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Nonadaptive Search Problem with Sets of Equal Sum

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Abstract: Consider the set $A = \{1, 2, 3, \ldots, 2^n\}$, $n \geq 3$ and let $x \in A$ be unknown element. For given natural number S we are allowed to ask whether x belongs to a subset B of A such that the sum of the elements of B equals S. We investigate for which S it is possible to find x using a nonadaptive search.

Keywords: search, nonadaptive search
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1 Introduction

We start with the general description of a search problem. Let x be an unknown number taken from a given set A. We want to find x by asking if x belongs to subset B of A, such that B satisfies some given conditions. By imposing different restrictions on B we obtain different search problems. Also, if every question is stated after the answer to the previous one has been received we say that this is an *adaptive search* [4], [5]. In this case one can make use of the information given by the previous answers. If all questions are asked simultaneously we say that this is a *nonadaptive search* [1], [2], [6].

In this paper we consider a nonadaptive search for an unknown element x in the set $A = \{1, 2, 3, \ldots, 2^n\}$, $n \geq 3$. Given a natural number S we are allowed to ask whether x belongs to a subset B of A if the sum of the elements of B equals S (see [3]). In this case we say that B is a *question set* of weight S or, when S is clear from the context, a just question set. Since $|A| = 2^n$, the minimum number of question sets needed to find

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the unknown element is n.

Call a natural number S *good* if for some m there exists a collection $B_1, B_2, \ldots, B_m$ of question sets of weight S which determines x. If $m = n$, i.e. x can be found by n question sets of weight S, then S is called *proper*. There are two problems of interest:

Problem A. Find all good numbers S.

Problem B. Find all proper numbers S.

2 Problem A

The following theorem gives necessary and sufficient condition for a natural number S to be good, thus solving problem A.

Theorem 2.1. A natural number S is good if and only if

$$S \in [2^n - 1; 2^{2n-1} - 2^{n-1} + 1].$$

Proof: If S is good, then for any two distinct a and b from A there exists a question set B of weight S such that $a \in B$ and $b \notin B$ or $a \notin B$ and $b \in B$. In other words B separates a and b. Denote by $B^S_{a,b}$ a question set of weight S that separates a and b. If $S < 2^n - 1$ then it is obvious that no question set of weight S separates $a = 2^n - 1$ and $b = 2^n$. If $S > 2^{2n-1} + 2^{n-1} - 2^n + 1$ then, since the sum of all elements from A is $2^{2n-1} + 2^{n-1}$, no question set of weight S separates $a = 2^n - 1$ and $b = 2^n$. Therefore $S \in [2^n - 1; 2^{2n-1} - 2^{n-1} + 1]$.

Suppose now that $S \in [2^n - 1; 2^{2n-1} - 2^{n-1} + 1]$. To show that there exists a question set of weight S that separates any two elements from A, we induct on S.

1. Let $S = 2^n - 1$ and $a, b \in A$ be such that $a < b$. It is clear that $a \leq 2^n - 1$. If $a = 2^n - 1$ then $B^S_{a,b} = \{2^n - 1\}$. Suppose $a \neq 2^n - 1$. If $a + b \neq 2^n - 1$ then take $B^S_{a,b} = \{a, 2^n - 1 - a\}$. If $a + b = 2^n - 1$ and $a \leq 2$ then $B^S_{a,b} = \{1, 2, 2^n - 4\}$ (note that $n \geq 3$). Finally, if $a + b = 2^n - 1$ and $a > 2$ then $B^S_{a,b} = \{1, a - 1, b\}$.

2. Suppose that for some $S \in [2^n - 1; 2^{2n-1} - 2^{n-1}]$ and for any two $a, b \in A$, $a < b$ there exists a set $B^S_{a,b}$.

3. We construct set $B^{S+1}_{a,b}$. In what follows we repeatedly make use of a simple observation. Suppose $l \neq 2^n$, $l \in B^S_{a,b}$, $l + 1 \notin B^S_{a,b}$, and $\{l, l + 1\} \cap \{a, b\} = \phi$. Then we can take $B^{S+1}_{a,b} = B^S_{a,b} \backslash \{l\} \cup \{l + 1\}$. Furthermore, if $a \neq 1$ and $1 \notin B^S_{a,b}$ then take $B^{S+1}_{a,b} = B^S_{a,b} \cup \{1\}$. It follows now that if $\{1, 2, \ldots, a - 1\} \not\subset B^S_{a,b}$ we are done. Suppose $\{1, 2, \ldots, a - 1\} \subset B^S_{a,b}$.

First consider the case $a > 2$, $b \neq 2^n$ and $a + 1 \neq b$. If $\{a + 1\} \notin B^S_{a,b}$ then $B^{S+1}_{a,b} = B^S_{a,b} \backslash \{1, a - 1\} \cup \{a + 1\}$. If $\{a + 1\} \in B^S_{a,b}$ then when $\{a + 2, a + 3, \ldots, b - 1\} \not\subset B^S_{a,b}$, using our observation, we construct $B^{S+1}_{a,b}$, so assume $\{a + 2, a + 3, \ldots, b - 1\} \subset B^S_{a,b}$. If $b + 1 \notin B^S_{a,b}$ then $B^{S+1}_{a,b} = B^S_{a,b} \backslash \{1, b - 1\} \cup \{b + 1\}$. If $b + 1 \in B^S_{a,b}$ then when $\{b + 2, \ldots, 2^n\} \not\subset B^S_{a,b}$, using our observation, we construct $B^{S+1}_{a,b}$, so assume $\{b + 2,$

$\ldots, 2^n\} \subset B^S_{a,b}$ which implies that $\{1, 2, \ldots, a-1, a+1, \ldots, b-1, b+1, \ldots, 2^n\} \subset B^S_{a,b}$. Since a or b belongs to $B^S_{a,b}$ we have that $S \geq 2^{2n-1} + 2^{n-1} - b$ which, together with $b \neq 2^n$, is a contradiction to $S \leq 2^{2n-1} - 2^{n-1}$.

