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Spin coherent states, Bell states, spin Hamilton operators, entanglement, Husimi distribution, uncertainty relation and Bell inequality

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Abstract: We investigate spin Hamilton operators and compare spin coherent states and Bell states concerning entanglement, Husimi distributions, uncertainty relation and Bell inequality. The distances between spin coherent states and Bell states are derived. The Rayleigh quotients of spin Hamilton operators for spin coherent states and Bell states are evaluated and compared.

Keywords: Bell inequality; entanglement; spin systems; uncertainty relation.

Bell states [1–7], Dicke states [6, 8, 9] and spin coherent states [10–23] play a central role in quantum computing. The Bell states are fully entangled whereas among quantum states the spin coherent states (also called atomic coherent states or Bloch coherent states) are the “most classical states”. Spin Hamilton operators are provided which admit the Bell states and the Dicke states as eigenvectors. We also show how the Bell states can be constructed from the spin coherent states in $\mathbb{C}^2$ and the Kronecker product. The measure of entanglement for these states is compared. The Husimi distributions are evaluated and discussed. The distances between the Bell states and spin coherent states are derived and shown that the distances cannot be 0. The uncertainty relations for the spin matrices S_1 and S_2 , Bell states and spin coherent states are derived and compared. Furthermore we look at the Bell inequality for Bell states and spin coherent states. We find for spin coherent states that the Bell inequality can be violated depending on the parameter values. The Bell matrix is expressed with spin matrices and the Rayleigh quotients for spin

coherent states and Bell states are derived and compared to the eigenvalues of the Bell matrix. We show that the spin coherent state cannot provide the ground state. With spin coherent states we can define a semi-classical density distribution. The Wehrl entropy for spin coherent states is evaluated.

Let $S_1^{(1/2)}$, $S_2^{(1/2)}$ and $S_3^{(1/2)}$ be the spin matrices for spin- $\frac{1}{2}$

$$S_1^{(1/2)} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$S_2^{(1/2)} = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$S_3^{(1/2)} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

obeying the commutation relations

$$[S_1^{(1/2)}, S_2^{(1/2)}] = iS_3^{(1/2)},$$

$$[S_2^{(1/2)}, S_3^{(1/2)}] = iS_1^{(1/2)},$$

$$[S_3^{(1/2)}, S_1^{(1/2)}] = iS_2^{(1/2)}$$

and

$$(S_1^{(1/2)})^2 + (S_2^{(1/2)})^2 + (S_3^{(1/2)})^2 = \frac{3}{4}I_2.$$

Let $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis in the Hilbert space $\mathbb{C}^2$. The four Bell states are given by

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle),$$

$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle),$$

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle),$$

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

where $|\Psi_-\rangle$ is the singlet state. The four Bell states form an orthonormal basis in the Hilbert space $\mathbb{C}^4$. Let $|e_1\rangle$, $|e_2\rangle$, $|e_3\rangle$ and $|e_4\rangle$ be the standard basis in the Hilbert space $\mathbb{C}^4$.

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Then the Bell states are given by

$$\begin{aligned} |\Phi_+\rangle &= B|e_1\rangle, & |\Phi_-\rangle &= B|e_4\rangle, \\ |\Psi_+\rangle &= B|e_2\rangle, & |\Psi_-\rangle &= B|e_3\rangle \end{aligned}$$

where B is the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$$= 2\sqrt{2}S_1^{(1/2)} \otimes S_1^{(1/2)} + \sqrt{2}S_3^{(1/2)} \otimes I_2$$

where I_2 is the 2×2 identity matrix. The Bell matrix is hermitian and unitary with $B^2 = I_4$, $\text{tr}(B) = 0$ and $\det(B) = 1$. Hence, the eigenvalues are -1 (twice) and $+1$ (twice). The Bell matrix cannot be written as a Kronecker product of two 2×2 matrices. The Bell matrix can be written as $B = \exp(K)$ with the skew hermitian matrix

$$K = i\pi \begin{pmatrix} \frac{1}{(2(\sqrt{2}+2))} & 0 & 0 & -\frac{(4+3\sqrt{2})}{(4(2\sqrt{2}+3))} \\ 0 & \frac{1}{(2(\sqrt{2}+2))} & -\frac{(4+3\sqrt{2})}{(4(2\sqrt{2}+3))} & 0 \\ 0 & -\frac{(4+3\sqrt{2})}{(4(2\sqrt{2}+3))} & \frac{(10+7\sqrt{2})}{(4(2\sqrt{2}+3))} & 0 \\ -\frac{(4+3\sqrt{2})}{(4(2\sqrt{2}+3))} & 0 & 0 & \frac{(10+7\sqrt{2})}{(4(2\sqrt{2}+3))} \end{pmatrix}$$

with the trace equal to $2\pi i$. With $K = iH$ we find a hermitian matrix H . The eigenvalues of K are 0 (twice) and $i\pi$ (twice). From the Bell matrix we can form the projection operators $\Pi_+ = (I_4 + B)/2$, $\Pi_- = (I_4 - B)/2$ with $\Pi_+\Pi_- = 0_4$ and $\Pi_+ - \Pi_- = B$.

