An Asset Allocation Model and Its Solving Method

Qingye ZHANG

School of Management, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail: zhangqingye123@163.com

Yan GAO

School of Management, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail: gaoyan@usst.edu.cn

Abstract Asset allocation is an important issue in finance, and both risk and return are its fundamental ingredients. Rather than the return, the measure of the risk is complicated and of controversy. In this paper, we propose an appropriate risk measure which is precisely a convex combination of mean semi-deviation and conditional value-at-risk. Based on this risk measure, investors can trade-off flexibly between the volatility and the loss to tackle the incurring risk by choosing different convex coefficients. As the presented risk measure contains nonsmooth term, the asset allocation model based on it is nonsmooth. To employ traditional gradient algorithms, we develop a uniform smooth approximation of the plus function and convert the model into a smooth one. Finally, an illustrative empirical study is given. The results indicate that investors can control risk efficiently by adjusting the convex coefficient and the confidence level simultaneously according to their perceptions. Moreover, the effectiveness of the smoothing function proposed in the paper is verified.

Keywords portfolio optimization; risk measure; mean semi-deviation; conditional value-at-risk; non-smooth optimization

1 Introduction

According to microeconomics, rational investors are return preferable and risk averse. They aim at making low-risk and high-return investments. And a portfolio allocation is considered to be efficient if it has the minimum risk for a given expected return or the maximum expected return for a given risk level. Different from the return, the measure of the risk is complicated. As different investors have different perceptions of the risk, choosing an appropriate risk measure is still a challenging problem. In fact, which risk measure is most appropriate is still a subject of controversy. As we know, the mean-variance (MV) model^[1], where variance was used as a risk measure, is regarded as the cornerstone of modern portfolio theory. Nevertheless, variance is not a good risk measure since it penalizes portfolio's return both below and above the mean without discrimination and calculating the covariance matrix is time-consuming. As its correlation, some risk measures have been proposed, such as semi-variance^[2], mean absolute deviation (MAD)^[3,4] and minimax model^[5,6], etc. All the risk measures in the framework of variance

calculate portfolio's volatility, where uncertainty is regarded as the risk. Meanwhile, another class of risk measures, based on the safety-first principle (SFP) proposed by Roy^[7], has also been extensively studied. Since SFP measures the downside loss of a portfolio with probability, value-at-risk (VaR)^[8,9] defined as a quantile of the loss under a specific confidence level, can be regarded as its development. Although VaR has been widely used because of its simplicity and clarity, it has some undesirable properties. On the one hand, the fact that VaR lacks subadditivity leads to the optimization problem with VaR as its objective function nonconvex. On the other hand, VaR, as a quantile, ignores the loss in excess of itself. As an alternative, conditional VaR (CVaR)^[10], also called mean excess loss, mean shortfall or tail VaR, was proposed. As CVaR satisfies positive homogeneity, subadditivity, monotonicity and translation invariance, it is a coherent measure^[11] and thought to be better than VaR.

Roughly speaking, risk measures can be divided into two categories. One measures the volatility of a portfolio with variance as its representative; the other measures the loss with VaR as its representative. In recent years, not only were the differences between the volatility and the loss noticed, but portfolio optimization models with both of them were considered^[12–14] as well. Zhu, et al.^[12] substituted CVaR of a portfolio for its loss (i.e., negative return) in the classical MV model and established a CVaR-variance model, where there was no constraint on the portfolio's terminal expected return. Zhao, et al.^[13] proposed a mean-CVaR-skewness portfolio optimization model in the case of allowing for short-selling. Alexander and Baptista^[14] compared the VaR and CVaR constraints on the mean-variance model. In this paper, we focus on proposing a risk measure which combines the volatility with the loss explicitly. And an asset allocation model based on this risk measure is formulated too. Here we take mean semi-deviation (MSD) to represent the volatility rather than variance. As everyone knows, besides the rationality of the model, the numerical method is an issue which can not be ignored. Since a nonsmooth term is contained in the risk measure, asset allocation problem based on it is nonsmooth. Moreover, we explore its different numerical methods, especially the smoothing method.