Suppose now that $a > 2$, $a+1 = b$ and $b \neq 2^n$. If $a \in B^S_{a,b}$ and $b = a+1 \notin B^S_{a,b}$ then $B^{S+1}_{a,b} = B^S_{a,b} \backslash \{a\} \cup \{a+1\}$. So, let $a \notin B^S_{a,b}$ and $b = a+1 \in B^S_{a,b}$. If $a+2 \notin B^S_{a,b}$ then $B^{S+1}_{a,b} = B^S_{a,b} \backslash \{1, a-1, a+1\} \cup \{a, a+2\}$. If $a+2 \in B^S_{a,b}$ then when $\{a+2, \ldots, 2^n\} \not\subset B^S_{a,b}$ we construct $B^{S+1}_{a,b}$ as above and when $\{1, 2, \ldots, a-1, a+2, \ldots, 2^n\} \subset B^S_{a,b}$ we get a contradiction to $S \leq 2^{2n-1} - 2^{n-1}$.

The cases $a = 1, a = 2$ or $b = 2^n$ are settled in similar manner and are left to the reader.

3 Problem B

In this section we present our main result. Theorem 3.3 gives necessary condition for S to be proper. Theorems 3.12 and 3.14 show that this condition is sufficient for n odd or n even but not a power of 2. We start with a simple lemma presenting some combinatorial identities needed for our further considerations.

Lemma 3.1. For any natural number n it is true that

(a) $\sum_{i=1}^{n} i \binom{n}{i}^2 = n \binom{2n-1}{n-1}$

(b) $\sum_{i=1}^{n} i \binom{n}{i} = n2^{n-1}$

(c) $\sum_{i=2}^{n} \binom{n}{i} \sum_{j=1}^{i-1} j \binom{n}{j} = n \left(2^{2n-2} - \binom{2n-1}{n-1} \right)$.

Definition 3.2. We say that a vector $V = (v_1, v_2, \ldots, v_{2^n})$ is *characteristic vector* for a subset B of A if $v_i = 1$ when $i \in B$ and $v_i = 0$ otherwise. It is clear that $\sum_{y \in B} y = \sum_{i=1}^{2^n} i v_i$. The *weight* of vector $V = (v_1, v_2, \ldots, v_n)$ is defined by $\mathrm{wt}(V) = |\{i | v_i \neq 0\}|$. An $n \times 2^n$ matrix G is called *characteristic matrix* for a collection $B_1, B_2, \ldots, B_n$ of subsets if the rows of G are the characteristic vectors of $B_1, B_2, \ldots, B_n$. The *weight* of a characteristic matrix G with vector columns $V_1, V_2, \ldots, V_{2^n}$ is defined by

$$\mathrm{wt}(G) = \frac{1}{n} \sum_{i=1}^{2^n} i \mathrm{wt}(V_i).$$

Consider a collection $B_1, B_2, \ldots, B_n$ of question sets of weight S. By asking whether x belongs to B_i for $i = 1, 2, \ldots, n$ we obtain as answers a sequence of "yes" and "no" of length n. In order to find x, every element from A should get a unique sequence of "yes" and "no". Note also that if the vector V_i is the $i-$th column of the characteristic matrix for this collection, then the element i gets as answer the transpose of V_i (1 meaning "yes" and 0 meaning "no"). Therefore, if the unknown element can be found by the collection $B_1, B_2, \ldots, B_n$ then the columns of the corresponding characteristic matrix are all binary vectors of length n. Thus, our problem is equivalent to finding a binary $n \times 2^n$ matrix G having as columns all binary vectors of length n and the scalar product of every row of G with $(1, 2, 3, \ldots, 2^n)$ equals S. Call such a matrix *proper*. It is clear that if a matrix G with vector columns $V_1, V_2, \ldots, V_{2^n}$ is proper then $\mathrm{wt}(G) = S$.

Denote by $\overline{G}$ the matrix obtained from G by interchanging 0 and 1. It is easy to see that $\overline{G}$ is proper matrix and $\mathrm{wt}\left(\overline{G}\right) = 2^{2n-1} + 2^{n-1} - \mathrm{wt}(G)$.