Let $\sigma_1 = 2S_1^{(1/2)}$, $\sigma_2 = 2S_2^{(1/2)}$ and $\sigma_3 = 2S_3^{(1/2)}$ be the Pauli spin matrices. Then we have the eigenvalue equations for the Bell states

$$(\sigma_1 \otimes \sigma_1)|\Phi_+\rangle = |\Phi_+\rangle, \quad (\sigma_1 \otimes \sigma_1)|\Phi_-\rangle = -|\Phi_-\rangle,$$

$$(\sigma_1 \otimes \sigma_1)|\Psi_+\rangle = |\Psi_+\rangle, \quad (\sigma_1 \otimes \sigma_1)|\Psi_-\rangle = -|\Psi_-\rangle,$$

$$(\sigma_2 \otimes \sigma_2)|\Phi_+\rangle = -|\Phi_+\rangle, \quad (\sigma_2 \otimes \sigma_2)|\Phi_-\rangle = |\Phi_-\rangle,$$

$$(\sigma_2 \otimes \sigma_2)|\Psi_+\rangle = |\Psi_+\rangle, \quad (\sigma_2 \otimes \sigma_2)|\Psi_-\rangle = -|\Psi_-\rangle,$$

$$(\sigma_3 \otimes \sigma_3)|\Phi_+\rangle = |\Phi_+\rangle, \quad (\sigma_3 \otimes \sigma_3)|\Phi_-\rangle = |\Phi_-\rangle,$$

$$(\sigma_3 \otimes \sigma_3)|\Psi_+\rangle = -|\Psi_+\rangle, \quad (\sigma_3 \otimes \sigma_3)|\Psi_-\rangle = -|\Psi_-\rangle.$$

Consider now entanglement for the Bell states. From the Bell states we can form the four density matrices

$$\rho_{\Phi_+} = |\Phi_+\rangle\langle\Phi_+|, \quad \rho_{\Phi_-} = |\Phi_-\rangle\langle\Phi_-|,$$

$$\rho_{\Psi_+} = |\Psi_+\rangle\langle\Psi_+|, \quad \rho_{\Psi_-} = |\Psi_-\rangle\langle\Psi_-|.$$

The reduced density matrices (taking the partial trace) is the same for all four Bell states, namely

$$\rho_R = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Hence the von Neumann entropy is given by $S(\rho) = 1$ which indicates that the Bell states are fully entangled. Using the 2-tangle as entanglement measure [24] we also find that the Bell states are completely entangled. Furthermore the Bell states cannot be written as the Kronecker product of two vectors in the Hilbert space $\mathbb{C}^2$.

The Dicke states in the Hilbert space $\mathbb{C}^4$ are given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence the second Dicke state is the Bell state $|\Psi_+\rangle$.

The hermitian matrix

$$\frac{\hat{H}}{\hbar\omega} = S_1^{(1/2)} \otimes S_1^{(1/2)} + \gamma S_3^{(1/2)} \otimes S_3^{(1/2)}$$

with $\gamma > 0$ admits the Bell states $|\Psi_-\rangle$, $|\Psi_+\rangle$, $|\Phi_-\rangle$ and $|\Phi_+\rangle$ as eigenvectors with the corresponding eigenvalues

$$\begin{aligned}\lambda_{\Psi-} &= \frac{1}{4}(-1 - \gamma), & \lambda_{\Psi+} &= \frac{1}{4}(1 - \gamma), \\ \lambda_{\Phi-} &= \frac{1}{4}(-1 + \gamma), & \lambda_{\Phi+} &= \frac{1}{4}(1 + \gamma).\end{aligned}$$

