New Results on Multiple Solutions for Intuitionistic Fuzzy Differential Equations

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Abstract The first-order fuzzy differential equations with intuitionistic fuzzy initial valued problem is studied. Firstly, with the help of (α, β) -cut set, the distance metric and the Hukuhara difference between intuitionistic fuzzy numbers are defined, and on this basis the concept of differentiability for the intuitionistic fuzzy number-valued functions is defined and a corresponding theorem is derived. Then, according to the selection of derivative the first order intuitionistic fuzzy differential equations is interpreted. Finally, an example of first-order linear intuitionistic fuzzy differential equations is solved, and the results show that there are four different solutions.

Keywords (α, β) -cut set; intuitionistic fuzzy number; Hukuhara difference of intuitionistic fuzzy number; differentiability of intuitionistic fuzzy number-valued functions; intuitionistic fuzzy differential equations

1 Introduction

As one of the research fields of fuzzy mathematics, fuzzy differential equations are a very important topic from the theoretical point of view as well as for their applications. There are several approaches to study fuzzy differential equations, the first one is based on the H-derivative, and the notion was introduced by Puri and Dan^[1], and later it was further explored by Kaleva^[2] and Seikkala^[3]. However, according to Diamond^[4], the H-derivative method has a shortcoming that a fuzzy differential equation may have only solutions with nondecreasing lengths of the diameter of the level sets. This shortcoming was solved by Hüllermeier^[5], who interpreted a fuzzy differential equation as a family of differential inclusions. Another approach can be found in literature [6, 7], who used the Zadeh's extension principle expanding the ordinary differential equations to the fuzzy cases, Mizukoshi, et al.^[8] pointed out that the solutions of the Cauchy problem obtained by both Zadeh's extension and differential inclusions are the same. The extension method and differential inclusions have the same weakness that the derivative of fuzzy-number-valued functions is missing. The forth approach was proposed in 2005 by

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Bede and Gal^[9], who developed the generalized differentiability concept of fuzzy interval-valued functions and studied in literature [10]. This approach overcame above weakness. Moreover, Wang and Guo^[11–12] studied the analytic solutions of fuzzy differential systems using fuzzy structuring element method.

The intuitionistic fuzzy set is an extension of fuzzy set introduced by Zadeh^[13]. The concept intuitionistic fuzzy set was first proposed by Atanassov^[14] in 1983. Fuzzy sets is only characterized by the degree of belongingness but intuitionistic fuzzy set is characterized by two functions expressing the degree of belongingness and the degree of nonbelongingness, respectively, and so that the sum of both values is less than one^[15–16]. As far as we know, however, there is only few investigation on the intuitionistic fuzzy differential equation, Melliani and Chadli^[17–18] studied differential and partial differential equations under the intuitionistic fuzzy case. Abbasbandy and Allahviranloo^[19] obtained numerical solution of intuitionistic fuzzy differential equation by Runge-Kutta method. Lata and Kumar^[20] studied N th-order time dependent intuitionistic fuzzy linear differential equation, where initial valued described by trapezoidal intuitionistic fuzzy number. Recently, Mondal, et al.^[21] studied strong and weak solution of first order homogeneous intuitionistic fuzzy differential equation, subsequently, who studied system of differential equation in literature [22]. Melliani, et al.^[23] discussed the existence and uniqueness of the solution of the intuitionistic fuzzy differential equation using the successive approximation method. Based on the α -cut of an intuitionistic fuzzy set, Nirmala and Chenthur Pandian^[24] discussed the numerical solution of intuitionistic fuzzy differential equation by Euler method.

In this paper, we define the differentiability of the intuitionistic fuzzy number-valued functions and study the solutions of the intuitionistic fuzzy differential equations. The rest of this paper is organized as follows, In Section 2, we recall some basic concepts of intuitionistic fuzzy sets. In Section 3, the H-difference and the differentiability of the intuitionistic fuzzy number-valued functions are defined, and the corresponding theorems are obtained. In Section 4, first order intuitionistic fuzzy differential equations with intuitionistic fuzzy initial valued is solved by a new method according to the selection of derivative type. In Section 5, the proposed method is illustrated by solving a linear intuitionistic fuzzy differential equations. In the last section, we give some conclusions.

2 Basic Concepts

Definition 1^[15,16] Let X be a fixed set, the Atanassov's intuitionistic fuzzy set defined on X can be given as

$$\tilde{A}_I = \{x, u_{\tilde{A}_I}(x), v_{\tilde{A}_I}(x) | x \in X\},$$

where the values $u_{\tilde{A}_I}(x)$ and $v_{\tilde{A}_I}(x)$ denote, respectively, the membership degree and the non-membership degree of the element x to X in $\tilde{A}_I$, with the conditions:

$$u_{\tilde{A}_I} : X \rightarrow [0, 1], \quad v_{\tilde{A}_I} : X \rightarrow [0, 1], \quad 0 \leq u_{\tilde{A}_I}(x) + v_{\tilde{A}_I}(x) \leq 1, \quad \forall x \in X.$$

Moreover, $\pi_{\tilde{A}_I}(x) = 1 - u_{\tilde{A}_I}(x) - v_{\tilde{A}_I}(x)$ is called a hesitancy degree of x . If $\pi_{\tilde{A}_I}(x) = 0$, $\forall x \in X$ then the intuitionistic fuzzy set is the same to the traditional fuzzy set^[13]. The symbol $IFS(X)$ denotes the set of all intuitionistic fuzzy set on X .