Theorem 3.3. If a natural number S is proper then

$$S \in \left[2^{2n-2} + 2^{n-2} - \frac{\binom{2n-1}{n-1}}{2}; 2^{2n-2} + 2^{n-2} + \frac{\binom{2n-1}{n-1}}{2}\right].$$

Proof: Let S be a proper number and G be a proper matrix of weight $\mathrm{wt}(G) = S$. We show first that $S \geq 2^{2n-2} + 2^{n-2} - \frac{\binom{2n-1}{n-1}}{2}$. Label the columns of G by $1, 2, \ldots, 2^n$ and denote by S_i, $i = 0, 1, \ldots, n$ the sum of the labels of the vector columns of G having weight i. Note that $nS = n\mathrm{wt}(G) = \sum_{i=0}^{n} iS_i$. Further, since there are $\binom{n}{i}$ vector columns of weight i we obtain

$$S_n \geq 1, \qquad S_n + S_{n-1} \geq 1 + 2 + \cdots + \left(\binom{n}{n} + \binom{n}{n-1}\right),$$

$$S_n + S_{n-1} + S_{n-2} \geq 1 + 2 + \cdots + \left(\binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2}\right)$$

and so on, up to

$$S_n + S_{n-1} + \cdots + S_1 \geq 1 + 2 + \cdots + \left(\binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{1}\right).$$

Adding the above inequalities gives

$$\begin{aligned}\sum_{i=0}^{n} iS_i \geq 1 &+ \frac{\left(\binom{n}{n} + \binom{n}{n-1}\right)\left(\binom{n}{n} + \binom{n}{n-1} + 1\right)}{2} \\ &+ \frac{\left(\binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2}\right)\left(\binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + 1\right)}{2} \\ &+ \cdots + \frac{\left(\binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{1}\right)\left(\binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{1} + 1\right)}{2}.\end{aligned}$$

A simple calculation shows that the latter inequality is equivalent to

$$2n\mathrm{wt}(G) \geq \sum_{i=1}^{n} i\binom{n}{i}^2 + \sum_{i=1}^{n} i\binom{n}{i} + 2\sum_{i=2}^{n}\binom{n}{i}\sum_{j=1}^{i-1} j\binom{n}{j}.$$

An application of Lemma 3.1 gives

$$\mathrm{wt}(G) \geq 2^{2n-2} + 2^{n-2} - \frac{\binom{2n-1}{n-1}}{2}.$$

Since $\mathrm{wt}(G) = S$ we have our assertion.

To prove the inequality $S \leq 2^{2n-2} + 2^{n-2} + \frac{\binom{2n-1}{n-1}}{2}$ recall that $\mathrm{wt}(G) = 2^{2n-1} + 2^{n-1} - \mathrm{wt}\left(\overline{G}\right)$ and use that $\mathrm{wt}\left(\overline{G}\right) \geq 2^{2n-2} + 2^{n-2} - \frac{\binom{2n-1}{n-1}}{2}$. ◇

Remark. It is not difficult to prove that the term $\frac{\binom{2n-1}{n-1}}{2}$ is an integer if and only if n is not power of 2.

We continue with the notation concerning our results. Let $V = (v_1, v_2, \ldots, v_n)^t$ be binary vector column of length n. Denote by π the cyclic shift of V by one position, i.e. $\pi(V) = (v_2, v_3, \ldots, v_n, v_1)^t$. It is well known that π partitions the set of all binary vectors of length n into orbits and the length of each orbit is a divisor of n. Also, the elements in one and the same orbit have equal weights. If the length of the orbit containing V where $\mathrm{wt}(V) = w$ equals l then call the matrix with consecutive columns $V, \pi(V), \pi^2(V), \ldots, \pi^{l-1}(V)$ *orbit matrix of weight w and length l*. Denote such a matrix by $C_{w,l}$. It is easy to see that n divides lw and there are $\frac{lw}{n}$ ones in every row of $C_{w,l}$. Note also that $\overline{C_{w,l}}$ is an orbit matrix of weight $n - w$.

3.1 The case $n = 2k + 1$

Throughout this section $n = 2k + 1$ is an odd integer. We prove that for

$$S \in \left[2^{2n-2} + 2^{n-2} - \frac{\binom{2n-1}{n-1}}{2}; 2^{2n-2} + 2^{n-2} + \frac{\binom{2n-1}{n-1}}{2}\right]$$

there always exists a proper matrix of weight S. We begin with some useful lemmas.

Lemma 3.4. The matrix $G = C_1 C_2 \ldots C_m$ where $C_1, C_2, \ldots, C_m$ is a permutation of all orbit matrices of weights $n, n-1, \ldots, k+1$ and their complements is proper.

Proof: It suffices to show that $C_{w,l}$ and $\overline{C_{w,l}}$ for $k + 1 \leq w \leq n$ add one and the same amount in the scalar product of every row with $(1, 2, \ldots, 2^n)$. Let the first column of $C_{w,l}$ be in position p and the first column of $\overline{C_{w,l}}$ be on position q. Using the fact that $C_{w,l}$ and $\overline{C_{w,l}}$ are complementary to each other it is easy to see that the amount added to the scalar product of each row with $(1, 2, \ldots, 2^n)$ equals

$$p + (p+1) + \ldots + (p+l-1) + (q-p)\frac{l(n-w)}{n} = \frac{l(l-1)}{2} + ql - (q-p)\frac{lw}{n}.$$

