Entire functions sharing a polynomial with their derivatives and normal families

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Received: April 18, 2007

Summary: Let f be a nonconstant entire function and Q be a nonconstant polynomial. Let M be a positive number and $k \geq 2$ be an integer. We prove that if f and f' share Q CM and $|f^{(k)}(z)| \leq M \cdot (1 + |Q(z)|)$ whenever $f(z) = Q(z)$, then

$$\frac{f' - Q}{f - Q}$$

is constant. Furthermore, we prove the corresponding normality criterion.

1 Introduction and main results

Let f, g be two non-constant meromorphic functions in a domain $D \subseteq \mathbb{C}$, and let A be a function meromorphic in D or a finite value. We say that f and g share A IM (ignoring multiplicities) provided that $f - A$ and $g - A$ have the same zeros. If $f - A$ and $g - A$ have the same zeros with the same multiplicities, then we say that f and g share A CM (counting multiplicities). By

$$f^\# := \frac{|f'|}{1 + |f|^2}$$

we denote the spherical derivative of f .

Furthermore, we denote the open disk with center c and radius r by $U_r(c)$ and set $\mathbb{D} := U_1(0)$ for the open unit disk.

In 1979, Mues and Steinmetz [8] have proved the following well-known result.

Theorem A: *Let f be a nonconstant entire function. If f and f' share two finite, distinct values IM, then $f \equiv f'$.*

The question what can be said when an entire function shares only one finite nonzero value with its first derivative was first studied by Jank, Mues and Volkmann [5]. They proved the following theorem.

Theorem B: *Let f be an entire function which shares some finite value $a \neq 0$ IM with f' . If, moreover, $f(z) = a$ implies $f''(z) = a$, then $f \equiv f'$.*

AMS 2000 subject classification: Primary: 30D35, 30D45

Key words and phrases: Shared values, sharing a polynomial, entire function, normal family, Zalcman's lemma

Further results in this direction are due to Gundersen and Yang [3] as well as Lin and Yi [7], among others. They can be summarized as follows.

Theorem C: *Let f be a nonconstant entire function such that f and f' share one finite nonzero value a CM. Assume that at least one of the following additional conditions holds:*

- (a) *f has finite order. [3]*
- (b) *There exists a constant $M > 0$ and an integer $k \geq 2$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = a$. [7]*

Then there exists a constant $c \neq 0$ such that

$$\frac{f' - a}{f - a} \equiv c.$$

Li and Yi studied the case that an entire function of finite order shares a polynomial with its first derivative. They have shown ([6, Corollary 1.4])

Theorem D: *Let f be a nonconstant entire function of finite order and let Q be a nonconstant polynomial. If f and f' share Q CM, then*

$$\frac{f' - Q}{f - Q} \equiv c$$

for some constant $c \neq 0$.

We prove the following extension of this result to functions of arbitrary order.

Theorem 1.1 *Let f be a nonconstant entire function and Q be a nonconstant polynomial. Let M be a positive number and $k \geq 2$ be an integer. If f and f' share Q CM and $|f^{(k)}(z)| \leq M \cdot (1 + |Q(z)|)$ whenever $f(z) = Q(z)$, then*

$$\frac{f' - Q}{f - Q}$$

is constant.

Of course, the additional assumption on the values of $f^{(k)}$ at the zeros of $f - Q$ cannot be completely skipped since for every polynomial Q there exist entire solutions of the linear differential equation

$$f'(z) = e^z \cdot f(z) + (1 - e^z) \cdot Q(z), \quad \text{i.e. of} \quad \frac{f' - Q}{f - Q} = \exp.$$

A similar result has been obtained by Wang [13]; he has proved

Theorem E: *Let f be a nonconstant entire function and Q be a polynomial of degree $q \geq 1$. Let $k \geq q + 1$ be an integer. If f and f' share Q CM and $f^{(k)}(z) = Q(z)$ whenever $f(z) = Q(z)$, then $f' \equiv f$.*

For $Q(z) \equiv z$ this goes back to Wang and Yi [14].

In fact, Theorem E is an easy consequence of Theorem 1.1 as we will show in Section 5; there, from the proof in [13] (which is based on intricate considerations involving Nevanlinna theory) we merely use the fact that under the assumptions of Theorem E, f has to be a transcendental function [13, Lemma 7]; this is, of course, rather elementary, though a little bit tedious to show.