Definition 2^[25] A set of (α, β) cut set, generated by intuitionistic fuzzy set, where α, β are fixed numbers such that $0 \leq \alpha + \beta \leq 1$ is defined as

$$[\tilde{A}_I]^{\alpha, \beta} = \{(x, u_{\tilde{A}_I}(x), v_{\tilde{A}_I}(x)) : x \in X, u_{\tilde{A}_I}(x) \geq \alpha, v_{\tilde{A}_I}(x) \leq \beta, \alpha, \beta \in [0, 1]\}.$$

denoted by $[\tilde{A}_I]^{\alpha, \beta} = \{\tilde{A}_1^\alpha, \tilde{A}_2^\beta\}$, which is defined as the crisp set of elements x which belong to at least to the degree α and which does belong to at most to the degree β .

Intuitionistic fuzzy number is the generalization of fuzzy number and so it can be represented in the following.

Definition 3^[25] An intuitionistic fuzzy subset of the real line R is called an intuitionistic fuzzy number if the following holds:

- i) There exist $m \in R$, $u_{\tilde{A}_I}(m) = 1$, and $v_{\tilde{A}_I}(m) = 0$ (m is called the mean value of $\tilde{A}_I$);
- ii) $u_{\tilde{A}_I}$ is a continuous mapping from R to the closed interval $[0, 1]$ and for all $x \in R$, the relation $0 \leq u_{\tilde{A}_I}(x) + v_{\tilde{A}_I}(x) \leq 1$ holds;
- iii) The membership and nonmembership function of $\tilde{A}_I$ is of the following form:

$$u_{\tilde{A}_I}(x) = \begin{cases} 0, & -\infty < x \leq a_1, \\ f_1(x), & x \in [a_1, m], \\ 1, & x = m, \\ h_1(x), & x \in [m, a_2], \\ 0, & a_2 \leq x < \infty, \end{cases}$$

where $f_1(x)$ and $h_1(x)$ are strictly increasing and decreasing functions in $[a_1, m]$ and $[m, a_2]$, respectively:

$$v_{\tilde{A}_I}(x) = \begin{cases} 1, & -\infty \leq x \leq a'_1, \\ f_2(x), & x \in [a'_1, m]; 0 \leq f_1(x) + f_2(x) \leq 1, \\ 0, & x = m, \\ h_2(x), & x \in [m, a'_2]; 0 \leq h_1(x) + h_2(x) \leq 1, \\ 1, & a'_2 \leq x < \infty, \end{cases}$$

where $a'_1 \leq a_1 \leq m \leq a_2 \leq a'_2$. The symbol $IFN(X)$ denotes the set of all intuitionistic fuzzy set on X .

The (α, β) -cut set representation of intuitionistic fuzzy number $\tilde{A}_I$ is given by

$$[\tilde{A}_I]^{\alpha, \beta} = \{\tilde{A}_1^\alpha, \tilde{A}_2^\beta | 0 \leq \alpha + \beta \leq 1\} = \{[\underline{A}_1^\alpha, \bar{A}_1^\alpha]; [\underline{A}_2^\beta, \bar{A}_2^\beta] | 0 \leq \alpha + \beta \leq 1\},$$

where $\tilde{A}^\alpha$ represents the α -cut set of membership function $u_{\tilde{A}_I}(x)$, $\tilde{A}^\beta$ represents the β -cut set of nonmembership function $v_{\tilde{A}_I}(x)$.

Definition 4 Let $\tilde{A}_I = \{x, u_{\tilde{A}_I}(x), v_{\tilde{A}_I}(x)\} \in IFN(X)$, in the Definition 3, if the function $f_1(x) = \frac{x-a_1}{m-a_1}, x \in [a_1, m]$, $h_1(x) = \frac{a_2-x}{a_2-m}, x \in [m, a_2]$, $f_2(x) = \frac{m-x}{m-a'_1}, x \in [a'_1, m]$ and $h_2(x) = \frac{x-m}{a'_2-m}, x \in [m, a'_2]$, then $\tilde{A}_I$ is called triangular intuitionistic fuzzy number, denoted by $\tilde{A}_{TRI} = (m; a_1, a_2; a'_1, a'_2)$, the symbol $TrIFS(X)$ denotes the set of all intuitionistic fuzzy set on X .

Definition 5 Let $T = [a, b] \subseteq R$. Then $\tilde{f}_I$ is called an intuitionistic fuzzy number-valued function on T if $\tilde{f}_I : [a, b] \rightarrow IFN(X)$, and $\tilde{f}_I$ is called an n dimensional vector of intuitionistic fuzzy number-valued function on T if $\tilde{f}_I : [a, b] \rightarrow (IFN(X))^n$.

3 Differentiability of Intuitionistic Fuzzy Number-Valued Functions

In this section, the concept of differentiability for intuitionistic fuzzy number-valued functions is defined.