This completes the proof. ◇

Example 3.5. Let $n = 3$. There are four orbit-matrices, namely

$$C_{3,1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad C_{2,3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \overline{C_{2,3}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \overline{C_{3,1}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The interval from Theorem 3.3 is $[13\ldots 23]$. It turns out that for all $S \in [13\ldots 23]$ there exists a proper matrix G of weight S such that G is formed by a permutation of $C_{3,1}, C_{2,3}, \overline{C_{2,3}}$ and $\overline{C_{3,1}}$. Indeed, if G_S is a proper matrix of weight S then

$$G_{13} = (C_{3,1}C_{2,3}\overline{C_{2,3}C_{3,1}}) = \begin{pmatrix} 1&0&1&1&1&0&0&0 \\ 1&1&0&1&0&1&0&0 \\ 1&1&1&0&0&0&1&0 \end{pmatrix}$$

$$G_{14} = (C_{3,1}C_{2,3}\overline{C_{3,1}C_{2,3}}) = \begin{pmatrix} 1&0&1&1&0&1&0&0 \\ 1&1&0&1&0&0&1&0 \\ 1&1&1&0&0&0&0&1 \end{pmatrix}$$

$$G_{15} = (C_{2,3}C_{3,1}\overline{C_{3,1}C_{2,3}}) = \begin{pmatrix} 0&1&1&1&0&1&0&0 \\ 1&0&1&1&0&0&1&0 \\ 1&1&0&1&0&0&0&1 \end{pmatrix}$$

$$G_{16} = (C_{2,3}\overline{C_{3,1}}C_{3,1}\overline{C_{2,3}}) = \begin{pmatrix} 0&1&1&0&1&1&0&0 \\ 1&0&1&0&1&0&1&0 \\ 1&1&0&0&1&0&0&1 \end{pmatrix}$$

$$G_{17} = (\overline{C_{3,1}}C_{3,1}C_{2,3}\overline{C_{2,3}}) = \begin{pmatrix} 0&1&0&1&1&1&0&0 \\ 0&1&1&0&1&0&1&0 \\ 0&1&1&1&0&0&0&1 \end{pmatrix}$$

$$G_{18} = (C_{3,1}\overline{C_{2,3}C_{3,1}}C_{2,3}) = \begin{pmatrix} 1&1&0&0&0&0&1&1 \\ 1&0&1&0&0&1&0&1 \\ 1&0&0&1&0&1&1&0 \end{pmatrix}$$

$$G_{19} = \overline{G_{17}}, \quad G_{20} = \overline{G_{16}}, \quad G_{21} = \overline{G_{15}}, \quad G_{22} = \overline{G_{14}}, \quad G_{23} = \overline{G_{13}}.$$

Let H_1 be submatrix of a matrix G. If H_2 is a matrix having the same dimensions as H_1 then denote by $G(H_1 \to H_2)$ the matrix obtained from G by replacing H_1 by H_2. The next four lemmas show how, given a proper matrix, one can obtain new proper matrices by transformations of the type $H_1 \to H_2$. Recall that $n = 2k + 1$.

Lemma 3.6. Consider a proper matrix G and let $C_{p,t}$ and $C_{q,h}$ be neighboring orbit matrices in G. Then $G_1 = G(C_{p,t}C_{q,h} \to C_{q,h}C_{p,t})$ is a proper matrix of weight $\mathrm{wt}(G_1) = \mathrm{wt}(G) + \frac{th(p-q)}{n}$.

Proof: By Lemma 3.4 the matrix $G_1 = G(C_{p,t}C_{q,h} \to C_{q,h}C_{p,t})$ is proper. There are tp one in $C_{p,t}$ and hq ones in $C_{q,h}$. Since $C_{p,t}C_{q,h} \to C_{q,h}C_{p,t}$ is equivalent to moving $C_{p,t}$ forwards by h positions and $C_{q,h}$ backwards by t positions we obtain that $n.\mathrm{wt}(G_1) = n.\mathrm{wt}(G) + tph - hqt$ giving $\mathrm{wt}(G_1) = \mathrm{wt}(G) + \frac{th(p-q)}{n}$. ⋄

Lemma 3.7. Let V be a vector-column of weight w, $k+1 \le w \le 2k+1$ and let

$$C_{w,l} = \left(V\pi(V)\pi^2(V)\dots\pi^{l-1}(V)\right)$$

be an orbit matrix of weight w and length l. Also, set

$$T_{w,n-w} = \left(V\overline{V}\pi\left(V\right)\pi\left(\overline{V}\right)\dots\pi^{l-1}\left(V\right)\pi^{l-1}\left(\overline{V}\right)\right)$$

and $T_{n-w,w} = \overline{T_{w,n-w}}$.

a) If G is a proper matrix having $C_{w,l}$ and $\overline{C_{w,l}}$ as neighboring matrices then $G_1 = G\left(C_{w,l}\overline{C_{w,l}} \to T_{w,n-w}\right)$ is proper and $\mathrm{wt}(G_1) = \mathrm{wt}(G) + (2w-n)\dfrac{l(l-1)}{2n}$;

b) If G is a proper matrix having $T_{w,n-w}$ as submatrix then $G_2 = G\left(T_{w,n-w} \to T_{n-w,w}\right)$ is proper and $\mathrm{wt}(G_2) = \mathrm{wt}(G) + (2w-n)\dfrac{l}{n}$.

Proof:

a) If the first column of $C_{w,l}$ is on position p then the first column of $\overline{C_{w,l}}$ is on position $p+l$. Applying Lemma 3.4 we obtain that the amount added to the scalar product of every row with $(1,2,\dots,2^n)$ by $C_{w,l}$ and $\overline{C_{w,l}}$ equals $X = l\left(\dfrac{(l-1)}{2} + p + l - \dfrac{lw}{n}\right)$.

