Multi-Period Mean-Absolute Deviation Fuzzy Portfolio Selection Model with Entropy Constraints

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Abstract This paper considers a multi-period fuzzy portfolio selection problem maximizing the terminal wealth imposed by risk control, in which the returns of assets are characterized by fuzzy numbers. A fuzzy absolute deviation is originally defined as the risk control of portfolio. Entropy constraints and borrowing constraints are added in the portfolio selection model. Based on the theories of possibility measures, a new multi-period portfolio optimization model with transaction costs is proposed. And then, the proposed model is transformed into a crisp nonlinear programming problem by using fuzzy programming approach. Because of the transaction costs, the multi-period portfolio selection is the dynamic optimization problem with path dependence. Through changing the cost function into a variable, the multi-period portfolio selection is approximately turned into the dynamic programming. Furthermore, the discrete approximate iteration method is designed to obtain the optimal portfolio strategy. Finally, an example is given to illustrate the behavior of the proposed model and the designed algorithm using real data from the Shanghai Stock Exchange.

Keywords multi-period fuzzy portfolio selection; mean absolute deviation; transaction costs; entropy constraints; discrete approximate iteration method

1 Introduction

In 1952, Markowitz^[1] published his pioneering work which laid the foundation of modern portfolio analysis. Markowitz's model has served as a basis for the development of modern financial theory over the past six decades. However, contrary to its theoretical reputation, it is not used extensively to construct large-scale portfolios. One of the most important reasons for this is the computational difficulty associated with solving a large-scale quadratic programming problem with a dense covariance matrix. Konno and Yamazaki^[2] used the absolute deviation risk function to replace the risk function in Markowitz's model and formulated a mean absolute deviation portfolio optimization model. It turns out that the mean absolute deviation model

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maintains the favorable properties of Markowitz's model and removes most of the principal difficulties in solving Markowitz's model. Simaan^[3] provided a thorough comparison of the mean variance model and the mean absolute deviation model. Furthermore, Speranza^[4] used the semi-absolute deviation to measure the risk and formulated a portfolio selection model. Feinstein and Thapa^[5] proposed the average absolute deviation to measure risk.

For those models above, they are assumed that the investment horizon is single-period. But, in real world, the portfolio strategies are usually multi-period, since the investor can reallocate his wealth from time to time. So, it is natural to extend the single-period portfolio selections to multi-period portfolio selections. Mossin^[6] presented optimal multiperiod portfolio selection policies by using dynamic programming approach. Hakansson^[7] analyzed the multi-period mean-variance by means of a general theory of portfolio choice. Li, Chan, and Ng^[8] employed dynamic programming approach to deal with the multi-period safety-first portfolio selection problem. Using the same approach, Li and Ng^[9] considered the mean-variance formulation for the multi-period portfolio selection problem and determined the optimal portfolio selection policy and an analytical expression of the mean-variance efficient frontier. Calafiore^[10] concerned with multi-period sequential decision problems for financial asset allocation and presented a multi-period optimization with linear control policies. Zhu, et al.^[11] proposed a dynamic mean-variance portfolio selection model with risk control over bankruptcy. Wei and Ye^[12] proposed a multi-period mean-variance portfolio selection model with bankruptcy control in stochastic market. Güpınar and Rustem^[13] constructed a multi-period mean-variance optimization framework for the stochastic aspects of the scenario tree. Yu, et al.^[14] proposed a dynamic portfolio selection optimization with bankruptcy control for absolute deviation model. Çlikyurt and Özekici^[15] introduced several multi-period portfolio optimization models in stochastic markets using the mean-variance approach. However, all the literatures mentioned above, they often used variance as a risk measure. Since in case when return distributions of assets are asymmetric, using the variance as the risk measure may have a potential danger to sacrifice too much expected return in eliminating both low and high return extremes. To express or measure the real investment risk in financial market, scholars have employed some new risk measures to replace variance. Such as, Yan and Li^[16] and Yan, et al.^[17] substituted variance with semi-variance as the risk measure to deal with the multi-period portfolio selection problem. Pınar^[18] used the downside-risk measure as risk measure to study the multi-period portfolio selection problem. Considering the linear transaction costs, diversification degree of portfolio and skewness, Zhang, et al.^[19,20] and Liu, et al.^[21,22] proposed multiperiod fuzzy portfolio selection, and genetic algorithm, hybrid intelligent algorithm and differential evolution algorithm were proposed to solve them.

There are many non-probabilistic factors that affect the financial markets such that the return of risky asset is fuzzy uncertainty. Recently, a number of researchers investigated fuzzy portfolio selection problem. Watada^[23] and León, et al.^[24] discussed portfolio selection using fuzzy decision theory. Tanaka and Guo^[25,26] proposed two kinds of portfolio selection models based on fuzzy probabilities and exponential possibility distributions, respectively. Inuiguchi and Tanino^[27] introduced a possibilistic programming approach to the portfolio selection problem based on the minimax regret criterion. Wang and Zhu^[28], Lai, et al.^[29] and Giove, et

al.^[30] constructed interval programming models of portfolio selection. Zhang and Nie^[31] and Zhang, et al.^[32] discussed the admissible efficient portfolio selection under the assumption that the expected return and risk of asset have admissible errors to reflect the uncertainty in real investment actions and gave an analytic derivation of admissible efficient frontier when short sales are not allowed on all risky assets. Dubois and Prade^[33] defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. They also showed that this expectation remains additive in the sense of addition of fuzzy numbers. Carlsson and Fullér^[34] introduced the notions of lower and upper possibilistic mean values of a fuzzy number, viewing them as possibility distributions. Huang^[35–37] proposed mean variance, mean semi-variance and mean risk curve portfolio selection models. Zhang, et al.^[38], Zhang^[39], Zhang and Xiao^[40] proposed the portfolio selection models based on the lower and upper possibilistic means and possibilistic variances of fuzzy numbers. Li, et al.^[41,42] proposed mean- variance and mean-variance-skewness fuzzy portfolio, and used hybrid intelligent algorithms to solve them. Carlsson, et al.^[43] introduced a possibilistic approach to selecting portfolios with highest utility score under the assumption that the returns of assets are trapezoidal fuzzy numbers.