Definition 6 Let $\tilde{A}_I = \{x, u_{\tilde{A}_I}(x), v_{\tilde{A}_I}(x)\}$, $\tilde{B}_I = \{x, u_{\tilde{B}_I}(x), v_{\tilde{B}_I}(x)\} \in IFN(X)$, and $[\tilde{A}_I]^{\alpha, \beta} = \{[\underline{A}_1^\alpha, \bar{A}_1^\alpha]; [\underline{A}_2^\beta, \bar{A}_2^\beta]\}$, $[\tilde{B}_I]^{\alpha, \beta} = \{[\underline{B}_1^\alpha, \bar{B}_1^\alpha]; [\underline{B}_2^\beta, \bar{B}_2^\beta]\}$ such that $0 \leq \alpha + \beta \leq 1$, then the distance between intuitionistic fuzzy number $\tilde{A}_I$ and $\tilde{B}_I$ is defined as

$$d(\tilde{A}_I, \tilde{B}_I) = \frac{1}{4} \left[\int_0^1 |\underline{A}_1^\alpha - \underline{B}_1^\alpha| d\alpha + \int_0^1 |\bar{A}_1^\alpha - \bar{B}_1^\alpha| d\alpha + \int_0^1 |\underline{A}_2^\beta - \underline{B}_2^\beta| d\beta + \int_0^1 |\bar{A}_2^\beta - \bar{B}_2^\beta| d\beta \right].$$

Obviously, for $\tilde{A}_I, \tilde{B}_I, \tilde{C}_I \in IFN(X)$, the following

- (a) $d(\tilde{A}_I, \tilde{A}_I) = 0$,
- (b) $d(\tilde{A}_I, \tilde{B}_I) = d(\tilde{B}_I, \tilde{A}_I)$,
- (c) $d(\tilde{A}_I, \tilde{C}_I) \leq d(\tilde{A}_I, \tilde{B}_I) + d(\tilde{B}_I, \tilde{C}_I)$.

hold, hence $(IFN(X), d)$ is a metric space.

Definition 7 Let $\tilde{x}_I, \tilde{y}_I \in IFN(X)$. If there exists $\tilde{z}_I \in IFN(X)$ such that $\tilde{x}_I = \tilde{y}_I + \tilde{z}_I$, then $\tilde{z}_I$ is called the IH-difference of $\tilde{x}_I$ and $\tilde{y}_I$, it is denoted by $\tilde{z}_I = \tilde{x}_I \overset{IH}{-} \tilde{y}_I$.

Theorem 1 If $\tilde{x}_I, \tilde{y}_I \in IFN(X)$, then the (α, β) -cut set of the IH-difference $\tilde{x}_I$ and $\tilde{y}_I$ is H-difference of membership function and nonmembership function of $\tilde{x}_I, \tilde{y}_I$.

Proof Suppose that the IH-difference $\tilde{x}_I$ and $\tilde{y}_I$ is $\tilde{z}_I$. Then $\tilde{x}_I = \tilde{y}_I + \tilde{z}_I$, and using the (α, β) -cut set, we have $[\tilde{x}_I]^{\alpha, \beta} = [\tilde{y}_I]^{\alpha, \beta} + [\tilde{z}_I]^{\alpha, \beta}$. It follows that $\{\tilde{x}_1^\alpha, \tilde{x}_2^\beta\} = \{\tilde{y}_1^\alpha, \tilde{y}_2^\beta\} + \{\tilde{z}_1^\alpha, \tilde{z}_2^\beta\}$, i.e., $\tilde{x}_1^\alpha = \tilde{y}_1^\alpha + \tilde{z}_1^\alpha$, $\tilde{x}_2^\beta = \tilde{y}_2^\beta + \tilde{z}_2^\beta$. Then, $\tilde{x}_1 \overset{H}{-} \tilde{y}_1 = \tilde{z}_1$, $\tilde{x}_2 \overset{H}{-} \tilde{y}_2 = \tilde{z}_2$.

Definition 8 Let $\tilde{f}_I : [a, b] \rightarrow IFN(X)$ and $t_0 \in [a, b]$. We say that $\tilde{f}_I$ is differentiable at t_0 , if there exists an element $\tilde{f}'_I(t_0) \in IFN(X)$, such that:

(i) For all $h > 0$ sufficiently small, $\exists \tilde{f}_I(t_0 + h) \overset{IH}{-} \tilde{f}_I(t_0)$, $\exists \tilde{f}_I(t_0) \overset{IH}{-} \tilde{f}_I(t_0 - h)$, and the limits (in the metric d)

$$\lim_{h \rightarrow 0} \frac{\tilde{f}_I(t_0 + h) \overset{IH}{-} \tilde{f}_I(t_0)}{h} = \lim_{h \rightarrow 0} \frac{\tilde{f}_I(t_0) \overset{IH}{-} \tilde{f}_I(t_0 - h)}{h} = \tilde{f}'_I(t_0).$$

(ii) For all $h > 0$ sufficiently small, $\exists \tilde{f}_I(t_0) \overset{IH}{-} \tilde{f}_I(t_0 + h)$, $\exists \tilde{f}_I(t_0 - h) \overset{IH}{-} \tilde{f}_I(t_0)$, and the limits (in the metric d)

$$\lim_{h \rightarrow 0} \frac{\tilde{f}_I(t_0) \overset{IH}{-} \tilde{f}_I(t_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{\tilde{f}_I(t_0 - h) \overset{IH}{-} \tilde{f}_I(t_0)}{-h} = \tilde{f}'_I(t_0).$$