Further, there are $\dfrac{lw}{n}$ ones in every row of $C_{w,l}$ and $\dfrac{l(n-w)}{n}$ ones in every row of $\overline{C_{w,l}}$. Therefore, if $(v_1, v_2, \dots, v_{2l})$ is a row of $T_{w,n-w}$, then there are $\dfrac{lw}{n}$ pairs $v_{2m+1}v_{2m+2} = 10$ and $\dfrac{l(n-w)}{n}$ pairs $v_{2m+1}v_{2m+2} = 01$. Thus, the scalar product of every row of $T_{w,n-w}$ with $(p, p+1, \dots, p+2l-1)$ equals $Y = p + p + 2 + p + 4 + \dots + p + 2l - 2 + \frac{l(n-w)}{n} = pl + l(l-1) + \frac{l(n-w)}{n}$. Thus, G_1 is proper and since $Y - X = (2w-n)\frac{l(l-1)}{2n}$ we are done.

b) By replacing $n-w$ by w in the expression for Y in a) we obtain that the scalar product of every row of $T_{n-w,w}$ with $(p, p+1, \dots, p+2l-1)$ equals $Z = p + p + 2 + p + 4 + \dots + p + 2l - 2 + \frac{l(n-w)}{n} = pl + l(l-1) + \frac{lw}{n}$. Since $Z - Y = (2w-n)\frac{l}{n}$ we have the result of b). ⋄

The proofs of the following lemmas are left to the reader.

Lemma 3.8. Let G be a proper matrix. If V is a vector column such that $V\overline{V}$ is submatrix of G then $G\left(V\overline{V}C_{w,l} \to \overline{C_{w,l}}V\overline{V}\right)$ is a proper matrix of weight $\mathrm{wt}(G) + (2w-n)\dfrac{l}{n}$.

Lemma 3.9. Let G be a proper matrix and V be vector column. Denote by $C_{n,1}$ the vector column of weight n. Then:

a) $G\left(V\overline{V\,C_{n,1}} \to \overline{C_{n,1}}V\overline{V}\right)$ is a proper matrix of weight $\mathrm{wt}(G)+1$;
b) $G\left(C_{n,1}V\overline{V} \to V\overline{V}\,C_{n,1}\right)$ is a proper matrix of weight $\mathrm{wt}(G)+1$;
c) $G\left(C_{n,1}\overline{C_{n,1}} \to \overline{C_{n,1}}C_{n,1}\right)$ is a proper matrix of weight $\mathrm{wt}(G)+1$.

Definition 3.10. Given a proper matrix G of weight S. A transformation $H_1 \to H_2$ is called *admissible* for G if:

1. the matrix $G_1 = G(H_1 \to H_2)$ is proper and
2. if $\mathrm{wt}(G_1) = \mathrm{wt}(G) + m$ then for any w, $\mathrm{wt}(G)+1 \geq w \leq \mathrm{wt}(G)+m-1$ there exists a proper matrix of weight w.

It is clear that if there exists a sequence of admissible transformations from G_1 to G_2 then for every $\mathrm{wt}(G_1) \leq S \leq \mathrm{wt}(G_2)$ there exists a proper matrix of weight S.

If G and $G_1 = G(H_1 \to H_2)$ are proper matrices and $w = \mathrm{wt}(G_1) - \mathrm{wt}(G)$ then we write $\mathrm{wt}^+(H_1 \to H_2) = w$.

Example 3.11. Let $n = 7$. Then the interval from Theorem 3.3 is $[2412 \ldots 5844]$. We show that for any $S \in [2412 \ldots 5844]$ there exists proper matrix of weight S.

Since 7 is a prime number all orbit matrices of weight w, $1 \leq w \leq 6$ are of length 7. Denote by s_w the number of orbits of weight w. Then $s_7 = 1$, $s_6 = 1$, $s_5 = 3$, $s_4 = 5$, $s_3 = 5$, $s_2 = 3$, $s_1 = 1$ and $s_0 = 1$. Let $C^t_{w,7}$ for $w = 6, 5$ or 4 and $t = 1, 2, \ldots, s_w$ be all orbit matrices of weight w. Denote by $C_{7,1}$ the only orbit matrix of weight 7 and for simplicity set $C_{6,7} = C^1_{6,7}$. Consider the following matrix

$$G_1 = C_{7,1}C_{6,7}C^1_{5,7}C^2_{5,7}C^3_{5,7}C^1_{4,7}C^2_{4,7}C^3_{4,7}C^4_{4,7}C^5_{4,7}\overline{C^5_{4,7}C^4_{4,7}C^3_{4,7}C^2_{4,7}C^1_{4,7}C^3_{5,7}C^2_{5,7}C^1_{5,7}C_{6,7}C_{7,1}}.$$

By Lemma 3.4 the matrix G_1 is a proper one. It follows from the proof of Theorem 3.3 that G_1 is of minimal weight, so $\mathrm{wt}(G_1) = 2412$. Note that $\overline{G_1}$ is of weight 5844. We show the existence of a sequence of admissible transformations from G_1 to $\overline{G_1}$.

