t} = \sum X_{kl}^{ij}(t+1), \quad \bar{y}_{t} = \sum X_{kl}^{ij}(t+1), \\
\bar{z}_{t} = \sum X_{kl}^{ij}(t+1), \\
\] (71)

(68) can be transformed into

\[
\bar{x}_{t+1} = a_{11}X_{kl}^{ij}(t) + a_{12}X_{kl}^{ij}(t) + \frac{r}{\delta}a_{11}d_{1}X_{kl}^{ij}\nabla_{i}x_{i}(t), \\
\bar{y}_{t+1} = a_{21}X_{kl}^{ij}(t) + a_{22}X_{kl}^{ij}(t) + \frac{r}{\delta}a_{21}d_{1}X_{kl}^{ij}\nabla_{j}y_{j}(t), \\
+ a_{23}X_{kl}^{ij}(t) + \frac{r}{\delta}a_{23}d_{3}X_{kl}^{ij}\nabla_{z}(z_{i},j), \\
\bar{z}_{t+1} = a_{32}X_{kl}^{ij}(t) + a_{33}X_{kl}^{ij}(t) + \frac{r}{\delta}a_{32}d_{2}X_{kl}^{ij}\nabla_{i}y_{i}(t), \\
+ a_{33}X_{kl}^{ij}(t) + \frac{r}{\delta}a_{33}d_{3}X_{kl}^{ij}\nabla_{z}(z_{i},j). \\
\] (72)

When the following condition is satisfied,

\[
Z_{m} = \max_{k,l=1}^{n} Z(k, l) > 1, \quad ((k, l) \neq (1, 1)), \quad (74)
\]

Turing instability occurs in the discrete food chain system and results in the formation of spatial heterogeneous patterns.

### 4 Numerical Simulations

Numerical simulations are carried out to verify the theoretical results obtained in the previous section and to demonstrate the complex dynamics of the discrete food chain system. Bifurcation diagrams are plotted to show the dynamic transition with the variation of bifurcation parameter \( r \). Lyapunov exponents are calculated to determine when the chaotic dynamics emerges. Turing patterns are plotted to show spatiotemporal self-organized structures of the populations as well as the evolution process of the population dynamics.

#### 4.1 Flip Bifurcation and Neimark–Sacker Bifurcation

To exhibit the flip bifurcation, we choose a group of biologically feasible parameter values as \( r = 2.5, a_{1} = 2, a_{2} = 0.5, b_{1} = 2, b_{2} = 0.25, c_{1} = 0.6, c_{2} = 0.15, d = 0.25 \). According to the calculations, the coexistent fixed point \( E^{*} \) is \((0.9337, 0.1524, 0.0096)\), and the critical point for flip bifurcation is determined as \( \tau^{*} = 0.9023 \). At this critical point, the three eigenvalues of \( E^{*} \) are \( \lambda_{1} = -1, \lambda_{2} = 0.8347, \) and \( \lambda_{3} = 0.9884 \). Via applying the centre manifold theorem, map (40), which describes the dynamics restricted to the centre manifold of map (34), is obtained as

\[
F : \tilde{u} \rightarrow -\tilde{u} - 3.4490\tilde{u}^{2} - 2.2165\tilde{u}\tilde{\tau} - 3.8213\tilde{u}^{2}\tilde{\tau} \\
- 3.4660\tilde{\tau}^{3} + O(\left(\tilde{u} + |\tilde{\tau}|\right)^{4}). \quad (75)
\]

On the basis of the one-dimensional map \( F \), the two discriminatory quantities \( \eta_{1} \) and \( \eta_{2} \) are determined as \( \eta_{1} = -2.2165 \) and \( \eta_{2} = 8.4222 \). Therefore, according to the flip bifurcation theorem, the discrete food chain system undergoes flip bifurcation at \( \tau = \tau^{*} \). Moreover, as \( \eta_{2} > 0 \), a stable period-2 orbit bifurcating from \( E^{*} \) and the fixed point \( E^{*} \) becomes unstable for \( \tau > \tau^{*} \). As shown in Figure 1a, the flip bifurcation indeed occurs under the above given parametric conditions, along with a period-doubling cascade and a dynamic transition from periodic orbits to chaotic behaviours. As determined by the
maximum Lyapunov exponents in Figure 1b, the chaotic dynamics first occurs at about $\tau = 1.122$. Figure 1c–e display three local magnifications of the flip bifurcation diagram. We explicitly see the period-doubling cascade and periodic windows with period-16 and period-6 orbits occurring in the chaotic zone.