Consider the spin $S_j^{(1/2)}$ matrices ($j = 1, 2, 3$) and the Hamilton operator

$$\begin{aligned}\hat{K} = \frac{\hat{H}}{\hbar\omega_1} &= \left(S_1^{(1/2)} \otimes S_1^{(1/2)} + S_2^{(1/2)} \otimes S_2^{(1/2)} \right. \\ &\quad \left. + S_3^{(1/2)} \otimes S_3^{(1/2)} \right) + \gamma(S_3^{(1/2)} \otimes I_2 + I_2 \otimes S_3^{(1/2)})\end{aligned}$$

with $\gamma = \omega_2/\omega_1$. The eigenvalues of $\hat{K}$ are given by

$$\lambda_1 = -\frac{3}{4}, \quad \lambda_2 = \frac{1}{4}, \quad \lambda_3 = \frac{1-4\gamma}{4}, \quad \lambda_4 = \frac{1+4\gamma}{4}$$

with the corresponding normalized eigenvectors

$$\begin{aligned}\mathbf{v}_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, & \mathbf{v}_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \\ \mathbf{v}_3 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, & \mathbf{v}_4 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\end{aligned}$$

where $\mathbf{v}_1$ is the singlet state, $\mathbf{v}_2$ is a Bell state and also a Dicke state. The eigenvectors $\mathbf{v}_3$ and $\mathbf{v}_4$ are Dicke states.

The Dicke states are eigenvectors of

$$S_3^{(1/2)} \otimes S_3^{(1/2)} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Starting point in the construction of the spin coherent states are the spin matrices $S_1^{(s)}$, $S_2^{(s)}$, $S_3^{(s)}$ and $S_+^{(s)} := S_1^{(s)} + iS_2^{(s)}$, $S_-^{(s)} := S_1^{(s)} - iS_2^{(s)}$. Then

$$\begin{aligned}K^{(s)}(\theta, \phi) &:= \frac{1}{2}\theta e^{i\phi} S_-^{(s)} - \frac{1}{2}\theta e^{-i\phi} S_+^{(s)} \\ &\equiv i\theta(S_1^{(s)} \sin(\phi) - S_2^{(s)} \cos(\phi))\end{aligned}$$

is a skew hermitian matrix with $\text{tr}(K^{(s)}(\theta, \phi)) = 0$ and $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$. It follows that $\exp(K^{(s)}(\theta, \phi))$ is a unitary matrix with $\det(\exp(K^{(s)}(\theta, \phi))) = 1$. We note that applying disentanglement we have the well-known identity

$$\begin{aligned}\exp\left(\frac{1}{2}\theta e^{i\phi} S_-^{(s)} - \frac{1}{2}\theta e^{-i\phi} S_+^{(s)}\right) \\ \equiv \exp(z S_-^{(s)}) \exp(-\ln(1 + |z|^2) S_3^{(s)}) \exp(-\bar{z} S_+^{(s)})\end{aligned}$$

where $z = e^{i\phi} \tan(\theta/2)$. The spin coherent states are given by

$$|\theta, \phi\rangle = \exp(K^{(s)}(\theta, \phi)) \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^T$$

with

$$\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^T \in \mathbb{C}^{2s+1}.$$

We set $|\Omega\rangle \equiv |\theta, \phi\rangle$ in the following. The unitary matrix $\exp(K^{(s)}(\theta, \phi))$ describes a rotation through an angle θ about an axis $\mathbf{n}(\phi) = (\sin(\phi), -\cos(\phi), 0)$. Note that the eigenvalues of $K^{(s)}(\theta, \phi)$ do not depend on ϕ and the eigenvectors do not depend on θ . This is true for all spin s . The overlap between two spin coherent states is given by

$$\begin{aligned}\langle \Omega_1 | \Omega_2 \rangle &= \left(\cos(\theta_1/2) \cos(\theta_2/2) + e^{i(\phi_1 - \phi_2)} \right. \\ &\quad \left. \times \sin(\theta_1/2) \sin(\theta_2/2) \right)^{2s}.\end{aligned}$$

For $s = 3/2$ the eigenvalues of $K^{(3/2)}(\theta, \phi)$ are $\lambda_1 = -3i\theta/2$, $\lambda_2 = -i\theta/2$, $\lambda_3 = i\theta/2$ and $\lambda_4 = 3i\theta/2$ with the corresponding normalized eigenvectors

$$\begin{aligned}\mathbf{v}_1 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{3}ie^{i\phi} \\ -\sqrt{3}e^{2i\phi} \\ -ie^{3i\phi} \end{pmatrix}, & \mathbf{v}_2 &= \frac{\sqrt{3}}{2\sqrt{2}} \begin{pmatrix} 1 \\ ie^{i\phi}/\sqrt{3} \\ e^{2i\phi}/\sqrt{3} \\ ie^{3i\phi} \end{pmatrix}, \\ \mathbf{v}_3 &= \frac{\sqrt{3}}{2\sqrt{2}} \begin{pmatrix} 1 \\ -ie^{i\phi}/\sqrt{3} \\ e^{2i\phi}/\sqrt{3} \\ -ie^{3i\phi} \end{pmatrix}, & \mathbf{v}_4 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ -\sqrt{3}ie^{i\phi} \\ -\sqrt{3}e^{2i\phi} \\ ie^{3i\phi} \end{pmatrix}.\end{aligned}$$