(iii) For all $h > 0$ sufficiently small, $\exists \tilde{f}_I(t_0 + h) \overset{IH}{-} \tilde{f}_I(t_0)$, $\exists \tilde{f}_I(t_0 - h) \overset{IH}{-} \tilde{f}_I(t_0)$, and the limits (in the metric d)

$$\lim_{h \rightarrow 0} \frac{\tilde{f}_I(t_0 + h) \overset{IH}{-} \tilde{f}_I(t_0)}{h} = \lim_{h \rightarrow 0} \frac{\tilde{f}_I(t_0 - h) \overset{IH}{-} \tilde{f}_I(t_0)}{-h} = \tilde{f}'_I(t_0).$$

(iv) For all $h > 0$ sufficiently small, $\exists \tilde{f}_I(t_0) \overset{IH}{-} \tilde{f}_I(t_0 + h)$, $\exists \tilde{f}_I(t_0) \overset{IH}{-} \tilde{f}_I(t_0 - h)$ and the limits (in the metric d)

$$\lim_{h \rightarrow 0} \frac{\tilde{f}_I(t_0) \overset{IH}{-} \tilde{f}_I(t_0 + h)}{-h} = \lim_{h \rightarrow 0} \frac{\tilde{f}_I(t_0) \overset{IH}{-} \tilde{f}_I(t_0 - h)}{h} = \tilde{f}'_I(t_0).$$

For the sake of simplicity, we say that the intuitionistic fuzzy-valued function $\tilde{f}_I$ is (i)-differentiable if it satisfies in Definition 8 case (i), and we say intuitionistic fuzzy-valued function $\tilde{f}_I$ is (ii)-differentiable if it satisfies in Definition 8 case (ii).

Theorem 2 Let $\tilde{f}_I : [a, b] \rightarrow IFN(X)$ be a intuitionistic fuzzy-valued function and denote $[\tilde{f}_I(t)]^{\alpha, \beta} = \{[\underline{f}_1(t, \alpha), \bar{f}_1(t, \alpha)]; [\underline{f}_2(t, \beta), \bar{f}_2(t, \beta)]\}$, for $0 \leq \alpha + \beta \leq 1$. Then

1) If $\tilde{f}_I$ is (i)-differentiable, then $\underline{f}_1(t, \alpha), \bar{f}_1(t, \alpha), \underline{f}_2(t, \beta)$ and $\bar{f}_2(t, \beta)$ are differentiable functions and $[\tilde{f}'_I(t)]^{\alpha, \beta} = \{[\underline{f}'_1(t, \alpha), \bar{f}'_1(t, \alpha)]; [\underline{f}'_2(t, \beta), \bar{f}'_2(t, \beta)]\}$.

2) If $\tilde{f}_I$ is (ii)-differentiable, then $\underline{f}_1(t, \alpha), \bar{f}_1(t, \alpha), \underline{f}_2(t, \beta)$ and $\bar{f}_2(t, \beta)$ are differentiable functions and $[\tilde{f}'_I(t)]^{\alpha, \beta} = \{[\bar{f}'_1(t, \alpha), \underline{f}'_1(t, \alpha)]; [\bar{f}'_2(t, \beta), \underline{f}'_2(t, \beta)]\}$.

Proof We present the details only for the case (1), since the other cases are analogous. For $h > 0, 0 \leq \alpha + \beta \leq 1$, let us consider $\tilde{f}_I$ to be IH-differentiable function in the case (1), then using Theorem 1,

$$[\tilde{f}_I(t_0 + h) \stackrel{IH}{\sim} \tilde{f}_I(t_0)]^{\alpha, \beta} = \{[\underline{f}_1(t_0 + h, \alpha) \stackrel{H}{\sim} \underline{f}_1(t_0, \alpha), \bar{f}_1(t_0 + h, \alpha) \stackrel{H}{\sim} \bar{f}_1(t_0, \alpha)]; [\underline{f}_2(t_0 + h, \beta) \stackrel{H}{\sim} \underline{f}_2(t_0, \beta), \bar{f}_2(t_0 + h, \beta) \stackrel{H}{\sim} \bar{f}_2(t_0, \beta)]\},$$

and

$$[\tilde{f}_I(t_0) \stackrel{IH}{\sim} \tilde{f}_I(t_0 - h)]^{\alpha, \beta} = \{[\underline{f}_1(t_0, \alpha) \stackrel{H}{\sim} \underline{f}_1(t_0 - h, \alpha), \bar{f}_1(t_0, \alpha) \stackrel{H}{\sim} \bar{f}_1(t_0 - h, \alpha)]; [\underline{f}_2(t_0, \beta) \stackrel{H}{\sim} \underline{f}_2(t_0 - h, \beta), \bar{f}_2(t_0, \beta) \stackrel{H}{\sim} \bar{f}_2(t_0 - h, \beta)]\},$$

multiplying two sides by $\frac{1}{h}$,

$$\left[\frac{\tilde{f}_I(t_0 + h) \stackrel{IH}{\sim} \tilde{f}_I(t_0)}{h} \right]^{\alpha, \beta} = \left\{ \left[\frac{\underline{f}_1(t_0 + h, \alpha) \stackrel{H}{\sim} \underline{f}_1(t_0, \alpha)}{h}, \frac{\bar{f}_1(t_0 + h, \alpha) \stackrel{H}{\sim} \bar{f}_1(t_0, \alpha)}{h} \right]; \left[\frac{\underline{f}_2(t_0 + h, \beta) \stackrel{H}{\sim} \underline{f}_2(t_0, \beta)}{h}, \frac{\bar{f}_2(t_0 + h, \beta) \stackrel{H}{\sim} \bar{f}_2(t_0, \beta)}{h} \right] \right\},$$