This paper is organized as follows. In Section 2, we review MSD and CVaR at first, then propose a risk measure. An asset allocation model is established in Section 3. In Section 4, we explore the method of solving the model proposed in Section 3 in two cases. At first, we assume the securities' rate of return follows a multidimensional normal distribution and its mean and covariance matrix are known; Then, we assume only a discrete sample of the securities' rate of return is known. In the first case, the model can be converted into a quadratic programming under some conditions. And in the second case, we construct a uniform smooth approximation of the nonsmooth function and propose a smoothing method. In Section 5, an empirical study is carried out. At last, we give a concluding remark.

2 A New Risk Measure

As mentioned, risk measures can be divided into two classes by and large. As one of the risk measures measuring the volatility, MSD is appealing since it measures the portion that portfolio's return is less than its mean and problem based on it can be converted into a linear programming. Meanwhile, the argument that CVaR as a risk measure measuring the loss is

better than others can be found in many literatures^[10–15]. So in this section, we recall MSD and CVaR at first and then propose a risk measure which is a convex combination of them.

2.1 Mean Semi-Deviation (MSD)

Consider a portfolio investment on the n given risky assets $s_1, s_2, \dots, s_n$. Let y_i be the rate of return of s_i and let x_i be the proportion of wealth invested in s_i , $i = 1, 2, \dots, n$. Then, MAD of the portfolio $x = (x_1, x_2, \dots, x_n)^T$ is

$$w(x) = E \left[\left| \sum_{i=1}^n y_i x_i - E \left(\sum_{i=1}^n y_i x_i \right) \right| \right].$$

If $E(y_i) = r_i$, then

$$w(x) = E \left[\left| \sum_{i=1}^n (y_i - r_i) x_i \right| \right].$$

As the correction of MAD, MSD is defined as

$$\omega(x) = E \left[\left(\sum_{i=1}^n (y_i - r_i) x_i \right)^- \right] = E \left[\left(\sum_{i=1}^n (r_i - y_i) x_i \right)^+ \right],$$

where $(x)^- = \max\{-x, 0\}$, $(x)^+ = \max\{x, 0\}$ and $(x)^+$ is the so-called plus function. Different from MAD, MSD measures the portfolio's return less than its mean and can reflect the investor's perception better. Let $y = (y_1, y_2, \dots, y_n)^T$ and $r = (r_1, r_2, \dots, r_n)^T$. Then

$$\omega(x) = E \left\{ [(r - y)^T x]^+ \right\}. \quad (2.1)$$

2.2 Conditional Value-at-Risk (CVaR)

CVaR, as a risk measure, is a correction of VaR, which describes the conditional expectation of the loss exceeding VaR. Let $f(x, y)$ be the loss associated with x and y . Then, $CVaR = E[f(x, y) | f(x, y) \geq VaR]$. Evidently, VaR is never greater than CVaR. For each x , the loss $f(x, y)$ is a random variable with some distribution induced by that of y . For the convenience of expression, we assume y has a density function $p(y)$. Denote the β -VaR and β -CVaR of the loss $f(x, y)$ for a given confidence level β by $\alpha_\beta(x)$ and $\phi_\beta(x)$ respectively, namely

$$\alpha_\beta(x) = \min_{\alpha \in R} \{ \alpha | P[f(x, y) \leq \alpha] \geq \beta \}$$

and

$$\phi_\beta(x) = \frac{1}{1 - \beta} \int_{f(x, y) \geq \alpha_\beta(x)} f(x, y) p(y) dy.$$

In order to calculate CVaR, Rockafellar and Uryasev^[10] proposed the following auxiliary function

$$F_\beta(x, \alpha) = \alpha + \frac{1}{1 - \beta} \int_{y \in R^n} [f(x, y) - \alpha]^+ p(y) dy,$$

where $\alpha = \alpha_\beta(x)$. And in the same paper, they showed that

$$\min_{x \in X} \phi_\beta(x) = \min_{(x, \alpha) \in X \times R} F_\beta(x, \alpha).$$