The four vectors form an orthonormal basis in the Hilbert space $\mathbb{C}^4$. The unitary matrix $\exp(K^{(3/2)})$ is given by

$$\exp(K^{(3/2)}) = e^{\lambda_1} \mathbf{v}_1 \mathbf{v}_1^* + e^{\lambda_2} \mathbf{v}_2 \mathbf{v}_2^* + e^{\lambda_3} \mathbf{v}_3 \mathbf{v}_3^* + e^{\lambda_4} \mathbf{v}_4 \mathbf{v}_4^*.$$

The spin coherent state for spin $s = \frac{3}{2}$ can also be written as

$$|z\rangle = \frac{1}{(1 + z\bar{z})^{3/2}} \begin{pmatrix} 1 \\ \sqrt{3}z \\ \sqrt{3}z^2 \\ z^3 \end{pmatrix}$$

where $z = e^{i\phi} \tan(\theta/2)$ ($0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$). With the standard basis

$$\begin{aligned}|3/2, -3/2\rangle &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T, \\ |3/2, -1/2\rangle &= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T,\end{aligned}$$

$$|3/2, 1/2\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T,$$

$$|3/2, 3/2\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T$$

we can write

$$|\Omega\rangle = \sum_{m=-3/2}^{3/2} \sqrt{\binom{3}{3/2+m}} (\cos(\theta/2))^{3/2+m} \times (\sin(\theta/2)e^{i\phi})^{3/2-m} |3/2, m\rangle$$

i.e.

$$|\Omega\rangle = \begin{pmatrix} \cos^3(\theta/2) \\ \sqrt{3}e^{i\phi} \cos^2(\theta/2) \sin(\theta/2) \\ \sqrt{3}e^{2i\phi} \cos(\theta/2) \sin^2(\theta/2) \\ e^{3i\phi} \sin^3(\theta/2) \end{pmatrix}.$$

Consider now the spin coherent states for $s = 3/2$ and entanglement. We note that the spin- $\frac{1}{2}$ coherent state is given by

$$\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}.$$

Then the Kronecker product [21] provides the normalized state in the Hilbert space $\mathbb{C}^4$

$$\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix} \otimes \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta/2) \\ e^{i\phi} \cos(\theta/2) \sin(\theta/2) \\ e^{i\phi} \sin(\theta/2) \cos(\theta/2) \\ e^{2i\phi} \sin^2(\theta/2) \end{pmatrix}.$$

Hence $|\Omega\rangle$ cannot be written as a Kronecker product of the spin coherent state of spin- $\frac{1}{2}$. Note, however, that the Bell

states can be constructed from the spin coherent states for $\mathbb{C}^2$ and the Kronecker product. Let

$$|z\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}, \quad |\bar{z}\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{-i\phi} \sin(\theta/2) \end{pmatrix}.$$

Then

$$|z\rangle \otimes |\bar{z}\rangle = \begin{pmatrix} \cos^2(\theta/2) \\ e^{-i\phi} \cos(\theta/2) \sin(\theta/2) \\ e^{i\phi} \sin(\theta/2) \cos(\theta/2) \\ \sin^2(\theta/2) \end{pmatrix}$$

and the first Bell states is given by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (|z\rangle \otimes |\bar{z}\rangle) d\Omega$$

with the measure $d\Omega$ given by

$$d\Omega = \frac{1}{2\pi} \sin(\theta) d\theta d\phi \equiv \frac{1}{\pi} \sin(\theta/2) \cos(\theta/2) d\theta d\phi.$$

Analogously we can find the other three Bell states.

Consider now entanglement of the spin coherent state via the density matrix and partial trace. Let $N = 1 + 3z\bar{z} + 3(z\bar{z})^2 + (z\bar{z})^3 \equiv (1 + x^2 + y^2)^3$ with $z = x + iy$ ($x, y \in \mathbb{R}$). The density matrices are given by

$$|z\rangle\langle z| = \frac{1}{N} \begin{pmatrix} 1 & \sqrt{3}\bar{z} & \sqrt{3}\bar{z}^2 & \bar{z}^3 \\ \sqrt{3}z & 3z\bar{z} & 3z\bar{z}^2 & \sqrt{3}z\bar{z}^3 \\ \sqrt{3}z^2 & 3z^2\bar{z} & 3(z\bar{z})^2 & \sqrt{3}z^2\bar{z}^3 \\ z^3 & \sqrt{3}z^3\bar{z} & \sqrt{3}z^3\bar{z}^2 & (z\bar{z})^3 \end{pmatrix}$$

