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On the secant varieties to the osculating variety of a Veronese surface

E. Ballico*, C. Fontanari†

Dept. of Mathematics
University of Trento
38050 Povo (TN), Italy

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Abstract: In this paper we study the k-th osculating variety of the order d Veronese embedding of $\mathbf{P}^n$. In particular, for $k = n = 2$ we show that the corresponding secant varieties have the expected dimension except in one case.

Keywords: tangent developable, secant variety, tangent space, fat point, zero-dimensional scheme, postulation
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1 Introduction

Let $X \subset \mathbf{P}^N$ be an integral n-dimensional projective variety and let h be a non-negative integer. The h-secant variety $\mathrm{Sec}_h(X) \subset \mathbf{P}^N$ of X is the closure in $\mathbf{P}^N$ of the union of all h-dimensional linear spaces containing $h+1$ distinct points of X. Thus $\mathrm{Sec}_h(X)$ is irreducible, $\mathrm{Sec}_0(X) = X$ and $\dim(\mathrm{Sec}_h(X)) \leq \min\{(h+1)(n+1)-1, N\}$. We say that X is h-defective if $\dim(\mathrm{Sec}_h(X)) < \min\{(h+1)(n+1)-1, N\}$.

Let $V_{2,d} \subset \mathbf{P}^N$, $N := (d^2+3d)/2$, be the order d Veronese embedding of $\mathbf{P}^2$. A well-known theorem due to Severi (see [14] and [9] for a modern revisitation) characterizes $V_{2,2} \subset \mathbf{P}^5$ as the unique projective surface which is not a curve and whose secant variety has dimension 4. This classical result has motivated further investigation about the defectivity of Veronese surfaces. The basic statement is the following (see [12] and [2]):

* E-mail: ballico@science.unitn.it
† E-mail: fontanar@science.unitn.it

Proposition 1.1. The Veronese surface $V_{2,d}$ is not h-defective unless $d = 2$ and $h = 1$, or $d = 4$ and $h = 4$.

We wish to mention also a more recent result (see [5] and [4]):

Proposition 1.2. The tangential variety to $V_{2,d}$ is not h-defective unless $d = 3$ and $h = 1$.

For a detailed analysis of the behaviour of Veronese surfaces from different but closely related points of view (Grassmann defectivity, deficiency of tangential envelopes) we refer to [10] and [6]. Here instead we are going to state the main result of the present paper:

Theorem 1.3. The 2-osculating variety of $V_{2,d}$ is not h-defective, unless $d = 4$ and $h = 1$.

The precise definition of osculating variety will be recalled in section 2, where we will also see that (at least in characteristic zero) theorem 1.3 is equivalent to the following result, to be proved in section 3:

Theorem 1.4. Fix an integer $d \neq 4$, a line $L \subset \mathbf{P}^2$ and $P \in L$. Let $A \subset \mathbf{P}^2$ be the length 8 subscheme with ideal sheaf $\mathcal{I}_A = ((\mathcal{I}_{\{P\}})^4 + (\mathcal{I}_L)^2) \cap (\mathcal{I}_{\{P\}})^3$. Let Γ be the set of all length 8 subschemes of $\mathbf{P}^2$ of the form $h(A)$ for some $h \in \mathrm{Aut}(\mathbf{P}^2)$. Then for every integer $x > 0$ the restriction map $\rho_{Z,d} : H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d)) \to H^0(Z, \mathcal{O}_Z(d))$ has maximal rank (i.e. it is either injective or surjective), where Z is the general disjoint union of x elements of Γ, i.e $h^0(\mathbf{P}^2, \mathcal{I}_Z(d)) = \max\{0, (d+2)(d+1)/2 - 8x\}$ and $h^1(\mathbf{P}^2, \mathcal{I}_Z(d)) = \max\{0, 8x - (d+2)(d+1)/2\}$.

We would like to raise the following questions, the second one being stronger.

Question 1.5. Fix an integer $n \geq 2$. Are there only finitely many pairs (d, k) such that the k-osculating variety of the Veronese embedding of order d of $\mathbf{P}^n$ has not the expected dimension ?

Question 1.6. Are there only finitely many triples (n, d, k) such that the k-osculating variety of the Veronese embedding of order d of $\mathbf{P}^n$ has not the expected dimension ?

The natural guess is that the answer to the stronger question, 1.6, is YES, but we do not know the answer to question 1.5 even for the case of $n = 2$.

The main ingredient in the proof of theorem 1.4 (and of theorem 1.3) is Horace's Lemma ([12]). In most lemmas we only apply an elementary form of it (i.e. remark 3.1). In several key points we have to use a refined and very powerful form of it introduced by J. Alexander and A. Hirschowitz in [2] (see remark 3.10). In the last part of the proof of lemma 3.24, in order to apply Horace's Lemma we make an elementary, but quite useful trick. The same trick is used in the proof of lemma 3.33.