In this study, we propose the dynamic optimization model for multi-period portfolio rebalancing with cardinality constraints that is mix integer nonlinear. This paper is organized as follows. In Section 2, we introduce the definitions of the possibilistic mean, the possibilistic absolute deviation, and some properties. In Section 3, we formulate the possibilistic return, the possibilistic absolute deviation, cardinality constraints, borrowing constraints and transaction costs into the multi-period portfolio. We construct a multi-period portfolio selection model in Section 4, then through changing the cost function into a variable, the model is converted into a dynamic programming model, the discrete approximate iteration method is proposed to solve it, and the method is proved convergent. In Section 5, we give the comparison analysis of different entropies to illustrate the idea of our model and the effectiveness of the designed algorithm. Finally, some conclusions are given in Section 6.

2 Preliminaries

Let us introduce some definitions, which we need in the following section. A fuzzy number A is a fuzzy set of the real line R with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by F .

Carlsson and Fullér^[34] defined the lower and upper possibilistic mean values of fuzzy number A with γ -level set $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ ($\gamma \in [0, 1]$) as

$$M_*(A) = \frac{\int_0^1 a_1(\gamma) \text{Pos}(A \leq a_1(\gamma)) d\gamma}{\int_0^1 \text{Pos}(A \leq a_1(\gamma)) d\gamma} = 2 \int_0^1 \gamma a_1(\gamma) d\gamma$$

and

$$M^*(A) = \frac{\int_0^1 a_2(\gamma) \text{Pos}(A \geq a_2(\gamma)) d\gamma}{\int_0^1 \text{Pos}(A \geq a_2(\gamma)) d\gamma} = 2 \int_0^1 \gamma a_2(\gamma) d\gamma,$$

where Pos denotes possibility, i.e.,

$$\text{Pos}(A \leq a_1(\gamma)) = \prod(-\infty, a_1(\gamma)) = \sup_{u \leq a_1(\gamma)} A(u) = \gamma,$$

$$\text{Pos}(A \geq a_2(\gamma)) = \prod (a_2(\gamma), +\infty) = \sup_{u \geq a_2(\gamma)} A(u) = \gamma.$$

Let $A, B \in F$, and let $\lambda \in R$, Then the following results can be found in [50].

$$M_*(A + B) = M_*(A) + M_*(B), M^*(A + B) = M^*(A) + M^*(B),$$

$$M_*(\lambda A) = \begin{cases} \lambda M_*(A), & \text{if } \lambda \geq 0, \\ \lambda M^*(A), & \text{if } \lambda < 0, \end{cases}$$

and

$$M^*(\lambda A) = \begin{cases} \lambda M^*(A), & \text{if } \lambda \geq 0, \\ \lambda M_*(A), & \text{if } \lambda < 0. \end{cases}$$

According to above results, it's easy to get the following theorem.

Theorem 1 Let $A_i \in F$ and let $\lambda_i \in R, i = 1, 2, \dots, n$. Then

$$M_*\left(\sum_{i=1}^n \lambda_i A_i\right) = \sum_{i=1}^n |\lambda_i| M_*(\phi(\lambda_i) A_i),$$

$$M^*\left(\sum_{i=1}^n \lambda_i A_i\right) = \sum_{i=1}^n |\lambda_i| M^*(\phi(\lambda_i) A_i),$$

where $\phi(\lambda_i)$ is a sign function.

Definition 1 (Carlsson and Fullér^[34]) Let A be a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ ($\gamma \in [0, 1]$). Then the crisp possibilistic mean is defined as

$$\overline{M}(A) = (M^*(A) + M_*(A))/2 = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma. \quad (1)$$

Definition 2 For any two given fuzzy numbers A with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ ($\gamma \in [0, 1]$) and B with $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$ ($\gamma \in [0, 1]$), the possibilistic absolute deviation between A and B is defined as

$$\omega(A, B) = \frac{1}{2}(\overline{M} | A + B - \overline{M}(A) - \overline{M}(B) |). \quad (2)$$

A popular fuzzy number is trapezoidal fuzzy number $A = (a_l, b_l, \alpha_l, \beta_l)$ with membership function $\mu_A(x)$ in the following form

$$\mu_A(x) = \begin{cases} \frac{x - (a_l - \alpha_l)}{\alpha_l}, & x \in [a_l - \alpha_l, a_l], \\ 1, & x \in [a_l, b_l], \\ \frac{b_l + \beta_l - x}{\beta_l}, & x \in [b_l, b_l + \beta_l], \\ 0, & \text{otherwise,} \end{cases}$$

where α_l and β_l are positive numbers, i.e., $\alpha_l, \beta_l > 0$. Thus, the γ -level set of trapezoidal fuzzy number $A = (a_l, b_l, \alpha_l, \beta_l)$ can be expressed as $[A_l]^\gamma = [a_l - (1 - \gamma)\alpha_l, b_l + (1 - \gamma)\beta_l]$, for all $\gamma \in [0, 1]$.