We also mention the following related result by Chang and Fang [1] which is an extension of the case $k = 2$ in Theorem E.

Theorem F: *Let f be a nonconstant entire function and let A be a meromorphic function such that $A \not\equiv A'$ and A is a small function relative to f in the sense of Nevanlinna theory (i.e. $T(r, A) = S(r, f)$). If f and f' share A IM and $f(z) = A(z)$ implies $f''(z) = A(z)$, then $f \equiv f'$.*

According to the heuristic principle known as “Bloch’s Principle”, one can expect that to each Picard type theorem there is a corresponding normality criterion. The normal family analogue belonging to Theorem B is due to Pang [9]. In fact, he has shown the following slightly stronger result.

Theorem G: *Let $\mathcal{F}$ be a family of functions analytic in the unit disk $\mathbb{D}$, $M > 0$ and $a \in \mathbb{C} \setminus \{0\}$. Suppose that for all $f \in \mathcal{F}$ and all $z \in \mathbb{D}$, $f(z) = 0$ implies $f'(z) = a$ and $f'(z) = a$ implies $|f''(z)| \leq M$. Then $\mathcal{F}$ is a normal family.*

A similar result for families of meromorphic functions was proved by Pang and Zalcman [11]. The normality result corresponding to case (b) of Theorem C is due to Lin and Yi [7].

We prove the following normality criterion belonging to Theorem 1.1.

Theorem 1.2 *Let $\mathcal{F}$ be a family of functions analytic in a domain $D \subseteq \mathbb{C}$ and A be an analytic function in D satisfying $A \not\equiv A'$. Let $k \geq 2$ be an integer and M be a positive number. If for each $f \in \mathcal{F}$, f and f' share A CM and $f(z) - A(z) = 0$ implies $|f^{(k)}(z)| \leq M$, then $\mathcal{F}$ is normal in D .*

The case $A(z) \equiv z$ of this Theorem can be found already in [15]; however, the proof given there seems to contain two major flaws.

Obviously, the condition $A \not\equiv A'$ in Theorem 1.2 is inevitable as the non-normal family of the functions $f_n(z) := e^{nz} + e^z$ which share $A(z) := e^z$ CM with their derivatives demonstrates. Furthermore, the non-normal family of the functions

$$f_n(z) := e^{nz} + \frac{n-1}{n^2} \cdot (nz+1)$$

which share $A(z) := z$ with f'_n CM shows that we cannot completely skip the assumption on $f^{(k)}$.

As a special case of Theorem 1.2 we obtain: *Let $\mathcal{F}$ be a family of functions analytic in a domain $D \subseteq \mathbb{C}$ and A be an analytic function in D satisfying $A \not\equiv A'$. If for each $f \in \mathcal{F}$, $f - A$ and $f' - A$ are nonvanishing in D , then $\mathcal{F}$ is normal in D . This has already been proved by Yang [16] and Schwick [12].*

2 Some Lemmas

The proof of our Theorems relies essentially on the following version of the well-known rescaling lemma by Zalcman and Pang [10, Lemma 1].

Lemma 2.1 *Let $\mathcal{F}$ be a family of functions meromorphic in $\mathbb{D}$, all of whose zeros have multiplicity at least m , and suppose there exists a $K \geq 1$ such that for all $f \in \mathcal{F}$ and all $z \in \mathbb{D}$, $f(z) = 0$ implies $|f^{(m)}(z)| \leq K$. If $\mathcal{F}$ is not normal in $z_0 \in \mathbb{D}$, then for each $\alpha \in [0; m]$ there exist sequences $(f_n)_n \subseteq \mathcal{F}$, $(z_n)_n \subseteq \mathbb{D}$ and $(\varrho_n)_n \subseteq (0; 1)$ such that $\lim_{n \rightarrow \infty} \varrho_n = 0$, $\lim_{n \rightarrow \infty} z_n = z_0$ and such that the sequence $(g_n)_n$ defined by*

$$g_n(\zeta) := \frac{1}{\varrho_n^\alpha} \cdot f_n(z_n + \varrho_n \zeta)$$

converges locally uniformly in $\mathbb{C}$ (with respect to the spherical metric) to a nonconstant function g meromorphic in $\mathbb{C}$ such that $g^\#(\zeta) \leq g^\#(0) = mK + 1$ for all $\zeta \in \mathbb{C}$.