and

$$|\Omega\rangle\langle\Omega| = \begin{pmatrix} \cos^6(.) & \sqrt{3}e^{-i\phi} \cos^5(.) \sin(.) & \sqrt{3}e^{-2i\phi} \cos^4(.) \sin^2(.) & e^{-3i\phi} \cos^3(.) \sin^3(.) \\ \sqrt{3}e^{i\phi} \cos^5(.) \sin(.) & 3 \cos^4(.) \sin^2(.) & 3e^{-i\phi} \cos^3(.) \sin^3(.) & \sqrt{3}e^{-2i\phi} \cos^2(.) \sin^4(.) \\ \sqrt{3}e^{2i\phi} \cos^4(.) \sin^2(.) & 3e^{i\phi} \cos^3(.) \sin^3(.) & 3 \cos^2(.) \sin^4(.) & \sqrt{3}e^{-2i\phi} \cos(.) \sin^5(.) \\ e^{3i\phi} \cos^3(.) \sin^3(.) & \sqrt{3}e^{2i\phi} \cos^2(.) \sin^4(.) & \sqrt{3}e^{i\phi} \cos(.) \sin^5(.) & \sin^6(.) \end{pmatrix}$$

with $(.) = (\theta/2)$. Utilizing the partial trace the reduced density matrices are

$$|z\rangle\langle z|_R = \frac{1}{N}$$

$$\times \begin{pmatrix} 1 + 3(x^2 + y^2) & \sqrt{3}(x - iy)^2(1 + x^2 + y^2) \\ \sqrt{3}(x + iy)^2(1 + x^2 + y^2) & (x^2 + y^2)^2(3 + x^2 + y^2) \end{pmatrix}$$

and

$$|\Omega\rangle\langle\Omega|_R = \begin{pmatrix} \sin^4(\theta/2)(2\sin^2(\theta/2) - 3) + 1 & \sqrt{3}e^{2i\phi}\sin^2(\theta/2)\cos^2(\theta/2) \\ \sqrt{3}e^{-2i\phi}\sin^2(\theta/2)\cos^2(\theta/2) & \sin^4(\theta/2)(-2\sin^2(\theta/2) + 3) \end{pmatrix}.$$

Obviously the trace is given by 1 for the two matrices. The determinant for $|\Omega\rangle\langle\Omega|_R$ is given by

$$\det(|\Omega\rangle\langle\Omega|_R) = \frac{\sin^6(\theta)}{16}.$$

Hence the determinant is equal to 0 for $\theta = 0$ and $1/16$ for $\theta = \pi/2$. The eigenvalues of $|\Omega\rangle\langle\Omega|_R$ are given by

$$\lambda_1(\theta) = \frac{1}{2} + \frac{\sqrt{4 - \sin^6(\theta)}}{4},$$

$$\lambda_2(\theta) = \frac{1}{2} - \frac{\sqrt{4 - \sin^6(\theta)}}{4}.$$

For $\theta = 0$ we have the eigenvalues $\lambda_1(0) = 1$ and $\lambda_2(0) = 0$ and the state is not entangled. For $\theta = \pi/2$ we have $\lambda_1(\pi/2) = 1/2 + \sqrt{3}/4$, $\lambda_2(\pi/2) = 1/2 - \sqrt{3}/4$ and the state is entangled but not fully entangled.

Consider now the Bell states and the spin coherent states for spin- $\frac{3}{2}$ and the Husimi distribution. We obtain

$$|\langle\Phi_+|\Omega\rangle|^2 = \frac{1}{2}(\sin^6(\theta/2) + \cos^6(\theta/2))$$

$$+ \cos(3\phi)\cos^3(\theta/2)\sin^3(\theta/2)$$

$$|\langle\Phi_-|\Omega\rangle|^2 = \frac{1}{2}(\sin^6(\theta/2) + \cos^6(\theta/2))$$

$$+ \cos(3\phi)\cos^3(\theta/2)\sin^3(\theta/2)$$

$$|\langle\Psi_+|\Omega\rangle|^2 = \frac{3}{2}\cos^2(\theta/2)\sin^2(\theta/2)$$

$$+ 3\cos(\phi)\cos^3(\theta/2)\sin^3(\theta/2)$$

$$|\langle\Psi_-|\Omega\rangle|^2 = \frac{3}{2}\cos^2(\theta/2)\sin^2(\theta/2) - 3\cos(\phi)$$

$$\times \cos^3(\theta/2)\sin^3(\theta/2).$$

Hence we find

$$0 \leq |\langle\Phi_+|\Omega\rangle|^2 \leq 1/2, \quad 0 \leq |\langle\Phi_-|\Omega\rangle|^2 \leq 1/2,$$

$$0 \leq |\langle\Psi_+|\Omega\rangle|^2 \leq 3/4, \quad 0 \leq |\langle\Psi_-|\Omega\rangle|^2 \leq 3/4.$$