By Definition 1, the lower and upper possibilistic means and the possibilistic mean value are respectively expressed as

$$\begin{aligned} M_*(A_i) &= a_i - \alpha_i/3, \\ M^*(A_i) &= b_i + \beta_i/3, \\ \overline{M}(A_i) &= \frac{a_i + b_i}{2} + \frac{\beta_i - \alpha_i}{6}. \end{aligned} \tag{3}$$

3 The Formulation of Multi-Period Portfolio Selection Problem

In this section, we discuss the multi-period portfolio selection problem with fuzzy returns. We first introduce the problem description and notations used in the following section. Then, we formulate the possibilistic return and risk of multi-period portfolio. Finally, we present cardinality constraints.

3.1 Problem Description and Notations

Let us consider a multi-period portfolio selection problem with n risky assets and a risk-free asset. The return rates of risky assets are denoted as fuzzy variables. Assume that an investor joins the market at the beginning of period 1 with initial wealth W_1 . The investor intends to allocate his/her wealth among the $n + 1$ assets for making T periods investment plan. His/her wealth can be reallocated among the $n + 1$ assets at the beginning of the following T consecutive time periods. To make it easier to follow our exposition, we put together all the notations that will be used hereafter.

- x_{it} the investment proportion of risky asset i at period t ;
- x_{i0} the initial investment proportion of risky asset i at period 1;
- x_t the portfolio at period t , where $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})$;
- x_{ft} the investment proportion of risk-free asset at period t , where $x_{ft} = 1 - \sum_{i=1}^n x_{it}$;
- x_{ft}^b the lower bound of the investment proportion of risk-free asset at period t , where $x_{ft} \geq x_{ft}^b$;
- R_{it} the return of risky asset i at period t ;
- r_{pt} the return rate of the portfolio x_t at period t ;
- r_{bt} the borrowing rate of the risk-free asset at period t ;
- r_{lt} the lending rate of the risk-free asset at period t ;
- u_{it} the upper bound constraints of x_{it} ;
- r_{Nt} the net return rate of the portfolio x_t at period t ;
- W_t the crisp form of the holding wealth at the beginning of period t ;
- c_{it} the unit transaction cost of risky asset i at period t ;
- K the desired number of assets in the portfolio at period t .

3.2 Possibilistic Return and Risk for Multi-Period Portfolio

As we know, in practical investment, different investors with different preferences may result from investment strategies. In order to achieve greater flexibility in portfolio selection, it is necessary to consider multiple criteria for expressing investors preferences. In the following subsections, we will introduce the criteria, including return, transaction cost, risk and the cardinality constraints of portfolio. We will quantify return by the possibilistic mean value and

risk by possibilistic absolute deviation about the fuzzy return of the asset. Assume that the whole investment process is self-financing, that is, the investor does not invest the additional capital during the portfolio selection. The return of risky asset, $R_{it} = (a_{it}, b_{it}, \alpha_{it}, \beta_{it})$ ($i = 1, 2, \dots, n; t = 1, 2, \dots, T$) are trapezoidal fuzzy numbers.

The borrowing constraints of portfolio selection are one of factors. Most of the brokerage houses provide the opportunity to make an acquisition on different assets by borrowing the money from the brokerage. Some researchers studied the borrowing constraints, for example, Deng and Li^[44] proposed a mean-variance fuzzy portfolio with borrowing constraint. Sadjadi, et al.^[45] proposed the fuzzy multi-period portfolio model with different rates for borrowing and lending. The borrowing constraints are taken account into the model. Then, derived from Equation (3), the possibilistic mean value of the portfolio $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})'$ at period t can be expressed as

$$\begin{aligned} r_{pt} &= \sum_{i=1}^n \overline{M}(R_{it})x_{it} + r_{ft} \left(1 - \sum_{i=1}^n x_{it}\right) \\ &= \sum_{i=1}^n \left(\frac{a_{it} + b_{it}}{2} + \frac{\beta_{it} - \alpha_{it}}{6}\right)x_{it} + r_{ft} \left(1 - \sum_{i=1}^n x_{it}\right), \quad t = 1, 2, \dots, T, \end{aligned} \quad (4)$$

where

$$r_{ft} = \begin{cases} r_{lt}, & 1 - \sum_{i=1}^n x_{it} \geq 0, \\ r_{bt}, & 1 - \sum_{i=1}^n x_{it} \leq 0, \end{cases} \quad r_{bt} \geq r_{lt}.$$

Transaction cost is one of the main concerns for portfolio managers. Arnott and Wagner^[46] found that ignoring transaction costs would result in an inefficient portfolio. Yoshimoto's empirical analysis^[47] also drew the same conclusion. Bertsimas and Pachamanova^[48] and Gulpinar, et al.^[49] incorporated transaction costs into consideration to study the multi-period portfolio selection problem. We also assume that the transaction cost is a V-shaped function of differences between the t th period portfolio $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})$ and the $(t-1)$ th period portfolio $x_{t-1} = (x_{1t-1}, x_{2t-1}, \dots, x_{nt-1})$. That's to say, the transaction cost for asset i at period t is $c_{it} |x_{it} - x_{it-1}|$. Hence, the total transaction cost of the portfolio $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})$ at period t can be expressed as

$$C_t = \sum_{i=1}^n c_{it} |x_{it} - x_{it-1}|, \quad t = 1, 2, \dots, T. \quad (5)$$