Furthermore, the following lemma of Gundersen and Yang [3, Lemma 1] plays a crucial role in our proofs.

Lemma 2.2 *Let Q be a nonconstant polynomial. Then every solution F of the differential equation*

$$F' - e^Q F = 1$$

is an entire function of infinite order.

Since the following easy consequence from Hurwitz's theorem is needed in the proof of both of our Theorems, we formulate it as a lemma.

Lemma 2.3 *Let $(g_n)_n$ and $(h_n)_n$ be sequences of functions analytic in a domain D which converge to limit functions $g \not\equiv 0$ and $h \not\equiv 0$ locally uniformly in D . Assume that for each $n \in \mathbb{N}$, g_n and h_n share the value 0 CM in D . Then g and h share 0 CM as well.*

Proof: Let us assume that ζ_0 is a zero of g of multiplicity $m \geq 0$ (which for $m = 0$ means that it isn't a zero of g at all). Then in view of $g \not\equiv 0$ there exists an $\varepsilon_0 > 0$ such that g has no zeros in the punctured disk $U_{\varepsilon_0}(\zeta_0) \setminus \{\zeta_0\}$. Let an $\varepsilon \in]0; \varepsilon_0]$ be given. By Hurwitz's Theorem we can find an $N \in \mathbb{N}$ such that for all $n \geq N$ the function g_n has exactly m zeros in the open disk $U_\varepsilon(\zeta_0)$, each zero counted according to its multiplicity. The same holds for h_n since g_n and h_n share 0 CM. Since $h \not\equiv 0$, again by Hurwitz's Theorem we can conclude that h has exactly m zeros (counting multiplicities) in $U_\varepsilon(\zeta_0)$. Since this holds for every $\varepsilon \in]0; \varepsilon_0]$, h has a zero of multiplicity m in ζ_0 . This shows the assertion. $\square$

Finally, we use the following well-known fact which is immediate from the representation of the order of a meromorphic function via the Ahlfors–Shimizu characteristic (cf. [17, p. 217]).

Lemma 2.4 *Let f be a function meromorphic in $\mathbb{C}$ with bounded spherical derivative. Then the order of f is at most two.*

3 Proof of Theorem 1.1

We consider the function

$$F := \frac{f}{Q}$$

which is meromorphic in $\mathbb{C}$.

Case 1: F has bounded spherical derivative.

Then by Lemma 2.4, F has finite order. Hence $f = F \cdot Q$ has finite order as well. From Theorem D we obtain the assertion in this case.

Case 2: F has unbounded spherical derivative.

Then there exists a sequence $(w_n)_n$ such that $\lim_{n \rightarrow \infty} F^\#(w_n) = \infty$. Since $F^\#$ is continuous, hence bounded in every compact set, we have $w_n \rightarrow \infty$ for $n \rightarrow \infty$.

Since Q is a polynomial, there exists an $R > 0$ such that for all $z \in \mathbb{C}$ satisfying $|z| \geq R$ we have

$$|Q(z)| \geq 1, \quad |Q'(z)| \leq |Q(z)|. \quad (3.1)$$

In particular, F is analytic in $D := \{z \in \mathbb{C} : |z| \geq R\}$. Since f and f' share Q CM and Q has no zeros in D , we can conclude that in D (not necessarily in $\mathbb{C}$!) the functions $\frac{f}{Q} = F$ and $\frac{f'}{Q}$ share the value 1 CM. From

$$F' = \frac{f'}{Q} - \frac{f}{Q} \cdot \frac{Q'}{Q} = \frac{f'}{Q} - F \cdot \frac{Q'}{Q} \quad (3.2)$$

we deduce that F and $F' + F \cdot \frac{Q'}{Q}$ share the value 1 CM.