As in [5], we work over an algebraically closed field K with $\mathrm{char}(K) = 0$.

2 The osculating variety

Let $X \subset \mathbf{P}^N$ be a projective variety of dimension n. Recall that the k-th osculating spaces to X at a smooth point $p \in X$ is the subspace of $\mathbf{P}^N$ spanned by p and by all the derivative points of degree less than or equal to k of a local parametrization of X centered at p (see for instance [13], Definition 1.1). We denote by $O^k(X,p)$ the k-th osculating space to X at p. Let $X_0 \subseteq X$ be the quasi-projective variety of smooth points where $O^k(X,p)$ has maximal dimension. The k-osculating variety to X is defined as

$$\mathrm{Osc}^k(X) := \overline{\bigcup_{p \in X_0} O^k(X,p)}.$$

Let $V_{n,d} \subset \mathbf{P}^N$, $N = \binom{n+d}{d} - 1$, be the n-Veronese embedding of $\mathbf{P}^n$. We are going to describe $O^h(\mathrm{Osc}^k(V_{n,d}), p)$ following the approach of [5]. Indeed, we consider the Veronese $V_{n,d}$ as the image of the map

$$\begin{aligned} \mathbf{P}S_1 &\longrightarrow \mathbf{P}S_d \\ L &\longmapsto L^d. \end{aligned}$$

Exactly as in [5], we pass to the affine, we choose a direction $L + \lambda M$ through L in S_1 and we check

$$\lim_{\lambda \to 0} \frac{d^k}{d\lambda^k}(L + \lambda M)^d$$

to be a multiple of $L^{d-k}M^k$. It follows that the affine cone over $O^k(V_{n,d}, L^d)$ is given by

$$< \{L^{d-k}M^k : M \in S_1\} >= \{L^{d-k}M : M \in S_k\}.$$

Next, we define the map

$$\begin{aligned} \phi^k : S_1 \times S_k &\longrightarrow S_d \\ (L, M) &\longmapsto L^{d-k}M. \end{aligned}$$

In order to determine the h-th osculating space to $\mathrm{Osc}^k(V_{n,d})$ at the point $(L, L^{d-k}M)$, we proceed as before. Namely, we pick a line $(L, M) + \lambda(A, B)$ through (L, M), we consider its image

$$\phi^k((L, M) + \lambda(A, B)) = (L + \lambda A)^{d-k}(M + \lambda B)$$

and we compute

$$\lim_{\lambda \to 0} \frac{d^h}{d\lambda^h}((L + \lambda A)^{d-k}(M + \lambda B))$$

as a linear combination of $L^{d-k-h+1}A^{h-1}B$ and $L^{d-k-h}A^hM$. It follows that the affine cone over $O^h(\mathrm{Osc}^k(V_{n,d}), (L, L^{d-k}M))$ is given by

$$W^{h,k} =< L^{d-k-h+1}S_{k+h-1}, L^{d-k-h}MS_h > .$$

Let now $L = x_0$ and $M = x_1^k$, so that $W^{h,k} =< x_0^{d-k-h+1} S_{k+h-1}, x_0^{d-k-h} x_1^k S_h >$. We point out that our choice of M is not generic; however, we stress that if the h-th osculating space to $\mathrm{Osc}^k(V_{n,d})$ has the expected dimension at a special point, by semicontinuity it is forced to have the same expected dimension also at the general point. Consider the homogeneous ideal in $K[y_0, \ldots, y_n]$ defined as

$$I^{h,k} = (y_1, \ldots, y_n)^{k+h+1} + (y_2, \ldots, y_n)^{h+1} \cap (y_1, \ldots, y_n)^{k+h}.$$

Geometrically, $I^{h,k}$ corresponds to the intersection of a fat point having multiplicity $k+h+1$ with a line of multiplicity $(h+1)$ passing through the point and fattened there with multiplicity $(k+h)$. If I^{-1} denotes the submodule of $K[x_0, \ldots, x_n]$ consisting of all elements annihilated by I via the natural action

$$y_i \circ x_j = \frac{\partial}{\partial x_i}(x_j)$$

(see [5], § 2), we have

$$(I^{h,k})_j^{-1} =< x_0^{d-k-h} S_{k+h} > \cap < (x_0, x_1)^{d-h} S_h, x_0^{d-k-h+1} S_{k+h-1} >= W^{h,k}.$$