Thus, the net return rate of the portfolio x_t at period t can be denoted as

$$\begin{aligned} r_{Nt} &= \sum_{i=1}^n \left(\frac{a_{it} + b_{it}}{2} + \frac{\beta_{it} - \alpha_{it}}{6}\right)x_{it} + r_{ft} \left(1 - \sum_{i=1}^n x_{it}\right) \\ &\quad - \sum_{i=1}^n c_{it} |x_{it} - x_{it-1}|, \quad t = 1, 2, \dots, T. \end{aligned} \quad (6)$$

Then, the crisp form of the holding wealth at the beginning of the period t can be written as

$$W_{t+1} = W_t(1 + r_{Nt})$$

$$= W_t \left(1 + \sum_{i=1}^n \left(\frac{a_{it} + b_{it}}{2} + \frac{\beta_{it} - \alpha_{it}}{6} \right) x_{it} + r_{ft} \left(1 - \sum_{i=1}^n x_{it} \right) - \sum_{i=1}^n c_{it} |x_{it} - x_{it-1}| \right), \quad t = 1, 2, \dots, T. \quad (7)$$

Derived from Equation (2), the possibilistic absolute deviation of the portfolio x_t can be expressed as

$$v_t = \overline{M} \left(\left| \sum_{i=1}^n (\overline{M}(R_{it}) - R_{it}) x_{it} \right| \right). \quad (8)$$

Theorem 2 Let $R_{it} = (a_{it}, b_{it}, \alpha_{it}, \beta_{it})$ be trapezoidal fuzzy numbers, $x_{it} \geq 0$ ($i = 1, 2, \dots, n; t = 1, 2, \dots, T$). Then

$$v_t = \sum_{i=1}^n \left(b_{it} - a_{it} + \frac{\beta_{it} + \alpha_{it}}{3} \right) x_{it}. \quad (9)$$

Proof Suppose $x_{it} \geq 0$, then

$$v_t = \overline{M} \left(\left| \sum_{i=1}^n (\overline{M}(R_{it}) - R_{it}) x_{it} \right| \right) = \sum_{i=1}^n \overline{M} |(\overline{M}(R_{it}) - R_{it})| x_{it}.$$

According to the define of the absolute deviation of the portfolio x_t , then

$$v_t = \sum_{i=1}^n \overline{M} (\max\{0, \overline{M}(R_{it}) - R_{it}\} - \min\{0, \overline{M}(R_{it}) - R_{it}\}). \quad (10)$$

Suppose $R_{it} = (a_{it}, b_{it}, \alpha_{it}, \beta_{it})$ be trapezoidal fuzzy numbers, then Equation (10) can be turned into

$$v_t = \sum_{i=1}^n \overline{M} \left(\left[0, b_{it} - a_{it} + \frac{\beta_{it}}{3}, 0, \alpha_{it} \right] + \left[b_{it} - a_{it} + \frac{\alpha_{it}}{3}, 0, \beta_{it}, 0 \right] \right). \quad (11)$$

Derived from Equation (3), then Equation (11) can be turned into $v_t = \sum_{i=1}^n (b_{it} - a_{it} + \frac{\beta_{it} + \alpha_{it}}{3}) x_{it}$, Which ends the proof.

In order to satisfy the requisition of decentralized investment, a novel possibilistic entropy will be developed to measure the diversification degree of portfolio. Before introducing the possibilistic entropy, let us first review the existing proportion entropy, which is employed to reflect the diversification degree of single-period portfolio selection problem in Fang, et al.^[50], Kapur^[51] and Jana, et al.^[52]. And the possibilistic entropy of the portfolio x_t can be expressed as follows:

$$En(x_t) = - \sum_{i=1}^n x_{it} \ln x_{it}. \quad (12)$$

From Equation (12), we have $x_{it} \geq 0$ ($i = 1, 2, \dots, n$), that is, every asset must be chosen for constructing a portfolio. Note that when $x_{1t} = x_{2t} = \dots = 1/n$, Equation (12) takes its maximum value. In other words, the diversification degree of the portfolio is the maximum. However, in the practical investment management, an investor often may not hope to distribute his/her wealth among every asset for constructing an extremely diversified portfolio. In particular, when the investor forecasts the rate of return on asset i , R_{it} , is less than the risk-free return rate, the investor may not support investment in asset i , namely, $x_{it} = 0$.

3.3 The Basic Multi-Period Portfolio Optimization Models

Assume that the objective of the investor wants to maximize terminal wealth over the whole T periods investment. At the same time, the desired number of assets in the portfolio at each period t must not achieve or exceed the given number. Thus, the multi-period portfolio selection problem can be formulated as the following problem (P1):

$$\begin{aligned} \max W_{T+1} = W_1 \prod_{t=1}^T & \left(\sum_{i=1}^n \left[\frac{(a_{it} + b_{it})}{2} + \frac{(\beta_{it} - \alpha_{it})}{6} \right] x_{it} + r_{ft} \left(1 - \sum_{i=1}^n x_{it} \right) - \sum_{i=1}^n c_{it} (|x_{it} - x_{it-1}|) \right) \\ \text{s.t.} \quad & \begin{cases} W_{t+1} = \left(1 + \left(\sum_{i=1}^n \left[\frac{(a_{it} + b_{it})}{2} + \frac{(\beta_{it} - \alpha_{it})}{6} \right] x_{it} \right. \right. & (a) \\ \quad \left. \left. + r_{ft} \left(1 - \sum_{i=1}^n x_{it} \right) - \sum_{i=1}^n c_{it} (|x_{it} - x_{it-1}|) \right) \right) W_t & (a) \\ \sum_{i=1}^n \left(b_{it} - a_{it} + \frac{\beta_{it} + \alpha_{it}}{3} \right) x_{it} \leq v_0 & (b) \\ 1 - \sum_{i=1}^n x_{it} \geq x_{ft}^b & (c) \\ \sum_{i=1}^n -x_{it} \ln x_{it} \geq H_t, \quad t = 1, 2, \dots, T & (d) \\ l_{it} \leq x_{it} \leq u_{it}, \quad i = 1, 2, \dots, n, \quad t = 1, 2, \dots, T & (e) \end{cases} \end{aligned} \quad (13)$$