and

$$\left[\frac{\tilde{f}_I(t_0) \stackrel{IH}{\sim} \tilde{f}_I(t_0 - h)}{h} \right]^{\alpha, \beta} = \left\{ \left[\frac{\underline{f}_1(t_0, \alpha) \stackrel{H}{\sim} \underline{f}_1(t_0 - h, \alpha)}{h}, \frac{\bar{f}_1(t_0, \alpha) \stackrel{H}{\sim} \bar{f}_1(t_0 - h, \alpha)}{h} \right]; \left[\frac{\underline{f}_2(t_0, \beta) \stackrel{H}{\sim} \underline{f}_2(t_0 - h, \beta)}{h}, \frac{\bar{f}_2(t_0, \beta) \stackrel{H}{\sim} \bar{f}_2(t_0 - h, \beta)}{h} \right] \right\}.$$

passing to the limit, we have

$$\lim_{h \rightarrow 0} \left[\frac{\tilde{f}_I(t_0 + h) \stackrel{IH}{\sim} \tilde{f}_I(t_0)}{h} \right]^{\alpha, \beta} = \lim_{h \rightarrow 0} \left\{ \left[\frac{\underline{f}_1(t_0 + h, \alpha) \stackrel{H}{\sim} \underline{f}_1(t_0, \alpha)}{h}, \frac{\bar{f}_1(t_0 + h, \alpha) \stackrel{H}{\sim} \bar{f}_1(t_0, \alpha)}{h} \right]; \left[\frac{\underline{f}_2(t_0 + h, \beta) \stackrel{H}{\sim} \underline{f}_2(t_0, \beta)}{h}, \frac{\bar{f}_2(t_0 + h, \beta) \stackrel{H}{\sim} \bar{f}_2(t_0, \beta)}{h} \right] \right\},$$

and

$$\lim_{h \rightarrow 0} \left[\frac{\tilde{f}_I(t_0) \stackrel{IH}{\sim} \tilde{f}_I(t_0 - h)}{h} \right]^{\alpha, \beta} = \lim_{h \rightarrow 0} \left\{ \left[\frac{\underline{f}_1(t_0, \alpha) \stackrel{H}{\sim} \underline{f}_1(t_0 - h, \alpha)}{h}, \frac{\bar{f}_1(t_0, \alpha) \stackrel{H}{\sim} \bar{f}_1(t_0 - h, \alpha)}{h} \right]; \left[\frac{\underline{f}_2(t_0, \beta) \stackrel{H}{\sim} \underline{f}_2(t_0 - h, \beta)}{h}, \frac{\bar{f}_2(t_0, \beta) \stackrel{H}{\sim} \bar{f}_2(t_0 - h, \beta)}{h} \right] \right\}.$$

Using Definition 3^[10] and Definitions 8 we have $[\tilde{f}'_I(t)]^{\alpha,\beta} = \{[\underline{f}'_1(t, \alpha), \bar{f}'_1(t, \alpha)]; [\underline{f}'_2(t, \beta), \bar{f}'_2(t, \beta)]\}$ and the proof is now completed.

Remark 1 Since $[\tilde{f}_I(t)]^{\alpha,\beta} = \{\tilde{f}_1(t, \alpha), \tilde{f}_2(t, \beta)\} = \{[\underline{f}_1(t, \alpha), \bar{f}_1(t, \alpha)]; [\underline{f}_2(t, \beta), \bar{f}_2(t, \beta)]\}$, from case (1) of Theorem 2, $[\tilde{f}'_I(t)]^{\alpha,\beta} = \{[\underline{f}'_1(t, \alpha), \bar{f}'_1(t, \alpha)]; [\underline{f}'_2(t, \beta), \bar{f}'_2(t, \beta)]\}$, i.e. if $\tilde{f}_I$ is (i)-differentiable, then $\tilde{f}_1$ and $\tilde{f}_2$ are differentiable in the sense of case 1) of Definition 3^[10]. We say if $\tilde{f}_1$ and $\tilde{f}_2$ are differentiable in the sense of case 1) of Definition 3^[10], then $\tilde{f}_I$ is (1,1)-differentiable, denoted by $D^{(1,1)}\tilde{f}_I$. General, we say if $\tilde{f}_1$ is differentiable in the sense of case (m), $m \in \{1, 2\}$ of Definition 3^[10] and $\tilde{f}_2$ is differentiable in the sense of case (n), $n \in \{1, 2\}$ of Definition 3^[10], then $\tilde{f}_I$ is (m,n)-differentiable, denoted by $D^{(m,n)}\tilde{f}_I$, $m, n \in \{1, 2\}$. So we have the following results.