W.l.o.g. we may assume $|w_n| \geq R + 1$ for all n . We define

$$F_n(z) := F(w_n + z) \quad \text{and} \quad Q_n(z) := Q(w_n + z).$$

In view of (3.1), all F_n are analytic in the unit disk $\mathbb{D}$ and $F_n^\#(0) = F^\#(w_n) \rightarrow \infty$ for $n \rightarrow \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal in $z = 0$.

In view of $\lim_{n \rightarrow \infty} w_n = \infty$ the sequence $\left(\frac{Q'_n}{Q_n}\right)_n$ converges to 0 uniformly in $\mathbb{D}$.

Furthermore, for each n the functions F_n and $F'_n + F_n \cdot \frac{Q'_n}{Q_n}$ share the value 1 CM in $\mathbb{D}$.

Assume that $F_n(z_0) = 1$ for some $z_0 \in \mathbb{D}$. Then in view of (3.2) and (3.1)

$$|F'_n(z_0)| = \left| 1 - F_n(z_0) \cdot \frac{Q'_n}{Q_n}(z_0) \right| = \left| 1 - \frac{Q'_n}{Q_n}(z_0) \right| \leq 2.$$

Therefore, we can apply Lemma 2.1 with $\alpha = 1$. Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n \subseteq \mathbb{D}$ and $(\varrho_n)_n \subseteq (0; 1)$ such that $\lim_{n \rightarrow \infty} z_n = 0$, $\lim_{n \rightarrow \infty} \varrho_n = 0$ and such that the sequence $(g_n)_n$ defined by

$$g_n(\zeta) := \frac{1}{\varrho_n} \cdot (F_n(z_n + \varrho_n \zeta) - 1)$$

converges locally uniformly in $\mathbb{C}$ to a nonconstant entire function g satisfying $g^\#(\zeta) \leq g^\#(0) = 3$ for all $\zeta \in \mathbb{C}$. Lemma 2.4 implies that the order of g is at most two. (In fact,

since g is entire, from a well-known result of Clunie and Hayman [2] we can even deduce that the order of g is at most one. But this is not needed for our purpose.)

If $g' \equiv 1$ then we would obtain $g^\#(0) \leq |g'(0)| = 1$ which contradicts $g^\#(0) = 3$. Therefore, $g' \not\equiv 1$. We define

$$h_n(\zeta) := \left(F'_n + F_n \cdot \frac{Q'_n}{Q_n} \right) (z_n + Q_n \zeta) - 1.$$

Then g_n and h_n share the value 0 CM, and from

$$h_n(\zeta) = g'_n(\zeta) + (Q_n g_n(\zeta) + 1) \cdot \frac{Q'_n}{Q_n} (z_n + Q_n \zeta) - 1$$

we see that $(h_n)_n$ converges to $g' - 1$ locally uniformly in $\mathbb{C}$.

From Lemma 2.3 we deduce that g and $g' - 1$ share the value 0 CM. Observing that g has finite order, we obtain

$$\frac{g' - 1}{g} = e^P, \quad \text{i.e.} \quad g' - e^P \cdot g = 1. \quad (3.3)$$

for some polynomial P . Lemma 2.2 implies that P is constant. Solving (3.3), we obtain

$$g(\zeta) = B e^{A\zeta} - \frac{1}{A} \quad (3.4)$$

where $A = e^P$ and B are nonzero constants. Obviously, g has a zero ζ_0 . By Hurwitz's Theorem, there exists a sequence $(\zeta_n)_n$ tending to ζ_0 such that for all n sufficiently large $g_n(\zeta_n) = 0$, hence $(f - Q)(w_n + z_n + Q_n \zeta_n) = 0$. By our assumption and in view of (3.1) we have

$$\begin{aligned} |f^{(k)}(w_n + z_n + Q_n \zeta_n)| &\leq M \cdot (1 + |Q(w_n + z_n + Q_n \zeta_n)|) \\ &\leq 2M \cdot |Q(w_n + z_n + Q_n \zeta_n)| \end{aligned} \quad (3.5)$$

for all n sufficiently large. From Leibniz's rule we obtain

$$F^{(k)} = \frac{f^{(k)}}{Q} + \sum_{\mu=0}^{k-1} \binom{k}{\mu} f^{(\mu)} \left(\frac{1}{Q} \right)^{(k-\mu)}$$