The model (P1) consists of an objective, namely, the maximization of the investors' terminal wealth. The details of the model (P1) are shown in Equation (13), where constraint (13)(a) denotes the wealth accumulation constraint; constraint (13)(b) states the absolute deviation of the portfolio x_t can't exceed the given minimum risk constraint at each period; constraint (13)(c) indicates the investment proportion of risk-free asset at period t must exceed the given lower bound; constraint (13)(d) represents the desired number of assets in the portfolio must not exceed the given value; constraint (13)(e) states the lower and upper bound constraints of x_{it} .

4 The Optimization on the Multi-Period Portfolio Selection Model

The sub-problem of period t of the Model (13) can be transformed into

$$\begin{aligned} \max \sum_{i=1}^n & \left[\frac{(a_{it} + b_{it})}{2} + \frac{(\beta_{it} - \alpha_{it})}{6} \right] x_{it} + r_{ft} \left(1 - \sum_{i=1}^n x_{it} \right) - \sum_{i=1}^n c_{it} (|x_{it} - x_{it-1}|) \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^n \left(b_{it} - a_{it} + \frac{\beta_{it} + \alpha_{it}}{3} \right) x_{it} \leq v_0 \\ 1 - \sum_{i=1}^n x_{it} \geq x_{ft}^b \\ \sum_{i=1}^n -x_{it} \ln x_{it} \geq H_t \\ l_{it} \leq x_{it} \leq u_{it}, \quad i = 1, 2, \dots, n \end{cases} \end{aligned} \quad (14)$$

We propose a forward dynamic programming method to solve Model (13).

Algorithm The forward dynamic programming method.

Step 1 When $t = 1$, because of W_1 and $x_0 = (x_{10}, \dots, x_{n0})$ being given, we use CPLEX to solve the Model (14), then the optimal solution of period 1 $x_1^{\max*} = (x_{11}^{\max*}, \dots, x_{n1}^{\max*})$ can be got. At the same time, we can get

$$W_2^{\max*} = \left(1 + \left(\sum_{i=1}^n \left[\frac{(a_{i1} + b_{i1})}{2} + \frac{(\beta_{i1} - \alpha_{i1})}{6} \right] x_{i1} + r_{f1} \left(1 - \sum_{i=1}^n x_{i1} \right) - \sum_{i=1}^n c_{i1} (|x_{i1} - x_{i0}|) \right) \right) W_1.$$

Step 2 When $t = 2$, because of W_2 and $x_1 = (x_{11}, x_{21}, \dots, x_{n1})$ being given, we use CPLEX to solve the model (14), then the optimal solution of period 1 $x_2^{\max*} = (x_{12}^{\max*}, \dots, x_{n2}^{\max*})$ can be got. At the same time, we can get

$$W_3^{\max*} = \left(1 + \left(\sum_{i=1}^n \left(\frac{(a_{i2} + b_{i2})}{2} + \frac{(\beta_{i2} - \alpha_{i2})}{6} \right) x_{i2}^* + r_{f2} \left(1 - \sum_{i=1}^n x_{i2}^* \right) - \sum_{i=1}^n c_{i2} (|x_{i2}^* - x_{i1}|) \right) \right) W_2^{\max*}.$$

Step 3 Using the same method, we can get $W_t^{\max*}, t = 4, 5, \dots, T + 1$.

5 Numerical Example

In this section, a numerical example is given to express the idea of the proposed model. Assume that an investor chooses thirty stocks from Shanghai Stock Exchange for his investment. The stocks codes are respectively S1 (600000), S2 (600005), S3 (600015), S4 (600016), S5 (600019), S6 (600028), S7 (600030), S8 (600036), S9 (600048), S10 (600050), S11 (600104), S12 (600362), S13 (600519), S14 (600900), S15 (601088), S16 (601111), S17 (601166), S18 (601168), S19 (601318), S20 (601328), S21 (601390), S22 (601398), S23 (601600), S24 (601601), S25 (601628), S26 (601857), S27 (601919), S28 (601939), S29 (601988), S30 (601998). He intends to make five periods of investment with initial wealth $W_1 = 1$ and his wealth can be adjusted at the beginning of each period. We assume that the returns, risk and turnover rates of the thirty stocks at each period are represented as trapezoidal fuzzy numbers. We collect historical data of them from April 2006 to December 2010 and set every three months as a period to handle the historical data. Using the simple estimation method in Vercher et al.^[53] to handle their historical data, the trapezoidal possibility distributions of the return rates of assets at each period can be obtained as shown in Table 1.1 to Table 1.10.