2.3 A Mixed Risk Measure (MRisk)

We take $h_{\lambda,\beta}(x, \alpha)$, a convex combination of MSD and CVaR, as a new risk measure, i.e.,

$$h_{\lambda,\beta}(x, \alpha) = \lambda\omega(x) + (1 - \lambda)F_\beta(x, \alpha),$$

where $\lambda \in [0, 1]$ is a convex coefficient, indicating how much investors emphasize on the volatility. For convenience, we denote this risk measure MRisk. Evidently, $h_{\lambda,\beta}(x, \alpha)$ happens to be CVaR if $\lambda = 0$ and MSD if $\lambda = 1$. The larger λ is, the more emphasis an investor pays on the volatility and vice versa.

3 Portfolio Optimization Model Based on MRisk

Noticing that risk management is usually not an investor's primary purpose, a rational investor prefers to limit his/her risk while maximizing the expected return or minimize his/her risk for a given level of expected return. Without loss of generality, we focus on the portfolio selection problem which acquires a portfolio of minimum MRisk with a desired expected return.

Suppose an investor allocates his/her wealth on the n given risky assets and short-selling is prohibited. Then, portfolio optimization model with MRisk as its objective function can be formulated as:

$$\begin{aligned} & \underset{(x, \alpha) \in R^{n+1}}{\text{minimize}} && h_{\lambda,\beta}(x, \alpha) \\ & \text{subject to} && \sum_{i=1}^n x_i = 1, \quad x \geq 0, \quad x^T r \geq r_0, \end{aligned} \quad (3.1)$$

where r_0 is the minimal rate of return required by the investor.

4 Solution Scheme for Model (3.1)

Let $f(x, y) = -x^T y$ be the loss. At first, we assume the assets' rate of return is multivariate normally distributed. In this case, we can convert (3.1) into a quadratic programming under some conditions. Unfortunately, the rate of return does not follow Gaussian distribution in most occasions, it is leptokurtic, fat-tail and skew. As a matter of fact, the distribution of the rate of return is extremely difficult to obtain and is impossible to get in most cases. So, methods based on scenario analysis are popular. In this case, we explore a smoothing method for Model (3.1).

4.1 Case 1

Let $y \sim N(r, \Sigma)$, where $r = E[y]$ is the average rate of return, $\Sigma = (\sigma_{ij})_{n \times n}$ is the covariance matrix, σ_{ij} is the covariance of y_i and y_j , $i, j = 1, 2, \dots, n$. Let $x = (x_1, x_2, \dots, x_n)^T$ be the optimal portfolio and $\sigma(x) = \sqrt{x^T \Sigma x}$. By deducing, we have $\omega(x) = \frac{1}{\sqrt{2\pi}}\sigma(x)$ and $F_\beta(x, \alpha) = c_2(\beta)\sigma(x) - x^T r$, where $c_2(\beta) = \frac{1}{(1-\beta)}\phi[c_1(\beta)]$, $c_1(\beta) = \Phi^{-1}(\beta)$, Φ is the standard normal cumulative distribution function and Φ^{-1} is the inverse of Φ , ϕ is the standard normal density function. Therefore,

$$\begin{aligned} h_{\lambda,\beta}(x, \alpha) &= \lambda \frac{1}{\sqrt{2\pi}}\sigma(x) + (1 - \lambda)[-r^T x + c_2(\beta)\sigma(x)] \\ &= \left[\frac{\lambda}{\sqrt{2\pi}} + (1 - \lambda)c_2(\beta) \right] \sigma(x) - (1 - \lambda)r^T x. \end{aligned}$$