From these results it follows that the distance between the Bell states and the spin coherent states cannot be 0

$$\| |\Phi_+\rangle - |\Omega\rangle \|^2 > 0, \quad \| |\Phi_-\rangle - |\Omega\rangle \|^2 > 0,$$

$$\| |\Psi_+\rangle - |\Omega\rangle \|^2 > 0, \quad \| |\Psi_-\rangle - |\Omega\rangle \|^2 > 0.$$

The shortest distance for $\| |\Phi_+\rangle - |\Omega\rangle \|^2$, $\| |\Phi_-\rangle - |\Omega\rangle \|^2$ is $2 - \sqrt{2}$, the shortest distance for $\| |\Psi_+\rangle - |\Omega\rangle \|^2$ is $2 - \sqrt{3}$ and the shortest distance for $\| |\Psi_-\rangle - |\Omega\rangle \|^2$ is $2 - 8/(3\sqrt{7})$.

Consider now the uncertainty relation for Bell states, spin coherent states for $s = 3/2$ and the spin matrices $S_1^{(3/2)}$, $S_2^{(3/2)}$. Let $S_1^{(s)}$, $S_2^{(s)}$ and $S_3^{(s)}$ be the spin matrices with spin- s and the commutation relations

$$[S_1^{(s)}, S_2^{(s)}] = iS_3^{(s)},$$

$$[S_2^{(s)}, S_3^{(s)}] = iS_1^{(s)},$$

$$[S_3^{(s)}, S_1^{(s)}] = iS_2^{(s)}.$$

Let $|\psi\rangle$ be a normalized state in the Hilbert space $\mathbb{C}^{2s+1}$. One defines the variance as

$$\Delta S_j^{(s)} := \sqrt{\langle\psi|(S_j^{(s)})^2|\psi\rangle - (\langle\psi|S_j^{(s)}|\psi\rangle)^2}, \quad j = 1, 2, 3.$$

The uncertainty relation for all s is

$$(\Delta S_1^{(s)})^2 \cdot (\Delta S_2^{(s)})^2 \geq \frac{1}{4} |\langle\psi|[S_1^{(s)}, S_2^{(s)}]|\psi\rangle|^2$$

$$\equiv \frac{1}{4} |\langle\psi|iS_3^{(s)}|\psi\rangle|^2.$$

With $s = 3/2$ we have $\langle\psi|S_3^{(3/2)}|\psi\rangle = 0$ for all four Bell states and obtain for $|\Phi_+\rangle$, $|\Phi_-\rangle$ that $3/4 > 0$. For $|\Psi_+\rangle$ and the singlet state $|\Psi_-\rangle$ we find $\sqrt{21}/4 > 0$. For $s = 3/2$ and the spin coherent state we obtain

$$\frac{9}{16}(\cos^2(\theta) + \sin^4(\theta)\sin^2(\phi)\cos^2(\phi)) \geq \frac{9}{16}\cos^2(\theta).$$

Note that the standard form of the uncertainty relation we obtain by taking the square root on both sides. The non-negative term $\sin^4(\theta)\sin^2(\phi)\cos^2(\phi)$ takes a maximum for $\theta = \pi/2$, $\phi = \pi/4$, namely $1/4$. For $\theta = 0$ and ϕ arbitrary we find an equality for the uncertainty relation.

Consider now the Bell inequality, Bell states and spin coherent states. Let σ_1 and σ_2 be the Pauli spin matrices

and

$$\begin{aligned}\hat{A}_1 &= \sigma_1, \\ \hat{B}_1 &= \sigma_2, \\ \hat{B}_1 &= \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_2), \\ \hat{B}_2 &= \frac{1}{\sqrt{2}}(\sigma_1 - \sigma_2).\end{aligned}$$

Note that $(\hat{A}_1)^2 = (\hat{A}_2)^2 = (\hat{B}_1)^2 = (\hat{B}_2)^2 = I_2$, where I_2 is the 2×2 identity matrix. We evaluate

$$\begin{aligned}|\langle \psi | \hat{A}_1 \otimes \hat{B}_1 | \psi \rangle + \langle \psi | \hat{A}_1 \otimes \hat{B}_2 | \psi \rangle \\ + \langle \psi | \hat{A}_2 \otimes \hat{B}_1 | \psi \rangle - \langle \psi | \hat{A}_2 \otimes \hat{B}_2 | \psi \rangle|\end{aligned}$$

with $|\psi\rangle$ replaced by the Bell states. We obtain the well-known results