Given parameter values as $r = 1$, $a_1 = 2$, $a_2 = 0.9$, $b_1 = 0.95$, $c_1 = 0.2$, $b_2 = 0.2$, $c_2 = 0.2$, $d = 0.25$, the discrete food chain system exhibits Neimark–Sacker bifurcation. In this case, the fixed point $E^*$ is $(0.8389, 0.0906, 0.1781)$, and the critical point for the Neimark–Sacker bifurcation is determined as $\tau_0 = 0.8465$. When we take $\tau = \tau_0$, the three eigenvalues of $E^*$ are $\lambda_{1,2} = 0.9680 \pm 0.2510i$, $\lambda_3 = 0.4515$, and $|\lambda_{1,2}| = 1$. Based on previous calculations, we have $\varphi = 0.0378$ and $a = -0.0215$. Hence, it is verified that the discrete system undergoes Neimark–Sacker bifurcation at $\tau = \tau_0$. Moreover, as $d > 0$ and $a < 0$, we know that an attracting invariant closed curve bifurcates from the fixed point $E^*$ for $\tau > \tau_0$.

Figure 2a exhibits the Neimark–Sacker bifurcation diagram with parameter $\tau$ ranging in [0.2, 2.1]. As the $\tau$ value increases around the critical point $\tau_0$, the system dynamics changes from a stable focus to an attracting invariant curve. Figure 2b shows the maximum Lyapunov exponents corresponding to Figure 2a, revealing that the occurrence of chaos begins at about $\tau_1 = 2.297$. When $\tau_0 < \tau < \tau_1$, the discrete food chain system is mainly attracted to invariant curves, with a few periodic windows occurring in between.

Figure 2c–d and 2f exhibit three types of periodic windows emerging suddenly in the zone of invariant curves. In Figure 2c, a period-20 orbit occurs all along the periodic window. In Figure 2d, the discrete system first presents a period-19 orbit, and then each branch of the periodic orbit undergoes sub–Neimark–Sacker bifurcation, resulting to the occurrence of multicycles. With the increase of $\tau$ value, the behaviour of multicycles then abruptly changes to period-38 orbit, after which the occurrence of inverse flip bifurcation makes the system dynamics return to the behaviour of period-19 orbit. The system dynamics exhibited in Figure 2f is far more complex than the above two cases. In this window, we find the dynamics of sub–Neimark–Sacker bifurcation, sub–inverse Neimark–Sacker bifurcation, sub–inverse flip bifurcation, sub–period-halving cascade, and sub–periodic windows. Subsequently after this periodic window, the dynamics of the discrete food chain system becomes chaotic.

The flip bifurcation and Neimark–Sacker bifurcation are two basic types of bifurcations emerging in the discrete food chain system. The combination of these two bifurcations can lead to very complex dynamics and dynamic transition with the variation of parameter $\tau$, as shown in Figures 3 and 4. In Figure 3, the system first undergoes flip bifurcation, generating an attracting period-2 orbit. In each branch of the periodic orbit,
Figure 2: (a) Bifurcation diagram with the occurrence of the Neimark–Sacker bifurcation; (b) maximum Lyapunov exponents corresponding to (a); (c–f) magnifications of (a) in local ranges; (e) magnification of one branch in (d); (g–h) magnifications of one branch in (f). $r = 1$, $a_1 = 2$, $a_2 = 0.9$, $b_1 = 0.95$, $c_1 = 0.2$, $b_2 = 0.2$, $c_2 = 0.2$, $d = 0.25$.

Figure 3: Bifurcation diagram and maximum Lyapunov exponents with parametric conditions given as $r = 1.7$, $a_1 = 2.5$, $a_2 = 0.6$, $b_1 = 1.5$, $c_1 = 0.1$, $b_2 = 0.08$, $c_2 = 0.5$, $d = 0.02$. 
Neimark–Sacker bifurcation occurs and induces the route to chaos, replacing the period-doubling cascades. In Figure 4, the Neimark–Sacker bifurcation also takes place at each branch of the period-2 orbit. However, as $\tau$ value rises, the occurrence of inverse Neimark–Sacker bifurcation forces the system dynamics to go out of the behaviours of attracting invariant curves and to return to period-doubling cascade. At about $\tau = 0.8472$, the discrete system goes into chaotic dynamics.

### 4.2 Turing Pattern Formation

When the conditions for Turing instability are satisfied, the discrete food chain system exhibits Turing pattern formation, which reveals the stable distribution of populations in space. With a test of the Turing instability conditions, we find that the parameter values can be fixed as $r = 1$, $a_1 = 2$, $b_1 = 0.95$, $c_1 = 0.2$, $a_2 = 0.9$, $b_2 = 0.2$, $c_2 = 0.2$, $d = 0.25$, and $\tau = 1.2$, and the values of $d_1$, $d_2$, $d_3$, and $h$ can be changed to investigate the pattern formation. On this basis, a large number of numerical simulations of Turing patterns for the discrete food chain system are carried out. In the pattern simulations, the initial conditions are given by introducing a small random perturbation on the stable state $E^*$. Simultaneously, periodic boundary conditions are applied in the simulations of the discrete food chain system.