Moreover, (3.1) can be converted into the following problem

$$\begin{aligned} & \underset{(x, \alpha) \in \mathbb{R}^{n+1}}{\text{minimize}} && \left[\frac{\lambda}{\sqrt{2\pi}} + (1 - \lambda)c_2(\beta) \right] \sigma(x) - (1 - \lambda)r^T x \\ & \text{subject to} && \sum_{i=1}^n x_i = 1, \quad x \geq 0, \quad x^T r \geq r_0. \end{aligned} \quad (4.1)$$

If the constraint $x^T r \geq r_0$ is active at optimality, then minimizing $h_{\lambda, \beta}(x, \alpha)$ is equivalent to minimizing $\left[\frac{\lambda}{\sqrt{2\pi}} + (1 - \lambda)c_2(\beta) \right] \sigma(x)$. Furthermore, since $\frac{\lambda}{\sqrt{2\pi}} + (1 - \lambda)c_2(\beta)$ is a positive constant for given $\lambda \in [0, 1]$ and $\beta \geq 0.5$, minimizing $h_{\lambda, \beta}(x, \alpha)$ is equivalent to minimizing $\sigma^2(x)$, i.e., $x^T \Sigma x$. In this case, Problem (4.1) is a quadratic programming and can be solved efficiently. In addition, we can generate a scenario matrix by Monte Carlo simulation according to the assumed normal distribution at first, and then employ methods mentioned in Section 4.2 to solve it.

4.2 Case 2

As we know, the actual rate of return does not follow Gaussian distribution in general. In this case, Model (3.1) can be solved by the method based on scenario analysis. Assume that we have a scenario matrix $y = (y_{ti})_{T \times n}$ generated by Monte Carlo simulation or sampling of historical data, where y_{ti} is the rate of return of asset s_i under scenario t , $i = 1, 2, \dots, n$, $t = 1, 2, \dots, T$. Let p_t be the probability of scenario t , where $t = 1, 2, \dots, T$ and $\sum_{t=1}^T p_t = 1$. Let $r_i = \sum_{t=1}^T y_{ti} p_t$ and let x_i be the proportion of capital invested in asset s_i , $i = 1, 2, \dots, n$. Then,

$$\omega(x) = \sum_{t=1}^T \left[\sum_{i=1}^n (r_i - y_{ti}) x_i \right]^+ p_t$$

and

$$\tilde{F}_\beta(x, \alpha) = \alpha + \frac{1}{1 - \beta} \sum_{t=1}^T \left[p_t \left(- \sum_{i=1}^n x_i y_{ti} - \alpha \right)^+ \right],$$

where $\tilde{F}_\beta(x, \alpha)$ is an approximation of $F_\beta(x, \alpha)$. For simplicity, we assume all scenarios are independent and with an equal probability, i.e., $p_t \equiv \frac{1}{T}$. Denote $r = (r_1, r_2, \dots, r_n)$, $y_t = (y_{t1}, y_{t2}, \dots, y_{tn})$ and $x = (x_1, x_2, \dots, x_n)^T$. Then, MRisk can be approximated by

$$\tilde{h}_{\lambda, \beta}(x, \alpha) = \frac{\lambda}{T} \sum_{t=1}^T [(r - y_t)x]^+ + (1 - \lambda) \left[\alpha + \frac{1}{(1 - \beta)T} \sum_{t=1}^T (-y_t x - \alpha)^+ \right].$$

Substituting the objective function of Model (3.1) with the above formula, we have

$$\begin{aligned} & \underset{(x, \alpha) \in \mathbb{R}^{n+1}}{\text{minimize}} && \frac{\lambda}{T} \sum_{t=1}^T [(r - y_t)x]^+ + (1 - \lambda) \left[\alpha + \frac{1}{(1 - \beta)T} \sum_{t=1}^T (-y_t x - \alpha)^+ \right] \\ & \text{subject to} && \sum_{i=1}^n x_i = 1, \quad x \geq 0, \quad rx \geq r_0. \end{aligned} \quad (4.2)$$

Obviously, the feasible set $X = \{(x, \alpha) \in \mathbb{R}^{n+1} | x^T e = 1, -x \leq 0, -rx \leq -r_0\}$ is convex and the objective function is piecewise linear, where $e = (1, 1, \dots, 1)^T$ is an n -dimensional vector.