Suppose that the transaction costs of assets of the two periods investment take the same value $c_{it} = 0.003 (i = 1, 2, \dots, 30; t = 1, 2, \dots, 5)$, investment preferences $v_0 = 0.1$, the given minimum entropy in the portfolio at period t $H_t = 0.6$ or $1.6, t = 1, 2, \dots, 5$, the lower bound of the investment proportion of risk-free asset $x_{ft}^b = -1$, the borrowing rate of the risk-free asset $r_{bt} = 0.017$, the lending rate of the risk-free asset $r_{lt} = 0.009, t = 1, 2, \dots, 5$, the lower $l_{it} = 0.05$ and upper bound constraints $u_{it} = 0.3 (i = 1, 2, \dots, 30; t = 1, 2, \dots, 5)$.

Table 1.1 The fuzzy return rates on assets of five periods investment

Asset t	Asset 1				Asset 2				Asset 3			
1	0.1300	0.1559	0.0919	0.1026	0.0556	0.0943	0.0463	0.1470	0.0921	0.1244	0.0670	0.0443
2	0.1339	0.1559	0.0771	0.1026	0.0603	0.1022	0.0498	0.1390	0.0925	0.1244	0.0521	0.0443
3	0.1357	0.1559	0.0699	0.1026	0.0645	0.1069	0.0454	0.1344	0.1034	0.1244	0.0620	0.0443
4	0.1449	0.1582	0.0553	0.1003	0.0742	0.1117	0.0391	0.1295	0.1059	0.1244	0.0467	0.0443
5	0.1480	0.1583	0.0557	0.1002	0.0943	0.1163	0.0552	0.1249	0.1099	0.1244	0.0497	0.0443

Table 1.2 The fuzzy return rates on assets of five periods investment

Asset t	Asset 4				Asset 5				Asset 6			
1	0.1044	0.1299	0.0604	0.0685	0.0611	0.0991	0.0601	0.0426	0.0899	0.1229	0.1470	0.0451
2	0.1106	0.1299	0.0646	0.0685	0.0702	0.0991	0.0627	0.0426	0.0916	0.1229	0.1477	0.0451
3	0.1210	0.1299	0.0704	0.0685	0.0809	0.0991	0.0412	0.0426	0.0936	0.1229	0.0946	0.0451
4	0.1249	0.1299	0.0708	0.0685	0.0820	0.0991	0.0421	0.0426	0.0952	0.1229	0.0624	0.0451
5	0.1250	0.1327	0.0594	0.0657	0.0860	0.0991	0.0429	0.0426	0.1029	0.1229	0.0627	0.0451

Table 1.3 The fuzzy return rates on assets of five periods investment

Asset t	Asset 7				Asset 8				Asset 9			
1	0.0675	0.0920	0.0439	0.1571	0.0981	0.1495	0.0558	0.0766	0.0513	0.0765	-0.1396	0.0825
2	0.0728	0.1085	0.0464	0.1406	0.1022	0.1495	0.0523	0.0766	0.0714	0.0866	0.0597	0.0790
3	0.0863	0.1120	0.0426	0.1371	0.1058	0.1495	0.0546	0.0766	0.0765	0.0870	0.0553	0.0786
4	0.0887	0.1171	0.0409	0.1320	0.1271	0.1495	0.0426	0.0766	0.0813	0.0908	0.0597	0.0748
5	0.0920	0.1217	0.0385	0.1274	0.1385	0.1528	0.0540	0.0733	0.0846	0.0921	0.0612	0.0735

Table 1.4 The fuzzy return rates on assets of five periods investment

Asset t	Asset 10				Asset 11				Asset 12			
1	0.0310	0.0443	0.0258	0.0347	0.0510	0.0639	0.0338	0.1217	0.1048	0.1438	0.0975	0.1645
2	0.0345	0.0475	0.0227	0.0314	0.0534	0.0650	0.0345	0.1206	0.1101	0.1504	0.0805	0.1579
3	0.0440	0.0497	0.0289	0.0292	0.0556	0.0781	0.0261	0.1075	0.1253	0.1506	0.0700	0.1577
4	0.0442	0.0518	0.0276	0.0271	0.0636	0.0811	0.0309	0.1045	0.1404	0.1577	0.0756	0.1506
5	0.0443	0.0540	0.0269	0.0249	0.0639	0.0842	0.0251	0.1014	0.1438	0.1641	0.0650	0.1442

Table 1.5 The fuzzy return rates on assets of five periods investment

Asset t	Asset 13				Asset 14				Asset 15			
1	0.1778	0.2319	0.0973	0.1060	0.0508	0.0746	0.0489	0.0364	0.1422	0.1550	0.0934	0.0826
2	0.1885	0.2319	0.0965	0.1060	0.0588	0.0746	0.0525	0.0364	0.1485	0.1550	0.0982	0.0826
3	0.2068	0.2319	0.1110	0.1060	0.0653	0.0746	0.0516	0.0364	0.1504	0.1571	0.0999	0.0806
4	0.2131	0.2319	0.1154	0.1060	0.0685	0.0746	0.0468	0.0364	0.1505	0.1624	0.0474	0.0752
5	0.2156	0.2319	0.0947	0.1060	0.0716	0.0746	0.0384	0.0364	0.1519	0.1680	0.0472	0.0696