It follows from Lemma 3.6 that $\mathrm{wt}^+\left(\overline{C_{6,7}C_{7,1}} \to \overline{C_{7,1}C_{6,7}}\right) = \mathrm{wt}^+(C_{7,1}C_{6,7} \to C_{6,7}C_{7,1})$ $= 1$ and $\mathrm{wt}^+\left(C_{7,1}C^t_{5,7} \to C^t_{5,7}C_{7,1}\right) = 2$ for $t = 1, 2$ or 3. Thus, we can get a proper matrix of weight $\mathrm{wt}(G_1) + t$ for $t = 1, 2, \ldots, 7$. It follows now that the transformation given by $C^5_{4,7}\overline{C^5_{4,7}} \to T^5_{43}$ (by Lemma 3.7 $\mathrm{wt}^+\left(C^5_{4,7}\overline{C^5_{4,7}} \to T^5_{43}\right) = 3$) is admissible for G_1. We obtain

$$G_2 = C_{7,1}C_{6,7}C^1_{5,7}C^2_{5,7}C^3_{5,7}C^1_{4,7}C^2_{4,7}C^3_{4,7}C^4_{4,7}T^5_{43}\overline{C^4_{4,7}C^3_{4,7}C^2_{4,7}C^1_{4,7}C^3_{5,7}C^2_{5,7}C^1_{5,7}C_{6,7}C_{7,1}}.$$

The matrix T^5_{43} consists of 7 pairs of the form $V\overline{V}$ where $\mathrm{wt}(V) = 4$. By Lemma 3.8 we have that $\mathrm{wt}^+\left(V\overline{V\,C^t_{w,7}} \to \overline{C^t_{w,7}}V\overline{V}\right) = 2w - 7$ for $w = 4, 5, 6$ and $t = 1, \ldots, s_w$. Since $2w - 7 < 7$ and $\mathrm{wt}^+\left(V\overline{V C_{7,1}} \to \overline{C_{7,1}}V\overline{V}\right) = 1$ we find that the transformation $V\overline{V C^t_{w,7}} \to \overline{C^t_{w,7}}V\overline{V}$ is admissible for any matrix containing $C_{7,1}C_{6,7}C^1_{5,7}C^2_{5,7}C^3_{5,7}$ and $\overline{C_{6,7}C_{7,1}}$ (or $V\overline{V C_{7,1}}$ instead of $\overline{C_{6,7}C_{7,1}}$).

Thus, by moving one by one all pairs $V\overline{V}$ from T^5_{43} to the right by skipping one by one the matrices $\overline{C^4_{4,7}}$, $\overline{C^3_{4,7}}$, $\overline{C^2_{4,7}}$, $\overline{C^1_{4,7}}$,$\overline{C^3_{5,7}}$, $\overline{C^2_{5,7}}$, $\overline{C^1_{5,7}}$ and $\overline{C_{6,7}}$ we obtain

$$G_3 = C_{7,1}C_{6,7}C^1_{5,7}C^2_{5,7}C^3_{5,7}C^1_{4,7}C^2_{4,7}C^3_{4,7}C^4_{4,7}\overline{C^4_{4,7}C^3_{4,7}C^2_{4,7}C^1_{4,7}C^3_{5,7}C^2_{5,7}C^1_{5,7}C_{6,7}}T^5_{43}\overline{C_{7,1}}.$$

It is clear now that repeating the above transformations consequently with $C^t_{4,7}\overline{C^t_{4,7}}$ for $t = 4, 3, 2$ and 1 we can obtain

$$G_4 = C_{7,1}C_{6,7}C^1_{5,7}C^2_{5,7}C^3_{5,7}\overline{C^3_{5,7}C^2_{5,7}C^1_{5,7}C_{6,7}}T^1_{43}T^2_{43}T^3_{43}T^4_{43}T^5_{43}\overline{C_{7,1}}$$

by sequence of admissible moves. Using the pairs of the form $V\overline{V}$ from the five matrices T^t_{43} (there are 35 such pairs altogether) and $\overline{C_{7,1}}$ we can obtain proper matrix of weight $\mathrm{wt}(G_4) + t$ for every $t \in [1 \dots 35]$. Thus, the transformation $C^3_{5,7}\overline{C^3_{5,7}} \to T^3_{52}$ (by Lemma 3.7 $\mathrm{wt}^+\left(C^3_{5,7}\overline{C^3_{5,7}} \to T^3_{52}\right) = 9$) is admissible for G_4. Further, by moving one by one all pairs $V\overline{V}$ from T^3_{52} to the right by skipping one by one the matrices $\overline{C^2_{5,7}C^1_{5,7}}$ and $\overline{C_{6,7}}$ (it is easy to see that all transformations are admissible) we obtain

$$G_5 = C_{7,1}C_{6,7}C^1_{5,7}C^2_{5,7}\overline{C^2_{5,7}C^1_{5,7}C_{6,7}}T^3_{52}T^1_{43}T^2_{43}T^3_{43}T^4_{43}T^5_{43}\overline{C_{7,1}}.$$

Repeating the above with $C^t_{5,7}\overline{C^t_{5,7}}$ for $t = 2, 1$ and $C_{6,7}\overline{C_{6,7}}$ we get

$$G_6 = C_{7,1}T_{61}T^1_{52}T^2_{52}T^3_{52}T^1_{43}T^2_{43}T^3_{43}T^4_{43}T^5_{43}\overline{C_{7,1}}$$

Recall that $\mathrm{wt}^+\left(V\overline{V}\overline{C_{7,1}} \to \overline{C_{7,1}}V\overline{V}\right) = 1$, $\mathrm{wt}^+\left(C_{7,1}V\overline{V} \to V\overline{V}C_{7,1}\right) = 1$ and $\mathrm{wt}^+\left(C_{7,1}\overline{C_{7,1}} \to \overline{C_{7,1}}C_{7,1}\right) = 1$. Since there are $2^6 - 1$ pairs of complement vector columns $V\overline{V}$ it follows that any transformation of a matrix of the form $C_{7,1}V_1\overline{V_1} \dots V_{2^6-1}\overline{V_{2^6-1}C_{7,1}}$ increasing the weight by at most $2^7 - 2$ is admissible for this matrix.