Therefore, Problem (4.2) is convex and its global optimal solution coincides with the local one. The only pity is that Problem (4.2) is nonsmooth in virtue of the existence of the plus function. So it cannot be solved by the conventional optimization algorithms directly.

By introducing auxiliary variables a_t, b_t ($t = 1, 2, \dots, T$), Problem (4.2) is equivalently converted into the following linear programming:

$$\begin{aligned} & \underset{(x, \alpha) \in R^{n+1}}{\text{minimize}} && \frac{\lambda}{T} \sum_{t=1}^T a_t + (1 - \lambda) \left[\alpha + \frac{1}{(1 - \beta)T} \sum_{t=1}^T b_t \right] \\ & \text{subject to} && a_t \geq 0, a_t \geq (r - y_t)x, \quad t = 1, 2, \dots, T, \\ & && b_t \geq 0, b_t \geq -y_t x - \alpha, \quad t = 1, 2, \dots, T, \\ & && x^T e = 1, \quad -x \leq 0, \quad -rx \leq -r_0, \end{aligned} \quad (4.3)$$

where there increases $2T$ decision variables and $4T$ constraints. Usually, the value of T is very large in the scenario-based method, which will give rise to the difficulty of computation. However, this method is still widely used because of its simplicity, see [4, 10]. Due to the nonsmoothness of the plus function, nonsmooth methods were adopted in some literatures, e.g., [16–19]. Though nonsmooth methods have been shown efficient, their applications are hindered to some extent because of their complexity. Recently, the smoothing method has attracted much attention of researchers, e.g., [20–22]. Smoothing method is a technique to approximate the nonsmooth function with a smooth one by introducing a smoothing parameter, where the smoothing parameter is a small positive number. In what follows, we intend to develop a smoothing method for Problem (4.2).

4.2.1 A Uniform Smooth Approximation of the Plus Function

For simplicity, we denote $g(x) = (x)^+$. Inspired by [23], we give a smooth function of the plus function $g(x)$ as follows

$$g(x, \mu) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{3\mu^2}(3\mu - x), & 0 \leq x < \mu, \\ x - \frac{1}{3}\mu, & x \geq \mu, \end{cases} \quad (4.4)$$

where $\mu > 0$ is the smoothing parameter. The properties of (4.4) are given in the following theorem.

Theorem 1 *For any $\mu > 0$, the smooth function $g(x, \mu)$ has the following properties:*

- 1) $g(x, \mu)$ is a smooth convex function of x ;
- 2) $|g(x, \mu) - g(x)| \leq \frac{\mu}{3}$, i.e., $g(x, \mu)$ is a uniform smooth approximation of $g(x)$.

Proof 1) By calculation, we have

$$g'_x(x, \mu) = \begin{cases} 0, & x < 0, \\ \frac{2x}{\mu} - \frac{x^2}{\mu^2}, & 0 \leq x < \mu, \\ 1, & x \geq \mu. \end{cases}$$

Since $\lim_{x \rightarrow 0^-} g'_x(x, \mu) = 0 = g'_x(0, \mu)$ and $\lim_{x \rightarrow \mu^-} g'_x(x, \mu) = \lim_{x \rightarrow \mu^-} (\frac{2x}{\mu} - \frac{x^2}{\mu^2}) = 1 = g'_x(\mu, \mu)$, $g'_x(x, \mu)$ is continuous everywhere with respect to x , i.e., $g(x, \mu)$ is a smooth function of x . Furthermore,

$$g''_x(x, \mu) = \begin{cases} \frac{2}{\mu^2}(\mu - x), & 0 \leq x < \mu, \\ 0, & \text{otherwise.} \end{cases}$$

So $g''_x(x, \mu) \geq 0$ and $g(x, \mu)$ is convex with respect to x .