Table 1.6 The fuzzy return rates on assets of five periods investment

Asset t	Asset 16				Asset 17				Asset 18			
1	0.0403	0.0833	0.5030	0.2098	0.1232	0.1621	0.0808	0.0701	0.0648	0.1183	0.0612	0.4231
2	0.0417	0.0833	0.2587	0.2098	0.1479	0.1621	0.0913	0.0701	0.0740	0.1625	0.0694	0.3789
3	0.0443	0.0868	0.0301	0.2063	0.1485	0.1621	0.0825	0.0701	0.0748	0.1949	0.0564	0.3465
4	0.0473	0.1020	0.0186	0.1911	0.1529	0.1621	0.0798	0.0701	0.0889	0.2044	0.0647	0.3369
5	0.0606	0.1064	0.0277	0.1867	0.1531	0.1626	0.0487	0.0696	0.1183	0.2144	0.0641	0.3269

Table 1.7 The fuzzy return rates on assets of five periods investment

Asset t	Asset 19				Asset 20				Asset 21			
1	0.0760	0.1000	0.0840	0.0550	0.1100	0.1284	0.0578	0.0552	0.0519	0.0833	0.0419	0.1091
2	0.0832	0.1000	0.0632	0.0550	0.1150	0.1284	0.0566	0.0552	0.0524	0.0884	0.0364	0.1040
3	0.0856	0.1000	0.0636	0.0550	0.1152	0.1284	0.0554	0.0552	0.0752	0.0923	0.0580	0.1001
4	0.0880	0.1000	0.0440	0.0550	0.1200	0.1285	0.0473	0.0550	0.0798	0.0961	0.0599	0.0963
5	0.0903	0.1000	0.0423	0.0550	0.1217	0.1320	0.0215	0.0516	0.0833	0.1001	0.0619	0.0923

Table 1.8 The fuzzy return rates on assets of five periods investment

Asset t	Asset 22				Asset 23				Asset 24			
1	0.1075	0.1205	0.0692	0.0703	0.0123	0.0439	0.1039	0.2054	0.0805	0.1082	0.0765	0.0257
2	0.1134	0.1205	0.0582	0.0703	0.0151	0.0756	0.0834	0.1737	0.0811	0.1082	0.0501	0.0257
3	0.1162	0.1238	0.0585	0.0670	0.0221	0.0840	0.0899	0.1653	0.0886	0.1082	0.0530	0.0257
4	0.1197	0.1272	0.0618	0.0636	0.0231	0.0916	0.0587	0.1577	0.0928	0.1082	0.0471	0.0257
5	0.1201	0.1307	0.0435	0.0601	0.0439	0.0996	0.0437	0.1498	0.0959	0.1082	0.0431	0.0257

Table 1.9 The fuzzy return rates on assets of five periods investment

Asset t	Asset 25				Asset 26				Asset 27			
1	0.0921	0.1100	0.0692	0.0511	0.1054	0.1440	0.0824	0.1098	0.0282	0.0455	0.2042	0.1811
2	0.0941	0.1100	0.0572	0.0511	0.1111	0.1440	0.0741	0.1098	0.0368	0.0508	0.1553	0.1758
3	0.0974	0.1100	0.0504	0.0511	0.1217	0.1440	0.0837	0.1098	0.0390	0.0622	0.1410	0.1644
4	0.0976	0.1112	0.0246	0.0499	0.1377	0.1487	0.0757	0.1050	0.0412	0.0712	0.1082	0.1554
5	0.1036	0.1144	0.0189	0.0467	0.1400	0.1490	0.0732	0.1048	0.0455	0.0783	0.0347	0.1483

Table 1.10 The fuzzy return rates on assets of five periods investment

Asset t	Asset 28				Asset 29				Asset 30			
1	0.1291	0.1388	0.0758	0.0962	0.1026	0.1201	0.0645	0.0420	0.0928	0.1101	0.0600	0.0373
2	0.1303	0.1460	0.0710	0.0890	0.1045	0.1201	0.0563	0.0420	0.0972	0.1101	0.0571	0.0373
3	0.1324	0.1465	0.0608	0.0885	0.1066	0.1201	0.0580	0.0420	0.0995	0.1101	0.0592	0.0373
4	0.1345	0.1507	0.0298	0.0843	0.1113	0.1201	0.0372	0.0420	0.1019	0.1101	0.0533	0.0373
5	0.1388	0.1552	0.0282	0.0798	0.1133	0.1217	0.0270	0.0404	0.1021	0.1101	0.0310	0.0373

For comparison, we also use the discrete approximate iteration method to solve the Model (13). We can obtain the corresponding results as follows. If $H_t=0.6$, we can get the optimal solution as the follows:

Table 2 The optimal solution when $H_t = 0.6$

t	The optimal investment proportions		
1	Asset 13	Asset 18	otherwise 0
	0.6	0.4	
2	Asset 13	Asset 18	otherwise 0
	0.6	0.4	
3	Asset 13	Asset 18	otherwise 0
	0.6	0.4	
4	Asset 13	Asset 18	otherwise 0
	0.6	0.4	
5	Asset 13	Asset 18	otherwise 0
	0.6	0.4	

When $H_t = 0.6$, the optimal investment strategy at period 1 is $x_{131} = 0.6, x_{181} = 0.4$ and otherwise 0, which means investor should allocate his initial wealth on asset 13, asset 18 and otherwise asset by the proportions of 60%, 40% and otherwise 0 among the thirty stocks, respectively. The optimal investment strategy at period 2 is $x_{132} = 0.6, x_{182} = 0.4$ and otherwise 0, which means investor should allocate his initial wealth on asset 13, asset 18 and otherwise asset by the proportions of 60%, 40% and otherwise 0 among the thirty stocks, respectively. The optimal investment strategy at period 3 is $x_{133} = 0.6, x_{183} = 0.4$ and otherwise 0, which means investor should allocate his initial wealth on asset 13, asset 18 and otherwise asset by the proportions of 60%, 40% and otherwise 0 among the thirty stocks, respectively. The optimal investment strategy at period 4 is $x_{134} = 0.6, x_{184} = 0.4$ and otherwise 0, which means investor should allocate his initial wealth on asset 13, asset 18 and otherwise asset by the proportions of 60%, 40% and otherwise 0 among the thirty stocks, respectively. The optimal investment strategy at period 5 is $x_{135} = 0.6, x_{185} = 0.4$ and otherwise 0, which means investor should allocate his initial wealth on asset 13, asset 18 and otherwise asset by the proportions of 60%, 40% and otherwise 0 among the thirty stocks, respectively. In this case, the available terminal wealth is 2.514198.