and

$$f^{(\mu)} = \sum_{v=0}^{\mu} \binom{\mu}{v} F^{(v)} Q^{(\mu-v)}.$$

Setting $a_{\mu,v} := \binom{k}{\mu} \binom{\mu}{v}$ this gives

$$\begin{aligned}
g_n^{(k)}(\zeta) &= \varrho_n^{k-1} F_n^{(k)}(z_n + \varrho_n \zeta) \\
&= \varrho_n^{k-1} F^{(k)}(w_n + z_n + \varrho_n \zeta) \\
&= \varrho_n^{k-1} \left(\frac{f^{(k)}}{\varrho} + \sum_{\mu=0}^{k-1} \sum_{v=0}^{\mu} a_{\mu,v} F^{(v)} \varrho^{(\mu-v)} \left(\frac{1}{\varrho} \right)^{(k-\mu)} \right) (w_n + z_n + \varrho_n \zeta) \\
&= \varrho_n^{k-1} \cdot \frac{f^{(k)}}{\varrho} (w_n + z_n + \varrho_n \zeta) \\
&\quad + \sum_{\mu=1}^{k-1} \sum_{v=1}^{\mu} a_{\mu,v} \varrho_n^{k-v} g_n^{(v)}(\zeta) \cdot \left[\varrho_n^{(\mu-v)} \left(\frac{1}{\varrho_n} \right)^{(k-\mu)} \right] (z_n + \varrho_n \zeta) \\
&\quad + \sum_{\mu=0}^{k-1} a_{\mu,0} \left(\varrho_n^k g_n(\zeta) + \varrho_n^{k-1} \right) \cdot \left[\varrho_n^{(\mu)} \left(\frac{1}{\varrho_n} \right)^{(k-\mu)} \right] (z_n + \varrho_n \zeta).
\end{aligned} \tag{3.6}$$

Observing $\lim_{n \rightarrow \infty} w_n = \infty$, it is easy to see that for all integers $j, l \geq 0$ with $j + l \geq 1$ the sequence $\left(\varrho_n^{(j)} \left(\frac{1}{\varrho_n} \right)^{(l)} \right)_n$ tends to zero uniformly in $\mathbb{D}$. Of course, $\left(\varrho_n^{k-v} g_n^{(v)} \right)_n$ tends to zero locally uniformly in $\mathbb{C}$ for all $v = 0, \dots, k-1$. So from (3.5) and (3.6) for $n \rightarrow \infty$ we obtain

$$g^{(k)}(\zeta_0) = \lim_{n \rightarrow \infty} g_n^{(k)}(\zeta_n) = 0.$$

But from (3.4) we know that $g^{(k)}$ is nonvanishing in $\mathbb{C}$. This contradiction shows that Case 2 cannot occur and hence completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Since normality is a local property, it suffices to restrict to the case that D is the disk $U_2(0)$ and to show that $\mathcal{F}$ is normal at the origin. Then $A' - A$ is bounded in the unit disk $\mathbb{D}$, so there exists a $K > 1$ such that

$$|A'(z) - A(z)| \leq K \quad \text{for all } z \in \mathbb{D}. \tag{4.1}$$

First of all, we consider the case that

$$A(0) \neq A'(0).$$

Let us assume that $\mathcal{F}$ is not normal at the origin. Then the family

$$\tilde{\mathcal{F}} := \{f - A : f \in \mathcal{F}\}$$

is not normal at the origin either. In view of (4.1) and the fact that the functions in $\mathcal{F}$ share A with their derivative, for each $F \in \tilde{\mathcal{F}}$ and each $z_0 \in \mathbb{D}$ the condition $F(z_0) = 0$ implies

$$|F'(z_0)| = |A(z_0) - A'(z_0)| \leq K.$$

So by Lemma 2.1 we find sequences $(f_n)_n \subseteq \mathcal{F}$, $(z_n)_n \subseteq \mathbb{D}$ and $(\varrho_n)_n \subseteq (0; 1)$ such that $\lim_{n \rightarrow \infty} z_n = 0$, $\lim_{n \rightarrow \infty} \varrho_n = 0$ and such that the sequence $(g_n)_n$ defined by

$$g_n(\zeta) := \frac{1}{\varrho_n} \cdot F_n(z_n + \varrho_n \zeta) \quad \text{where} \quad F_n := f_n - A$$

converges locally uniformly in $\mathbb{C}$ to a nonconstant entire function g satisfying $g^\#(\zeta) \leq g^\#(0) = 1 + K$ for all $\zeta \in \mathbb{C}$. Lemma 2.4 implies that the order of g is at most two.