Theorem 3 Let $\tilde{f}_I : [a, b] \rightarrow IFN(X)$ be a intuitionistic fuzzy-valued function and denote $[\tilde{f}_I(t)]^{\alpha,\beta} = \{[\underline{f}_1(t, \alpha), \bar{f}_1(t, \alpha)]; [\underline{f}_2(t, \beta), \bar{f}_2(t, \beta)]\}$, for $0 \leq \alpha + \beta \leq 1$. Then:

- 1) If $\tilde{f}_I$ is (1,1)-differentiable, then $\underline{f}_1(t, \alpha)$, $\bar{f}_1(t, \alpha)$, $\underline{f}_2(t, \beta)$ and $\bar{f}_2(t, \beta)$ are differentiable functions and $[\tilde{f}'_I(t)]^{\alpha,\beta} = \{[\underline{f}'_1(t, \alpha), \bar{f}'_1(t, \alpha)]; [\underline{f}'_2(t, \beta), \bar{f}'_2(t, \beta)]\}$.
- 2) If $\tilde{f}_I$ is (1,2)-differentiable, then $\underline{f}_1(t, \alpha)$, $\bar{f}_1(t, \alpha)$, $\underline{f}_2(t, \beta)$ and $\bar{f}_2(t, \beta)$ are differentiable functions and $[\tilde{f}'_I(t)]^{\alpha,\beta} = \{[\underline{f}'_1(t, \alpha), \bar{f}'_1(t, \alpha)]; [\underline{f}'_2(t, \beta), \bar{f}'_2(t, \beta)]\}$.
- 3) If $\tilde{f}_I$ is (2,1)-differentiable, then $\underline{f}_1(t, \alpha)$, $\bar{f}_1(t, \alpha)$, $\underline{f}_2(t, \beta)$ and $\bar{f}_2(t, \beta)$ are differentiable functions and $[\tilde{f}'_I(t)]^{\alpha,\beta} = \{[\underline{f}'_1(t, \alpha), \bar{f}'_1(t, \alpha)]; [\underline{f}'_2(t, \beta), \bar{f}'_2(t, \beta)]\}$.
- 4) If $\tilde{f}_I$ is (2,2)-differentiable, then $\underline{f}_1(t, \alpha)$, $\bar{f}_1(t, \alpha)$, $\underline{f}_2(t, \beta)$ and $\bar{f}_2(t, \beta)$ are differentiable functions and $[\tilde{f}'_I(t)]^{\alpha,\beta} = \{[\underline{f}'_1(t, \alpha), \bar{f}'_1(t, \alpha)]; [\underline{f}'_2(t, \beta), \bar{f}'_2(t, \beta)]\}$.

Proof From Theorem 2 and Theorem 5^[26], the conclusion is easy obtained, we omitted here.

4 Solving Intuitionistic Fuzzy Differential Equations

In this section, we study the intuitionistic fuzzy initial value problem for a first-order intuitionistic fuzzy differential equation:

$$\begin{cases} \tilde{x}'_I(t) = f(t, \tilde{x}_I(t)), \\ \tilde{x}_I(t_0) = \tilde{x}_0, \end{cases} \quad (1)$$

where $\tilde{x}_0 \in IFN(R)$.

Our method of solving (1) is based on the selection of derivative type in the intuitionistic fuzzy differential equation. We first give the following definition for the solutions of (1).

Definition 9 Let $\tilde{x}_I : [a, b] \rightarrow IFN(R)$ be an intuitionistic fuzzy-valued function and $m, n = 1, 2$. We say x is an (m,n)-solution for problem (1), if $D^{(m,n)}\tilde{x}_I(t)$ exist and $D^{(m,n)}\tilde{x}_I(t) = f(t, \tilde{x}_I(t))$, $\tilde{x}_I(t_0) = \tilde{x}_0 \in IFN(R)$.

Remark 2 Let $[\tilde{x}_I(t)]^{\alpha,\beta} = \{[\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]; [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]\}$, and

$$\begin{aligned} [f(t, \tilde{x}_I(t))]^{\alpha,\beta} &= f(t, \{[\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]; [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]\}) \\ &= \{f(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]), f(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)])\} \\ &= \{[\underline{f}_1(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]), \bar{f}_1(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)])]; \\ &\quad [\underline{f}_2(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]), \bar{f}_2(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)])]\}, \end{aligned}$$

and $[\tilde{x}_I(t_0)]^{\alpha,\beta} = [\tilde{x}_0]^{\alpha,\beta} = \{[\underline{x}_1(t_0, \alpha), \bar{x}_1(t_0, \alpha)]; [\underline{x}_2(t_0, \beta), \bar{x}_2(t_0, \beta)]\} = \{[l_1, l_2]; [r_1, r_2]\}$. Using Theorem 3 and considering the intuitionistic initial values. Then problem (1) can be translated into a system of first-order ordinary differential equations, which called corresponding (m, n) -system for problem (1).

Therefore, four ordinary differential equations systems are possible for problem (1), as follows:

(1,1)-system

$$\left\{ \begin{array}{l} \underline{x}'_1(t, \alpha) = \underline{f}_1(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]), \\ \bar{x}'_1(t, \alpha) = \bar{f}_1(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]), \\ \underline{x}'_2(t, \beta) = \underline{f}_2(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]), \\ \bar{x}'_2(t, \beta) = \bar{f}_2(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]), \\ \underline{x}_1(t_0, \alpha) = l_1, \bar{x}_1(t_0, \alpha) = l_2, \\ \underline{x}_2(t_0, \beta) = r_1, \bar{x}_2(t_0, \beta) = r_2. \end{array} \right. \quad (2)$$

(1,2)-system

$$\left\{ \begin{array}{l} \underline{x}'_1(t, \alpha) = \underline{f}_1(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]), \\ \bar{x}'_1(t, \alpha) = \bar{f}_1(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]), \\ \underline{x}'_2(t, \beta) = \bar{f}_2(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]), \\ \bar{x}'_2(t, \beta) = \underline{f}_2(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]), \\ \underline{x}_1(t_0, \alpha) = l_1, \bar{x}_1(t_0, \alpha) = l_2, \\ \underline{x}_2(t_0, \beta) = r_1, \bar{x}_2(t_0, \beta) = r_2. \end{array} \right. \quad (3)$$