2) The proof is divided into two parts.

Case (i) If $x < 0$, then $g(x, \mu) = g(x) \equiv 0$. Thus, $|g(x, \mu) - g(x)| = 0 \leq \frac{\mu}{3}$ always holds.

Case (ii) If $x \geq 0$, we have

$$g(x, \mu) - g(x) = g(x, \mu) - x = \begin{cases} \frac{x^2}{\mu} - x - \frac{x^3}{3\mu^2}, & 0 \leq x < \mu, \\ -\frac{1}{3}\mu, & x \geq \mu. \end{cases}$$

If $0 \leq x < \mu$, then $(\frac{x^2}{\mu} - x - \frac{x^3}{3\mu^2})'_x = -\frac{(x-\mu)^2}{\mu^2} \leq 0$. Consequently, $\frac{x^2}{\mu} - x - \frac{x^3}{3\mu^2}$ is monotone decreasing and $-\frac{\mu}{3} < \frac{x^2}{\mu} - x - \frac{x^3}{3\mu^2} \leq 0$; If $x \geq \mu$, then $g(x, \mu) - g(x) = -\frac{1}{3}\mu$. Evidently, $|g(x, \mu) - g(x)| \leq \frac{\mu}{3}$ holds.

In conclusion, for any x , we have $|g(x, \mu) - g(x)| \leq \frac{\mu}{3}$, i.e., $g(x, \mu)$ is a uniform smooth approximation of the plus function.

We compare the plus function $g(x)$ with its smooth approximation $g(x, \mu)$ in Figure 1. Obviously, as the smoothing parameter tends to zero, $g(x, \mu)$ and $g(x)$ are close more and more.

Figure 1 Smooth approximation of the plus function

4.2.2 A Smoothing Method

Let $Y(x, t) = [(r - y_t)x]^+$. We obtain its smooth approximation $Y(x, t, \mu)$ as the following

$$Y(x, t, \mu) = \begin{cases} 0, & (r - y_t)x < 0, \\ \frac{[(r - y_t)x]^2}{3\mu^2} [3\mu - (r - y_t)x], & 0 \leq (r - y_t)x < \mu, \\ (r - y_t)x - \frac{1}{3}\mu, & (r - y_t)x \geq \mu. \end{cases}$$

Similarly, let $Z(x, \alpha, t) = [-y_tx - \alpha]^+$. Then, its approximation $Z(x, \alpha, t, \mu)$ can be formulated as

$$Z(x, \alpha, t, \mu) = \begin{cases} 0, & -y_tx - \alpha < 0, \\ \frac{(-y_tx - \alpha)^2}{3\mu^2} (3\mu + y_tx + \alpha), & 0 \leq -y_tx - \alpha < \mu, \\ -y_tx - \alpha - \frac{1}{3}\mu, & -y_tx - \alpha \geq \mu. \end{cases}$$

Replacing $Y(x, t)$ with $Y(x, t, \mu)$ and $Z(x, \alpha, t)$ with $Z(x, \alpha, t, \mu)$ respectively, we convert Problem (4.2) into the following smooth one,

$$\begin{aligned} & \underset{(x, \alpha) \in R^{(n+1)}}{\text{minimize}} && \frac{\lambda}{T} \sum_{t=1}^T Y(x, t, \mu) + (1 - \lambda) \left\{ \alpha + \frac{1}{(1 - \beta)T} \sum_{t=1}^T Z(x, \alpha, t, \mu) \right\} \\ & \text{subject to} && x^T e = 1, \quad x \geq 0, \quad x^T r \geq r_0, \end{aligned} \quad (4.5)$$

where the smoothing parameter $\mu > 0$ is small enough. Now Problem (4.5) can be solved efficiently by classical algorithms based on gradient information. Here we use ‘fmincon’, an internal function of Matlab, to solve it. Rather than LP method, the biggest advantage of the smoothing method is that the number of variables and constraints is the same as the original problem. To choose an appropriate smoothing parameter μ , we give the following algorithm for Problem (4.5).