If $g' \equiv A(0) - A'(0)$ then in view of (4.1) we would obtain $g^\#(0) \leq |g'(0)| \leq K$ which contradicts $g^\#(0) = 1 + K$. Therefore, $g' \not\equiv A(0) - A'(0)$. We define

$$h_n(\zeta) := (f'_n - A)(z_n + \varrho_n \zeta).$$

Then g_n and h_n share the value 0 CM. From

$$h_n(\zeta) = g'_n(\zeta) + (A' - A)(z_n + \varrho_n \zeta)$$

we see that $(h_n)_n$ converges to $g' + A'(0) - A(0) \not\equiv 0$ locally uniformly in $\mathbb{C}$.

From Lemma 2.3 we deduce that g and $g' + A'(0) - A(0)$ share the value 0 CM. Observing that g has finite order, we obtain

$$g' + A'(0) - A(0) = e^P \cdot g, \quad \text{i.e.} \quad g' - e^P \cdot g = A(0) - A'(0) \quad (4.2)$$

for some polynomial P . Now in view of $A'(0) - A(0) \neq 0$, Lemma 2.2 implies that P is constant. Solving (4.2), we obtain

$$g(\zeta) = Be^{C\zeta} + \frac{A'(0) - A(0)}{C} \quad (4.3)$$

where $C = e^P$ and B are nonzero constants. Obviously, g has a zero ζ_0 . By Hurwitz's Theorem, there exists a sequence $(\zeta_n)_n$ tending to ζ_0 such that for all n sufficiently large $g_n(\zeta_n) = 0$, hence $(f_n - A)(z_n + \varrho_n \zeta_n) = 0$. By our assumption we have

$$|f_n^{(k)}(z_n + \varrho_n \zeta_n)| \leq M$$

for all n sufficiently large. From this and from

$$g_n^{(k)}(\zeta) = \varrho_n^{k-1} \left(f_n^{(k)} - A^{(k)} \right) (z_n + \varrho_n \zeta)$$

we obtain

$$g^{(k)}(\zeta_0) = \lim_{n \rightarrow \infty} g_n^{(k)}(\zeta_n) = 0.$$

But from (4.3) we know that $g^{(k)}$ is nonvanishing in $\mathbb{C}$. This contradiction shows that $\mathcal{F}$ is normal at 0 provided that $A(0) \neq A'(0)$.

Now we consider the case that

$$A(0) = A'(0).$$

Then in view of $A \not\equiv A'$ there exists an $r > 0$ such that

$$A(z) - A'(z) \neq 0 \quad \text{for all } z \in U_r(0) \setminus \{0\} =: G.$$

From the case considered so far we deduce that $\mathcal{F}$ is normal in G . Therefore every sequence in $\mathcal{F}$ contains a subsequence $(f_n)_n$ which converges locally uniformly in G to an analytic limit function f or possibly to $f \equiv \infty$. If f is analytic, the convergence of $(f_n)_n$ carries over to the whole of $U_r(0)$ by the maximum principle.

Therefore it only remains to consider the case $f \equiv \infty$. Then the sequence $(f_n - A)_n$ tends to ∞ uniformly on the circle $\{z \in \mathbb{C} : |z| = \frac{r}{2}\} =: \Gamma$, so there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $f_n(z) - A(z) \neq 0$ for all $z \in \Gamma$ and

$$\left| \int_{|z|=\frac{r}{2}} \frac{A' - A}{f_n - A}(z) dz \right| \leq \pi. \quad (4.4)$$