(2,1)-system

$$\left\{ \begin{array}{l} \underline{x}'_1(t, \alpha) = \bar{f}_1(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]), \\ \bar{x}'_1(t, \alpha) = \underline{f}_1(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]), \\ \underline{x}'_2(t, \beta) = \underline{f}_2(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]), \\ \bar{x}'_2(t, \beta) = \bar{f}_2(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]), \\ \underline{x}_1(t_0, \alpha) = l_1, \bar{x}_1(t_0, \alpha) = l_2, \\ \underline{x}_2(t_0, \beta) = r_1, \bar{x}_2(t_0, \beta) = r_2. \end{array} \right. \quad (4)$$

(2,2)-system

$$\left\{ \begin{array}{l} \underline{x}'_1(t, \alpha) = \bar{f}_1(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]), \\ \bar{x}'_1(t, \alpha) = \underline{f}_1(t, [\underline{x}_1(t, \alpha), \bar{x}_1(t, \alpha)]), \\ \underline{x}'_2(t, \beta) = \bar{f}_2(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]), \\ \bar{x}'_2(t, \beta) = \underline{f}_2(t, [\underline{x}_2(t, \beta), \bar{x}_2(t, \beta)]), \\ \underline{x}_1(t_0, \alpha) = l_1, \bar{x}_1(t_0, \alpha) = l_2, \\ \underline{x}_2(t_0, \beta) = r_1, \bar{x}_2(t_0, \beta) = r_2. \end{array} \right. \quad (5)$$

5 Example

In this section, we present an example, let us consider the following first-order linear intuitionistic fuzzy initial value problem

$$\begin{cases} \tilde{x}'_I(t) = k\tilde{x}'_I(t), \\ \tilde{x}_I(t_0) = \tilde{x}_0, \end{cases} \quad (6)$$

where $\tilde{x}_0$ is a triangular intuitionistic fuzzy number $(4; 3, 5; 1, 7)$ with (α, β) -cut set $\{[3 + \alpha, 5 - \alpha]; [4 - 3\beta, 4 + 3\beta]\}$, and $k \in \mathbb{R}$. Then, we have the following alternatives for solving the problem (6):

Case I $k > 0$

For getting (1,1)-solution, we obtain the following solutions of (1,1)-system:

$$\underline{x}_1(t, \alpha) = (3 + \alpha)e^{kt}, \quad \bar{x}_1(t, \alpha) = (5 - \alpha)e^{kt}, \quad \underline{x}_2(t, \beta) = (4 - 3\beta)e^{kt}, \quad \bar{x}_2(t, \beta) = (4 + 3\beta)e^{kt},$$

and

$$[\tilde{x}_1(t)]^\alpha = [(3 + \alpha)e^{kt}, (5 - \alpha)e^{kt}], \quad [\tilde{x}_2(t)]^\beta = [(4 - 3\beta)e^{kt}, (4 + 3\beta)e^{kt}]$$

are valid α and β -cut set, respectively, for $t \geq 0$. Take $\alpha = 0, [\tilde{x}_1(t)]^0 = [3e^{kt}, 5e^{kt}]$, take $\beta = 1, [\tilde{x}_2(t)]^1 = [e^{kt}, 7e^{kt}]$, take $\alpha = 1, \beta = 0, [\tilde{x}_1(t)]^1 = [\tilde{x}_2(t)]^0 = 4e^{kt}$, we have $e^{kt} < 3e^{kt} < 4e^{kt} < 5e^{kt} < 7e^{kt}$ for $t \geq 0$. Next, we consider the left sides membership and nonmembership function of $\tilde{x}_I(t)$, let $(3 + \alpha)e^{kt} = z_1, (4 - 3\beta)e^{kt} = z_2$, so $\alpha = z_1e^{-kt} - 3, \beta = \frac{4}{3} - \frac{z_2}{3}e^{-kt}$, let $z_1 = z_2 = z$, hence $\alpha + \beta = \frac{2z}{3}e^{-kt} - \frac{5}{3}, z \in [3e^{kt}, 4e^{kt}]$, we have $\frac{1}{3} \leq \alpha + \beta \leq 1$ satisfy $\alpha + \beta \in [0, 1]$. Similarly, we consider the right sides membership and nonmembership function of $\tilde{x}_I(t)$, let $(5 - \alpha)e^{kt} = y_1, (4 + 3\beta)e^{kt} = y_2$, so $\alpha = 5 - y_1e^{-kt}, \beta = \frac{y_2}{3}e^{-kt} - \frac{4}{3}$, let $y_1 = y_2 = y$, hence $\alpha + \beta = \frac{11}{3} - \frac{2y}{3}e^{-kt}, y \in [4e^{kt}, 5e^{kt}]$, we have $\frac{1}{3} \leq \alpha + \beta \leq 1$ satisfy $\alpha + \beta \in [0, 1]$. Hence

$$[\tilde{x}_I(t)]^{\alpha, \beta} = \{[(3 + \alpha)e^{kt}, (5 - \alpha)e^{kt}]; [(4 - 3\beta)e^{kt}, (4 + 3\beta)e^{kt}]\} \quad (7)$$

are valid (α, β) -cut set to (1,1)-solution, for $t \geq 0$. Based on the extension principle, the (1,1)-solution to problem (6) for $t \geq 0$ is

$$\tilde{x}_I(t) = \bigcup_{(\alpha, \beta) \in I^2} (\alpha, \beta)[\tilde{x}_I(t)]^{\alpha, \beta}. \quad (8)$$