Algorithm 4.1

Step 1 Choose an initial smoothing parameter $\mu > 0$, zoom factor $\tau \in (0, 1)$, error precision $\varepsilon > 0$, convex coefficient $\lambda \in [0, 1]$, confidence level $\beta \in (0.5, 1)$ and threshold of expected rate of return r_0 . Input the rate of return matrix y . Compute the average rate of return r . Let $k = 1$.

Step 2 Solve Problem (4.5).

Step 3 Terminate if a prescribed stopping criterion is satisfied. Otherwise, let $\mu = \tau\mu$, $k = k + 1$, and loop to Step 2.

5 An Empirical Study

The model and the solution methods will be illustrated in this section by an empirical study. We choose 13 large-cap stocks from Shanghai and Shenzhen stock market as the research object. They are from different industry groups, went public earlier and have great influences in the industry. The codes of stocks are listed in Table 1. We collect the daily closing prices of each stock from January 4th, 2000 to November 27th, 2015 by Flush, a stock’s software. If a stock’s datum is missing some day, we delete all the other stocks’ data of the day. In this way, we

obtain a price matrix with 3001 rows and 13 columns. By calculating the simple rate of return, we get the rate of return matrix with 3000 rows and 13 columns. Furthermore, we can obtain the average rate of return per stock, see Table 2.

Table 1 The codes of stocks

No.	Code	No.	Code	No.	Code	No.	Code	No.	Code
1	600118	4	600796	7	000550	10	600000	13	600029
2	600054	5	000538	8	000910	11	000021		
3	600611	6	000619	9	600887	12	600359		

Table 2 The average rate of return

No.	r	No.	r	No.	r	No.	r	No.	r
1	0.00150	4	0.00049	7	0.00117	10	0.00041	13	0.00076
2	0.00069	5	0.00099	8	0.00083	11	0.00032		
3	0.00068	6	0.00051	9	0.00053	12	0.00080		

In what follows, we verify the efficiency of the model by varying parameters, especially λ and β at first. Then, we show the superiority of the uniform smooth function proposed in this paper by comparing accuracy and CPU time with LP method and classical maximum entropy function. All the computations are performed in a PC using Matlab R2012a.

Figure 2 Efficient frontier

5.1 Model Analysis

For simplicity, we fix the smoothing parameter $\mu = 0.01$ and $\beta = 0.95$. Take the scenario matrix y into Problem (4.5) and let $\lambda=0, 0.5, 1$ respectively. Consider 10 levels of expected rate of return, which divide the interval $[\min(r), \max(r)]$ into 9 equal parts. In this way, we obtain ten pairs $\{(r_0^i, x^i, \text{Mrisk}^i), i = 1, 2, \dots, 10\}$ for each λ , where $r_0^i = \min(r) + \frac{i-1}{9}[\max(r) - \min(r)]$,