If $H_t = 1.6$, we can get the optimal solution as Table 3.

Table 3 The optimal solution when $H_t = 1.6$

t	The optimal investment proportions								
1	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9
	0.0567	0.0027	0.0045	0.0084	0.0004	0.0010	0.0020	0.0159	0.0023
	Asset 11	Asset 12	Asset 13	Asset 14	Asset 15	Asset 17	Asset 18	Asset 19	Asset 20
	0.0003	0.0289	0.6	0.0001	0.0658	0.0428	0.0951	0.0006	0.0086
	Asset 21	Asset 22	Asset 24	Asset 25	Asset 26	Asset 28	Asset 29	Asset 30	otherwise
	0.0005	0.0062	0.0008	0.0019	0.0184	0.0331	0.0038	0.0019	0
2	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9
	0.0411	0.0021	0.0039	0.0084	0.0009	0.0015	0.0037	0.0140	0.0009
	Asset 11	Asset 12	Asset 13	Asset 14	Asset 15	Asset 17	Asset 18	Asset 19	Asset 20
	0.0006	0.0293	0.6	0.0003	0.0411	0.0467	0.1308	0.0015	0.0087
	Asset 21	Asset 22	Asset 24	Asset 25	Asset 26	Asset 28	Asset 29	Asset 30	otherwise
	0.0009	0.0075	0.0015	0.0028	0.0171	0.0001	0.0046	0.0027	0
3	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9
	0.0322	0.0027	0.0045	0.0091	0.0017	0.0027	0.0055	0.0122	0.0013
	Asset 11	Asset 12	Asset 13	Asset 14	Asset 15	Asset 16	Asset 17	Asset 18	Asset 19
	0.0010	0.0343	0.6	0.0005	0.0311	0.0021	0.0355	0.1522	0.0018
	Asset 20	Asset 21	Asset 22	Asset 24	Asset 25	Asset 26	Asset 28	Asset 29	Asset 30
	0.0077	0.0017	0.0076	0.0020	0.0032	0.0162	0.0227	0.0045	0.0028
4	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9
	0.0327	0.0030	0.0041	0.0072	0.0013	0.0027	0.0048	0.0161	0.0012
	Asset 12	Asset 13	Asset 15	Asset 16	Asset 17	Asset 18	Asset 19	Asset 20	Asset 21
	0.0367	0.5972	0.0359	0.0024	0.0281	0.1536	0.0017	0.0068	0.0015
	Asset 22	Asset 23	Asset 24	Asset 25	Asset 26	Asset 28	Asset 29	Asset 30	otherwise
	0.0062	0.0006	0.0018	0.0031	0.0200	0.0231	0.0044	0.0024	0
5	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9
	0.0255	0.0033	0.0032	0.0061	0.0011	0.0024	0.0042	0.0148	0.0010
	Asset 12	Asset 13	Asset 15	Asset 16	Asset 17	Asset 18	Asset 19	Asset 20	Asset 21
	0.0346	0.5257	0.0295	0.0024	0.0264	0.2558	0.0013	0.0067	0.0013
	Asset 22	Asset 23	Asset 24	Asset 25	Asset 26	Asset 28	Asset 29	Asset 30	otherwise
	0.0056	0.0009	0.0015	0.0015	0.0154	0.0202	0.0039	0.0021	0

The available terminal wealth is 2.410746.

To display the influence of H_t on the portfolio decision-making, we set its value as 0.6 and 1.6, respectively. Then, we use the Model (13) for portfolio decision-making. After using the discrete approximate iteration method, the corresponding optimal investment strategies can be obtained as shown in Table 2 and Table 3. From Table 2 and Table 3, it can be seen that when

the preset the desired number of assets in the portfolio become larger, the terminal wealth also becomes larger, which reflects the influence of the desired number of assets on portfolio selection.

From the above calculation, we can get the following solution when the entropy becomes bigger and bigger, the portfolio selection becomes more and more diversified and the terminal wealth becomes smaller and smaller.

6 Conclusions

In this paper, we discuss a multi-period portfolio selection problem in fuzzy environment, in which the returns, risk of risky assets are characterized by trapezoidal fuzzy variables rather than single values. We use fuzzy analysis approach to handle the imprecise data in financial markets and propose a multi-period fuzzy portfolio optimization model. Since the proposed model is a fuzzy programming problem, we use the fuzzy decision-making technique to convert it into a crisp form dynamic optimization with path dependence. Through changing the cost function into a variable, the multi-period portfolio selection is approximately turned into the dynamic programming. Furthermore, the discrete approximate iteration method is designed to obtain the optimal portfolio strategy. A numerical example is given to illustrate the application of the proposed model and demonstrate the effectiveness of the designed algorithm for solving our model.

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