$$\begin{aligned}|\langle \Phi_+ | \hat{A}_1 \otimes \hat{B}_1 | \Phi_+ \rangle + \langle \Phi_+ | \hat{A}_1 \otimes \hat{B}_2 | \Phi_+ \rangle \\ + \langle \Phi_+ | \hat{A}_2 \otimes \hat{B}_1 | \Phi_+ \rangle - \langle \Phi_+ | \hat{A}_2 \otimes \hat{B}_2 | \Phi_+ \rangle| = 0\end{aligned}$$

$$\begin{aligned}|\langle \Phi_- | \hat{A}_1 \otimes \hat{B}_1 | \Phi_- \rangle + \langle \Phi_- | \hat{A}_1 \otimes \hat{B}_2 | \Phi_- \rangle \\ + \langle \Phi_- | \hat{A}_2 \otimes \hat{B}_1 | \Phi_- \rangle - \langle \Phi_- | \hat{A}_2 \otimes \hat{B}_2 | \Phi_- \rangle| = 0\end{aligned}$$

$$\begin{aligned}|\langle \Psi_+ | \hat{A}_1 \otimes \hat{B}_1 | \Psi_+ \rangle + \langle \Psi_+ | \hat{A}_1 \otimes \hat{B}_2 | \Psi_+ \rangle \\ + \langle \Psi_+ | \hat{A}_2 \otimes \hat{B}_1 | \Psi_+ \rangle - \langle \Psi_+ | \hat{A}_2 \otimes \hat{B}_2 | \Psi_+ \rangle| = \sqrt{2} \cdot 2\end{aligned}$$

$$\begin{aligned}|\langle \Psi_- | \hat{A}_1 \otimes \hat{B}_1 | \Psi_- \rangle + \langle \Psi_- | \hat{A}_1 \otimes \hat{B}_2 | \Psi_- \rangle \\ + \langle \Psi_- | \hat{A}_2 \otimes \hat{B}_1 | \Psi_- \rangle - \langle \Psi_- | \hat{A}_2 \otimes \hat{B}_2 | \Psi_- \rangle| = \sqrt{2} \cdot 2.\end{aligned}$$

If we find θ, ϕ such that

$$\begin{aligned}|\langle \Omega | \hat{A}_1 \otimes \hat{B}_1 | \Omega \rangle + \langle \Omega | \hat{A}_1 \otimes \hat{B}_2 | \Omega \rangle \\ + \langle \Omega | \hat{A}_2 \otimes \hat{B}_1 | \Omega \rangle - \langle \Omega | \hat{A}_2 \otimes \hat{B}_2 | \Omega \rangle| > 2\end{aligned}$$

then the Bell inequality is violated. For the absolute value we obtain

$$12 \cdot \sqrt{2} \sqrt{\cos^2(\phi)} \cos^3(\theta/2) \sin^3(\theta/2).$$

With $\phi = 0, \theta/2 = \pi/4$ we obtain $3/\sqrt{2} \approx 2.121$ and the Bell inequality is violated for this value. The normalized state for $\phi = 0, \theta/2 = \pi/4$ is given by

$$\begin{pmatrix} 1/(2\sqrt{2}) \\ \sqrt{3}/(2\sqrt{2}) \\ \sqrt{3}/(2\sqrt{2}) \\ 1/(2\sqrt{2}) \end{pmatrix}.$$

This vector cannot be written as a Kronecker product of two vectors in $\mathbb{C}^2$. Note that the eigenvalues of the reduced density matrix for this vector are $1/2 + \sqrt{3}/4$ and $1/2 - \sqrt{3}/4$.

Consider the Bell matrix B , Bell states, spin coherent state for $\text{spin-}\frac{3}{2}$ and the Rayleigh quotient. For the Bell states we have

$$\langle \Phi_+ | B | \Phi_+ \rangle = \frac{1}{\sqrt{2}}, \quad \langle \Phi_- | B | \Phi_- \rangle = -\frac{1}{\sqrt{2}},$$

$$\langle \Psi_+ | B | \Psi_+ \rangle = \frac{1}{\sqrt{2}}, \quad \langle \Psi_- | B | \Psi_- \rangle = -\frac{1}{\sqrt{2}}.$$

Next we find the minimum of $f(z, \bar{z}) = f(x, y) = \langle z | B | z \rangle$ for the spin coherent state $|z\rangle$ for $\text{spin-}\frac{3}{2}$. Then

$$f(x, y) = \frac{1 + 8x^3 - x^6 + 3y^2 - 3x^4(1 + y^2) - y^4(3 + y^2) - 3x^2(-1 + 2y^2 + y^4)}{\sqrt{2}(1 + x^2 + y^2)^3}.$$

We find two critical points. The first critical point is $x = \frac{1}{2}(\sqrt{5} - 1), y = 0$. The second critical point is $x = -\frac{1}{2} - \frac{\sqrt{5}}{2}, y = 0$. Then for the first critical point we obtain $\frac{3}{\sqrt{10}}$ and for the second critical point we obtain $-\frac{3}{\sqrt{10}}$. Together with the two eigenvalues of the Bell matrix B , namely -1 and $+1$ we have

$$-1 < -\frac{3}{\sqrt{10}} < -\frac{1}{\sqrt{2}} < \frac{1}{\sqrt{2}} < \frac{3}{\sqrt{10}} < +1.$$

Hence the spin-coherent state does not provide the ground state for the Bell matrix.