Since $f_n - A$ and $f'_n - A$ share 0 CM, $\frac{f'_n - A}{f_n - A}$ is analytic in $\mathbb{D}$, and by Cauchy's theorem we conclude that

$$\int_{|z|=\frac{r}{2}} \frac{f'_n - A}{f_n - A}(z) dz = 0. \quad (4.5)$$

By $n(\varrho, h)$ we denote the number of zeros of an analytic function h in $U_\varrho(0)$ counting multiplicity. Then from the argument principle, (4.5) and (4.4) we get for $n \geq N$

$$\begin{aligned} 0 \leq n\left(\frac{r}{2}, f_n - A\right) &= \left| \frac{1}{2\pi i} \int_{|z|=\frac{r}{2}} \frac{f'_n - A'}{f_n - A}(z) dz \right| \\ &= \frac{1}{2\pi} \cdot \left| \int_{|z|=\frac{r}{2}} \frac{A - A'}{f_n - A}(z) dz \right| \leq \frac{1}{2}, \end{aligned}$$

hence $n\left(\frac{r}{2}, f_n - A\right) = 0$. So $f_n - A$ has no zeros in $U_{r/2}(0)$, i.e. $\frac{1}{f_n - A}$ is analytic there. Therefore the uniform convergence of $\left(\frac{1}{f_n - A}\right)_n$ to 0 extends from $U_{r/2}(0) \setminus \{0\}$ to $U_{r/2}(0)$ by the maximum principle once more. But this means that $(f_n)_n$ tends to ∞ locally uniformly in $U_{r/2}(0)$. This shows that $\mathcal{F}$ is normal in 0 and hence completes the proof of Theorem 1.2.

5 Deduction of Theorem E from Theorem 1.1

Assume that f satisfies the assumptions of Theorem E. Then the assumptions of Theorem 1.1 are satisfied as well, and we obtain

$$f' = Q + c \cdot (f - Q) \quad (5.1)$$

for some constant $c \in \mathbb{C} \setminus \{0\}$. Differentiating this k times and using $Q^{(k)} \equiv 0$, we deduce

$$f^{(k+1)} = c \cdot f^{(k)}, \quad \text{hence} \quad f(z) = Ae^{cz} + P(z) \quad \text{for all } z \in \mathbb{C}$$

for some constant $A \in \mathbb{C}$ and a polynomial P of degree at most $k - 1$. As mentioned above, from [13, Lemma 7], we use the fact that f is transcendental. Therefore, $A \neq 0$.

Assume that $P \equiv Q$. Then we would obtain

$$cAe^{cz} + Q'(z) = f'(z) = Q(z) + c \cdot (f(z) - P(z)) = Q(z) + cAe^{cz}$$

for all $z \in \mathbb{C}$, hence $Q \equiv Q'$. But this would imply $Q \equiv 0$, a contradiction. Therefore, $Q \not\equiv P$.

Now from the version of the Second Fundamental Theorem for three small exceptional functions ([4, p. 47]) we can deduce that $f - Q$ has infinitely many zeros. (Otherwise $z \mapsto Ae^{cz} \not\equiv 0$ would have $Q - P \not\equiv 0$ as an exceptional function besides the two exceptional values 0 and ∞ .) But $f(z_0) = Q(z_0)$ implies $f^{(k)}(z_0) = Q(z_0)$, hence

$$c^k(Q - P)(z_0) = c^k(f - P)(z_0) = c^k A e^{cz_0} = f^{(k)}(z_0) = Q(z_0).$$

Therefore, the polynomial $c^k(Q - P) - Q$ has infinitely many zeros, and we can conclude $c^k(Q - P) - Q \equiv 0$, hence

$$f(z) = Ae^{cz} + \left(1 - \frac{1}{c^k}\right) \cdot Q.$$

Inserting this into (5.1) gives

$$cAe^{cz} + \left(1 - \frac{1}{c^k}\right) \cdot Q'(z) = cAe^{cz} + c \left(1 - \frac{1}{c^k}\right) Q(z) + (1 - c)Q(z)$$

for all $z \in \mathbb{C}$, hence

$$\left(1 - \frac{1}{c^k}\right) \cdot \frac{Q'}{Q} = 1 - \frac{1}{c^{k-1}}.$$

This implies $1 - \frac{1}{c^k} = 0$ and $1 - \frac{1}{c^{k-1}} = 0$, hence $c = 1$. Using (5.1), we arrive at $f' \equiv f$, and this is the conclusion of Theorem E.

Acknowledgements. The first author has received partial support by the German Israeli Foundation for Scientific Research and Development (No. G 809–234.6/2003).

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