With the parameter variations, many types of patterns are found for the discrete food chain system under Turing instability. Figure 5 shows the formation of a labyrinth pattern, which is typical as described in the literature of

![Turing Pattern Formation](image)

**Figure 5:** Turing pattern formation of the discrete food chain system when the parametric conditions are given as $r = 1$, $a_1 = 2$, $b_1 = 0.95$, $c_1 = 0.2$, $a_2 = 0.9$, $b_2 = 0.2$, $c_2 = 0.2$, $d = 0.25$, $\tau = 1.2$, $d_1 = 2.5$, $d_2 = 0.1$, $d_3 = 0.1$, $n = 100$, $h = 5.5$. 

![Bifurcation Diagram](image)

**Figure 4:** Bifurcation diagram and maximum Lyapunov exponents with parametric conditions given as $r = 3.2$, $a_1 = 1.4$, $a_2 = 0.9$, $b_1 = 0.5$, $c_1 = 0.2$, $b_2 = 0.2$, $c_2 = 0.2$, $d = 0.25$. 

![Maximum Lyapunov Exponent](image)
population dynamics. As the time grows, the spatial distribution of the populations in the food chain changes from random pattern to labyrinth pattern. As shown in Figure 6, with the increase of time, the patterns for the three populations in the food chain system experience a complicated change from random pattern, to irregular labyrinth pattern and then striped-spiral pattern, and finally to striped pattern, where the main characteristic of pattern configuration maintains unchanged after $t > 20,000$. It means the system dynamics becomes stable at the striped pattern, which is composed of a series of parallel stripes that are winding in a few directions.

Change of parameter values may lead to the variation of pattern configurations, i.e. the transition between different pattern types. As demonstrated in Figure 7, with the increase of parameter $d_1$, the food chain system undergoes a pattern transition process between irregular patterns and regular striped patterns. The variation of $d_1$ means the change of the spatial diffusion capacity of prey population. Therefore, the pattern transition reflects that the spatial diffusion capacity of populations can strongly affect the type of spatial distribution.

5 Conclusions

In this research, we have studied the dynamics of a discrete three-species food chain system with Beddington–DeAngelis response. To our best knowledge, there is little literature that theoretically analyses the bifurcations
and complex pattern formation of the spatiotemporally discrete food chain system. With the work of this research, the concluding remarks can be stated as follows:

1. The discrete food chain system shows a stable fixed point where the three species stably coexist and maintain steady population densities. Around the fixed point, the system can undergo flip bifurcation and Neimark–Sacker bifurcation, which both induce the route to chaos. Moreover, combination of the two bifurcations can lead to very complex and rich dynamics of the system.

2. Many periodic windows are found in the zones of invariant curves and chaos. In the periodic windows, we find many periodic orbits, as well as sub–(inverse) Neimark–Sacker bifurcation, sub–(inverse) flip bifurcation, chaotic interior crisis, sub–period-doubling/halving cascade, and sub–periodic windows.

3. The discrete food chain system also exhibits the formation of Turing patterns, which reveal important ways for the coexistence of species in space as regular and irregular self-organized structure. The self-organized patterns are rich in types, including stripes, labyrinth, gaps, and spiral waves. With the parameter variations, the system can display complicated transition between different patterns.

In comparison with the corresponding continuous food chain system, the dynamic behaviours of the discrete system show more complicated characteristics. First, when the spatial diffusion of the populations is not considered, the discrete system can exhibit Neimark–Sacker bifurcation and complex dynamic transition in combination of flip bifurcation and Neimark–Sacker bifurcation, which are not present in the continuous system. Second, with the consideration of population diffusion, the discrete system can exhibit more complex spatiotemporal dynamics than the continuous system. This spatiotemporal complexity may reflect in the richness of Turing pattern structures. At least, the patterns in Figures 5–7 cannot emerge in the continuous system if the same parametric conditions are provided.

The investigation demonstrates the diversity and complexity of dynamic behaviours in a discrete three-species food chain system. The system exhibits different bifurcations and pattern types, which are ecologically important and, especially, reveal how the dynamics of the food chain system transits between order and disorder and how the spatial distribution of populations changes between different self-organized structures. The complex patterns reflect diverse possible ways for species coexistence in the food chain. The results obtained may be helpful for the researchers who will further explore the dynamics of discrete-time food chain systems.

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