By Lemma 3.7 $\mathrm{wt}^+\left(T^t_{w,7-w} \to T^t_{7-w,w}\right) = 2w - 7$ for $w = 6, 5$ or 4 (not that both $T^t_{w,7-w}$ and $T^t_{7-w,w}$ are formed by 7 pairs of the form $V\overline{V}$) and therefore is admissible for any matrix of the above form. Thus, we obtain

$$G_7 = \overline{C_{7,1}T_{16}T^1_{25}T^2_{25}T^3_{25}T^1_{34}T^2_{34}T^3_{34}T^4_{34}T^5_{34}}C_{7,1}.$$

Notice that $G_7 = \overline{G_6}$. Since for every S, $\mathrm{wt}(G_1) \le S \le \mathrm{wt}(G_6)$ there exists a proper matrix of weight S, it is clear that for any S, $\mathrm{wt}(\overline{G_1}) \ge S \ge \mathrm{wt}(\overline{G_6})$ there exists a proper matrix of weight S. So, we have shown that for any $S \in [2412 \dots 5844]$ there exists a proper matrix of weight S.

Following the reasonings from Example 3.11 we prove the following Theorem.

Theorem 3.12. If $n = 2k + 1$ is odd and

$$S \in \left[2^{2n-2} + 2^{n-2} - \frac{\binom{2n-1}{n-1}}{2}; 2^{2n-2} + 2^{n-2} + \frac{\binom{2n-1}{n-1}}{2}\right]$$

then S is proper.

Proof: Since $\gcd(2k+1, k) = \gcd(2k+1, k+1) = \gcd(2k+1, 2) = 1$ all orbit matrices of weights $k + 1$, k and 2 are of length $n = 2k + 1$. It is clear that there is one orbit matrix for each $w = n, n-1, 1, 0$ and k orbit matrices for each $w = n-2$ and $w = 2$. Denote by $C_{n,1}$ the only orbit matrix of weight n and for simplicity set $C_{n-1,n} = C^1_{n-1,n}$.

Consider the matrix obtained by arranging all orbit matrices in decreasing order of their weights:

$$G_1 = C_{n,1}C_{n-1,n}C^1_{n-2,n}\dots C^k_{n-2,n}\dots C^k_{k+1,n}\overline{C^k_{k+1,n}}\dots\overline{C^k_{n-2,n}}\dots\overline{C^1_{n-2,n}C_{n-1,n}C_{n,1}}.$$

By Lemma 3.4 the matrix G_1 is proper and by Theorem 2.1 it is of minimal weight. By Lemma 3.6 $\mathrm{wt}^+(C_{n,1}C_{n-1,n} \to C_{n-1,n}C_{n,1}) = 1$, $\mathrm{wt}^+\left(\overline{C_{n-1,n}C_{n,1}} \to \overline{C_{n,1}C_{n-1,n}}\right) = 1$ and $\mathrm{wt}^+\left(C_{n,1}C^t_{n-2,n} \to C^t_{n-2,n}C_{n,1}\right) = 2$ for $t = 1, 2, \dots, k$. It is easy to see that using transformations of the type given above one can obtain proper matrix of weight $\mathrm{wt}(G_1)+w$ for all $w = 1, 2, \dots, 2k+2$. Note that by Lemma 3.9 $\mathrm{wt}^+\left(V\overline{V}\overline{C_{n,1}} \to \overline{C_{n,1}}V\overline{V}\right) = 1$.

This implies that if a matrix contains as submatrices $C_{n,1}C_{n-1,n}C^1_{n-2,n}\dots C^k_{n-2,n}$ and $\overline{C_{n-1,n}C_{n,1}}$ (or $V\overline{V}\overline{C_{n,1}}$ instead $\overline{C_{n-1,n}C_{n,1}}$) then any transformation $H_1 \to H_2$ such that $\mathrm{wt}^+(H_1 \to H_2) \le 2k+3$ is admissible for this matrix.

We proceed as follows:

1. Apply the transformation $C^k_{k+1,n}\overline{C^k_{k+1,n}} \to T^k_{k+1,k}$. This transformation is admissible for G_1 since by Lemma 3.7 $\mathrm{wt}^+\left(C^k_{k+1,n}\overline{C^k_{k+1,n}} \to T^k_{k+1,k}\right) = k$.
2. Move one by one all pairs of complement columns from $T^k_{k+1,k}$ by skipping one by one the matrices $\overline{C^t_{w,l}}$ for $k+1 \le w \le n$ to the left of $\overline{C_n}$. By Lemma 3.8 we have that $\mathrm{wt}^+\left(V\overline{V}\overline{C^t_{w,l}} \to \overline{C^t_{w,l}}V\overline{V}\right) = (2w-n)\frac{l}{n}$. But since $l \le n$ and $w \le n$ we have that $(2w-n)\frac{l}{n} \le n$ which means that all such transformations are admissible. We obtain the matrix
$$G_2 = C_{n,1}C_{n-1,n}C^1_{n-2,n}\dots C^{k-1}_{k+1,n}\dots\overline{C^{k-1}_{k+1,n}}\dots\overline{C^1_{n-2,n}C_{n-1,n}}T^k_{k+1,k}\overline{C_{n,1}}$$
3. Repeat steps 1. and 2. for all pairs $C^t_{k+1,n}\overline{C^t_{k+1,n}}$ for $t = k-1, k-2, \dots, 1$. Denote the resulting matrix by G_3.
$$G_3 = C_{n,1}C_{n-1,n}C^1_{n-2,n}\dots C^t_{k+2,l}\overline{C^t_{k+2,l}}\dots\overline{C^1_{n-2,n}C_{n-1,n}}T^1_{k+1,k}\dots T^k_{k+1,k}\overline{C_{n,1}}.$$
Since there are $\binom{2k+1}{k}$ pairs of complementary vector columns in $T^1_{k+1,k}T^2_{k+1,k}\dots T^k_{k+1,k}$ and $\mathrm{wt}^+\left(V\overline{V}\overline{C_{n,1}} \to \overline{C_{n,1}}V\overline{V}\right) = 1$, all transformations for a matrix containing $T^1_{k+1,k}T^2_{k+1,k}\dots T^k_{k+1,k}\overline{C_{n,1}}$ and increasing the weight by at most $\binom{2k+1}{k}+1$ are admissible for this matrix.
4. Proceed with steps 1, 2 and 3 for the middle two matrices $C^t_{w,l}\overline{C^t_{w,l}}$. They are always complimentary to each other. It is easy to see that $C^t_{w,l}\overline{C^t_{w,l}} \to T^t_{w,n-w}$ is admissible. Also, moving all pairs of compliments (one by one) from $T_{w,n-w}$ to the left of $T_{w-1,n-w+1}$ are admissible moves.
5. Finally, we get a proper matrix
$$G_4 = C_{n,1}T_{n-1,1}T^1_{n-2,2}\dots T^k_{n-2,2}\dots T^1_{k+1,k}\dots T^k_{k+1,k}\overline{C_{n,1}}.$$

Recall that $\mathrm{wt}^+\left(V\overline{V}\overline{C_{n,1}} \to \overline{C_{n,1}}V\overline{V}\right) = 1$, $\mathrm{wt}^+\left(C_{n,1}V\overline{V} \to V\overline{V}C_{n,1}\right) = 1$ and $\mathrm{wt}^+\left(C_{n,1}\overline{C_{n,1}} \to \overline{C_{n,1}}C_{n,1}\right) = 1$. Since there are $2^{n-1}-1$ pairs $V\overline{V}$, $\mathrm{wt}(V) \ne n$ of complement vector columns, it follows that any transformation of a matrix of the form $C_{n,1}V_1\overline{V_1}\ldots V_{2^{n-1}-1}\overline{V_{2^{n-1}-1}C_{n,1}}$ increasing the weight by at most 2^n is admissible for this matrix.

It follows from $\mathrm{wt}^+\left(T^t_{w,n-w}\to T^t_{n-w,w}\right)=(2w-n)\frac{l}{n}<2^n$, where l is the length of $C^t_{w,l}$, that $T_{w,n-w}\to T_{n-w,w}$ is admissible transformation. We obtain

$$G_5=\overline{C_{n,1}}T_{1,n-1}T^1_{2,n-2}\ldots T^k_{2,n-2}\ldots T^1_{k,k+1}\ldots T^k_{k,k+1}C_{n,1}.$$

Notice that $G_5=\overline{G_4}$. Since for every S, $\mathrm{wt}(G_1)\le S\le\mathrm{wt}(G_4)$ there exists a proper matrix of weight S, it is clear that for any S, $\mathrm{wt}\left(\overline{G_1}\right)\ge S\ge\mathrm{wt}\left(\overline{G_4}\right)$ there also exists a proper matrix of weight S. This completes the proof. ⋄

3.2 The case $n=2k$

In this case the matrices C_k and $\overline{C_k}$ do not always add one and the same amount to the scalar product of each row with $(1,2,\ldots,2^n)$. The simplest example is $C_k=(1,0,1,0\ldots1,0)^t$ having k ones and k zeroes.

Lemma 3.13. Let $n=2k$ is not power of 2. There exists a $2k\times\binom{2k}{k}$ matrix G of the form $V_1\overline{V_1}\,V_2V_2\,\ldots V_t\overline{V_t}$ where $t=2^{\binom{2k}{k}/2}$, all columns of which are the vectors of weight k such that the scalar product of every row of G with $\left(1,2,\ldots,\binom{2k}{k}\right)$ equals $\dfrac{\binom{2k}{k}\left(\binom{2k}{k}+1\right)}{4}$.

Proof: There are $\binom{2k-1}{k}$ vector columns of weight k having 0 in the last position. Since $\binom{2k-1}{k}=\frac{1}{2}\binom{2k}{k}$ and since $\binom{2k-1}{k}$ is an even number if and only if n is not power of 2 we can place these vectors on positions $1,3,5,\ldots,\binom{2k-1}{k}-1,\binom{2k-1}{k}+2,\binom{2k-1}{k}+4,\ldots,\binom{2k}{k}$. If a vector is in the first half put on its right its complement, and if a vector is in the second half then put on its left its complement. It is easy to see that the matrix obtained has the desired property. ⋄

Using Lemma 3.13 and similar arguments to those in Theorem 3.12 one can prove

Theorem 3.14. If n is even $n\ne2^k$ and

$$S\in\left[2^{2n-2}+2^{n-2}-\frac{\binom{2n-1}{n-1}}{2};2^{2n-2}+2^{n-2}+\frac{\binom{2n-1}{n-1}}{2}\right]$$

then S is proper.

Research Problem: Prove Theorem 3.14 without restriction $n\ne2^k$.

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