Taking $k = 2$, we get $[\tilde{x}_I(t)]^{\alpha, \beta} = \{[(3 + \alpha)e^{2t}, (5 - \alpha)e^{2t}]; [(4 - 3\beta)e^{2t}, (4 + 3\beta)e^{2t}]\}$, it is a strong solution [18], Figure 1 shows the (1,1)-solution to problem (6) for $t \in [0, 2]$ and $k = 2$.

Figure 1 The (1,1)-solution of problem (6) with $k = 2$

Similar manner, we get the following (1,2)-solution for (1,2)-system:

$$[\tilde{x}_I(t)]^{\alpha,\beta} = \{[(3 + \alpha)e^{kt}, (5 - \alpha)e^{kt}]; [4e^{kt} - 3\beta e^{-kt}, 4e^{kt} + 3\beta e^{-kt}]\} \quad (9)$$

for $t \in [0, \frac{1}{2k} \ln 3]$. Take $k = 2$, the results are plotted in Figure 2 for $t \in [0, \frac{1}{4} \ln 3]$.

Figure 2 The (1,2)-solution of problem (6) with $k = 2$

The (2,1)-solution for (2,1)-system is given by

$$[\tilde{x}_I(t)]^{\alpha,\beta} = \{[4e^{kt} - (1 - \alpha)e^{-kt}, 4e^{kt} + (1 - \alpha)e^{-kt}]; [(4 - 3\beta)e^{kt}, (4 + 3\beta)e^{kt}]\} \quad (10)$$

for $t \geq 0$. Take $k = 2$, see Figure 3.

Figure 3 The (2,1)-solution of problem (6) with $k = 2$

The (2,2)-solution for (2,2)-system is given by

$$[\tilde{x}_I(t)]^{\alpha, \beta} = \{[4e^{kt} - (1-a)e^{-kt}, 4e^{kt} + (1-a)e^{-kt}]; [4e^{kt} - 3\beta e^{-kt}, 4e^{kt} + 3\beta e^{-kt}]\} \quad (11)$$

for $t \geq 0$. Take $k = 2$, the results are shown in Figure 4.

Figure 4 The (2,2)-solution of problem (6) with $k = 2$

Case II $k < 0$

The (1,1)-solution for (1,1)-system is given by

$$[\tilde{x}_I(t)]^{\alpha,\beta} = \{[4e^{kt} - (1-a)e^{-kt}, 4e^{kt} + (1-a)e^{-kt}]; [4e^{kt} - 3\beta e^{-kt}, 4e^{kt} + 3\beta e^{-kt}]\} \quad (12)$$

for $t \geq 0$. Take $k = -2$, Figure 5 shows the (1,1)-solution to problem (6) for $t \in [0, 2]$.

Figure 5 The (1,1)-solution of problem (6) with $k = -2$

The (1,2)-solution for (1,2)-system is given by

$$[\tilde{x}_I(t)]^{\alpha,\beta} = \{[4e^{kt} - (1-a)e^{-kt}, 4e^{kt} + (1-a)e^{-kt}]; [(4-3\beta)e^{kt}, (4+3\beta)e^{kt}]\} \quad (13)$$

for $t \geq 0$. Take $k = -2$, the results are plotted in Figure 6.

Figure 6 The (1,2)-solution of problem (6) with $k = -2$

The (2,1)-solution for (2,1)-system is given by

$$[\tilde{x}_I(t)]^{\alpha,\beta} = \{[(3 + \alpha)e^{kt}, (5 - a)e^{kt}]; [4e^{kt} - 3\beta e^{-kt}, 4e^{kt} + 3\beta e^{-kt}]\} \quad (14)$$

for $t \in [0, \frac{1}{-2k} \ln 3]$. Take $k = -2$, see Figure 7.

Figure 7 The (2,1)-solution of problem (6) with $k = -2$

The (2,2)-solution for (2,2)-system is given by

$$[\tilde{x}_I(t)]^{\alpha,\beta} = \{[(3 + \alpha)e^{kt}, (5 - a)e^{kt}]; [(4 - 3\beta)e^{kt}, (4 + 3\beta)e^{kt}]\} \quad (15)$$

for $t \geq 0$. Take $k = -2$, The results are shown in Figure 8.

Figure 8 The (2,2)-solution of problem (6) with $k = -2$

6 Conclusion

We have studied first-order intuitionistic fuzzy differential equations with intuitionistic fuzzy initial valued, based on the concept of differentiability of the intuitionistic fuzzy number-valued functions and the selection of derivative type. The first-order intuitionistic fuzzy differential equations can be translated into four ordinary different equations systems, and the results of illustrated example show that first-order intuitionistic fuzzy differential equations have four different solutions and extend the previous results in literature [20–21]. The uniqueness of solution is lost but according to literature [10], we can choose the solution that can reflects better the behavior of the modeled real-world problems through these solutions. For further study we can consider the higher-order intuitionistic fuzzy differential equations and boundary value problems for intuitionistic fuzzy differential equations.

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