MRisk^i and x^i are the optimal value and optimal portfolio of Problem (4.5) with respect to r_0^i , respectively. Firstly, we trace out the efficient frontier, as shown in Figure 2. From Figure 2, we can easily find that: 1) MRisk increases as r_0 increases, which is coincident with the principle of high yield inevitably accompanied by high risk; 2) $\text{CVaR} \geq \text{MSD}$ at the same expected rate of return, so CVaR is a more conservative measure than MSD; 3) For the given $\lambda \in (0, 1)$, the difference between MRisk and CVaR decreases with the increasing of r_0 , which is contrary to the difference of MRisk and MSD. Secondly, as the main purpose of asset allocation is to find the optimal proportion of each asset, we draw two area charts of r_0 versus optimal portfolios with different λ , as shown in Figure 3 and Figure 4. From Figure 3 and Figure 4, we can find that the number of invested assets decreases with the increasing of λ . As we know, diversification reduces risk. But investors can invest only on assets with high rate of return to obtain a desired high yield level. Further, comparing Figure 3 with Figure 4, we find that the number of invested assets with MSD as the risk measure is greater than CVaR as the risk measure, while Figure 2 is more regular than Figure 4. At last, we fix $r_0 = 0.0009$ and consider 10 values of β , which divide the interval $[0.9, 0.99]$ into 9 equal parts. Let λ be 0 and 0.5 respectively. By solving Problem (4.5), we obtain 10 pairs $\{(\beta^i, \text{MRisk}^i), i = 1, 2, \dots, 10\}$. Moreover, we plot Figure 5. Observing the blue curve and the red one in Figure 5, we find that MRisk increases with β increases and the difference between the blue curve and the red one increases with the increase of β . By definition, MRisk is CVaR if $\lambda = 0$ and MRisk is a combination of CVaR and MSD if $0 < \lambda < 1$. In fact, if $\lambda = 1$, β has no effect on MRisk. In this regard, the confidence level β can be regarded as a loss aversion factor. To sum up, the confidence level β and the convex parameter λ together affect MRisk: MRisk and β have positive correlation for fixed $\lambda (\neq 1)$; given β , investors are more averse to volatility with the increase of λ and vice versa. In a word, a rational investor should choose appropriate λ and β to mitigate risk according to his/her perception.

Figure 3 Portfolio Allocation with $\lambda = 0$

Figure 4 Portfolio allocation with $\lambda = 1$

Figure 5 Confidence level vs MRisk

5.2 Smoothing Method Analysis

To illustrate the efficiency of smoothing method, especially the uniform smooth approximation (4.4), we solve Problem (4.2) by three methods: LP method, smoothing method with approximation (4.4) (denoted by SM1) and classical maximum entropy function (denoted by SM2) of the plus function, where $g(x, \mu) = \mu \ln[1 + \exp(\frac{x}{\mu})]$ is the so called maximum entropy function, $\mu > 0$ is the smoothing parameter. As for smoothing methods, we perform Algorithm 4.1, where parameters are chosen to be $\mu_0 = 0.01$, $\tau = 0.5$, $\varepsilon = 1e - 6$, $\lambda = 0.5$, $\beta = 0.95$, $r_0 = 0.0008$. Since Problem (4.3) is equivalent to Problem (4.2), we solve Problem (4.3) at first to obtain the optimal objective value, denoted by MRisk^* . And we use the following stopping rule in Algorithm 4.1 in our implementation $|\text{Risk}^* - \text{MRisk}^*| \leq \varepsilon$, where Risk^* is the optimal

value of (4.5). We obtain the optimal value $\text{MRisk}^*=0.025222$ after 34.13908 seconds by using the LP method. As for SM1, we obtain the optimal value $\text{Risk}^*=0.025223$ after 17.1500 seconds, where the iteration times is 11 and the final smoothing parameter is $9.7656\text{e}-6$. As the smoothing parameter μ tends to zero, $\exp(\frac{x}{\mu})$ goes to infinity if x is positive. SM2 is running without stopping, in other words, SM2 fails to get a solution. In this regard, SM1 is better than SM2, which means the uniform smooth approximation proposed in this paper is better than the maximum entropy function for our asset allocation model. In terms of CPU time, SM1 is better than LP method.

6 Conclusions

A risk measure which takes into account both the volatility and the loss is proposed in this paper. By using this measure, investors can reflect their perceptions more flexibly. Thus, they can control the risk by choosing a suitable convex coefficient and the confidence level. Furthermore, as the plus function is contained in the risk measure, asset allocation problem based on it is nonsmooth. To employ classical algorithms based on gradient, we put forward a uniform smooth approximation of the plus function. And an empirical study was illustrated to verify the rationality of the model and validity of the smoothing method. As MSD is used to measure the volatility, other risk measures, such as variance, semi-variance, minimax measure, etc, can also be used in MRisk. Meanwhile, as CVaR is used to measure the loss, other risk measures which take the loss as risk can also be used too.

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