Finally consider the Wehrl entropy and spin coherent states (Lieb [12], Schupp [14], Lieb and Solovej [15]). Let s be the spin $s = 1/2, 1, 3/2, 2, \dots$ and $|\Omega\rangle \equiv |\theta, \phi\rangle$ be a spin coherent state in $\mathbb{C}^{2s+1}$. Let ρ be a density matrix in $\mathbb{C}^{2s+1}$. With the spin coherent states one can define the semi-classical density distribution

$$\rho_{cl}(\Omega) := \langle \Omega | \rho | \Omega \rangle.$$

The measure is

$$d\Omega = \frac{2s+1}{4\pi} \sin(\theta) d\theta d\phi \equiv \frac{2s+1}{2\pi} \sin(\theta/2) \cos(\theta/2) d\theta d\phi.$$

The Wehrl entropy of $\rho_{cl}(\Omega)$ is defined as

$$\begin{aligned} S_W(\rho_{cl}) &= - \int_{\mathbb{S}^2} \rho_{cl}(\Omega) \ln(\rho_{cl}(\Omega)) d\Omega \\ &= - \frac{1}{\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \langle \Omega | \rho | \Omega \rangle \\ &\quad \times \sin(\theta) \ln(\langle \Omega | \rho | \Omega \rangle) d\theta d\phi \\ &= -2 \int_{\theta=0}^{\pi} \langle \Omega | \rho | \Omega \rangle \sin(\theta) \ln(\langle \Omega | \rho | \Omega \rangle) d\theta. \end{aligned}$$

For spin s the conjecture is

$$S_W(\rho) \geq \frac{2s}{2s+1}.$$

Let $s = 3/2$ and

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

With $\sin(\theta) \equiv 2 \sin(\theta/2) \cos(\theta/2)$ we have

$$\begin{aligned} S_W(\rho) &= -2 \int_{\theta=0}^{\pi} \cos^6(\theta/2) 2 \sin(\theta/2) \\ &\quad \times \cos(\theta/2) \ln(\cos^6(\theta/2)) d\theta \\ &= \frac{3}{4} = \frac{2s}{2s+1} \end{aligned}$$

for $s = 3/2$. When we consider the mixed state $\rho = \frac{1}{4} I_4$ we obtain

$$S_W(\rho) = 2 \ln(2) > \frac{3}{4}.$$

Extensions to higher dimensions are obvious with spin coherent states are elements in the Hilbert space $\mathbb{C}^{2s+1}$ with s the spin. For example $\mathbb{C}^6 = \mathbb{C}^{2s+1}$ ($s = 5/2$) and the orthonormal basis

$$\begin{aligned} &\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}^T, \\ &\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}^T, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}^T, \\ &\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}^T, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}^T, \end{aligned}$$

and the spin coherent states for $s = 5/2$. Here Hamilton operators [22] can be written as Kronecker products of spin

matrices $S_j^{(1/2)}$ and $S_k^{(1)}$ together with the 2×2 and 3×3 identity matrices. Another extension for the four Bell states with spin s ($s = 1/2, 1, 3/2, 2, \dots$) is given by

$$\begin{aligned} |B_1\rangle &= \frac{1}{\sqrt{2s+1}} \sum_{k=0}^{2s} |k\rangle \otimes |k\rangle \\ |B_2\rangle &= \frac{1}{\sqrt{2s+1}} \sum_{k=0}^{2s} (-1)^k |k\rangle \otimes |k\rangle \\ |B_3\rangle &= \frac{1}{\sqrt{2s+1}} \sum_{k=0}^{2s} |k\rangle \otimes |2s-k\rangle \\ |B_4\rangle &= \frac{1}{\sqrt{2s+1}} \sum_{k=0}^{2s} (-1)^k |k\rangle \otimes |2s-k\rangle. \end{aligned}$$

For $s = 1/2$ we obtain the standard Bell basis discussed above. Note that for spin $s = 1$ the Bell states are linearly dependent.

In the Hilbert space $\mathbb{C}^8 = \mathbb{C}^{2s+1}$ ($s = 7/2$) the GHZ-state and spin coherent states can be studied.

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