In particular, if we fix $n = 2$, $k = 3$, and $h = 1$, we see that theorem 1.3 and theorem 1.4 are equivalent. Moreover, it is easy to check that for $d = 4$ the thesis of theorem 1.4 is no longer true. Namely, if

$$I := (y_0, y_1)^4 + y_1^2 \cap (y_0, y_1)^3 \cap (y_0, y_2)^4 + y_2^2 \cap (y_0, y_2)^3,$$

then $y_0^4 \in I_4$, hence $h^0(\mathbf{P}^2, \mathcal{I}(4)) = \dim_K I_4 > 0$. We stress that the previous calculation works for the union of two general elements of $h(A)$. Indeed, take points $P_i \in \mathbf{P}^2$ and lines $L_i \subset \mathbf{P}^2$ such that $P_i \in L_i, P_i \notin L_j$ for $i, j = 1, 2$. Thus $L_1 \neq L_2$ and $P_0 := L_1 \cap L_2 \in< \{P_1, P_2\} >$. Conversely, given any three non-collinear points P_0, P_1, P_2, the lines $L_i :=< \{P_0, P_i\} >$, $i = 1, 2$, are as above. Finally, $\mathrm{Aut}(\mathbf{P}^2)$ acts transitively on the set of all triples of non-collinear distinct points of $\mathbf{P}^2$.

3 Proofs and examples

Remark 3.1. Let X be an integral scheme, $L \in \mathrm{Pic}(X)$, $Z \subset X$ a closed subscheme and $D \subset X$ an effective Cartier divisor. The residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of X with $Hom(\mathcal{I}_D, \mathcal{I}_Z)$ as ideal sheaf. We have an exact sequence on X:

$$0 \to \mathcal{I}_{\mathrm{Res}_D(X)} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z \cap D, D} \otimes (L|D) \to 0 \tag{1}$$

From (1) we obtain $h^0(X, \mathcal{I}_Z \otimes L) \leq h^0(X, \mathcal{I}_{\mathrm{Res}_D(X)} \otimes L(-D)) + h^0(D, \mathcal{I}_{Z \cap D, D} \otimes (L|D))$ and $h^1(X, \mathcal{I}_Z \otimes L) \leq h^1(X, \mathcal{I}_{\mathrm{Res}_D(X)} \otimes L(-D)) + h^1(D, \mathcal{I}_{Z \cap D, D} \otimes (L|D))$. We will call Horace's Lemma this simple observation. This observation is a very particular case of

Horace Method (see ([12], [2], [3]). Set $\mathrm{Res}_D^0(Z) := Z$. For every integer $k > 0$ define inductively the scheme $\mathrm{Res}_D^k(Z) \subseteq Z \subseteq X$ by the formula $\mathrm{Res}_D^k(Z) := \mathrm{Res}_D(\mathrm{Res}_D^{k-1}(Z))$.

Definition 3.2. Let Γ be a set of zero-dimensional subschemes of $\mathbf{P}^n$. We will say that Γ is invariant if $h(Z) \in \Gamma$ for every $Z \in \Gamma$ and every $h \in \mathrm{Aut}(\mathbf{P}^n)$. We will say that Γ is irreducible if it is parametrized by an irreducible algebraic variety, i.e. if its irreducible zero-dimensional schemes form an irreducible constructible subset of the Hilbert scheme of $\mathbf{P}^n$. An invariant class Γ will be called of type $(a_1, \ldots, a_s)$ for some integer $s > 0$ and some integers $a_1 \geq \cdots \geq a_s \geq 0$ if each Z is connected and for every $Z \in \Gamma$ there is a hyperplane D containing the point Z_{red} and such that $\mathrm{length}(\mathrm{Res}_D^k(Z) \cap D) = a_{k+1}$ for every integer k such that $0 \leq k \leq s$, with the convention $a_{s+1} := 0$. Hence if Γ has type $(a_1, \ldots, a_s)$ every $Z \in \Gamma$ has length $a_1 + \cdots + a_s$. In this definition we will identify the string of integers $(a_1, \ldots, a_s)$ with the string of integers $(a_1, \ldots, a_s, 0)$.

Notation 3.3. In the sequel, Γ will always denote the invariant set of the zero-dimensional schemes considered in the corresponding statement.

Example 3.4. Fix $P \in \mathbf{P}^n$ and an integer $x > 0$. Let xP or $P^{(x-1)}$ be the infinitesimal neighborhood of order $x - 1$ in $\mathbf{P}^n$, i.e. the fat point of order x with $\{P\}$ as support, i.e. the closed subscheme of $\mathbf{P}^n$ with $(\mathcal{I}_{\{P\}})^x$ as ideal sheaf. Thus xP has length $\binom{n+x-1}{n}$. For any hyperplane $H \subset \mathbf{P}^n$ such that $P \in H$, xP has $(x, x-1, x-2, \ldots, 2, 1)$ as associated sequence. By varying x we obtain in this way all invariant classes Γ admitting a unique associated sequence, independent from the choice of a hyperplane H_Z containing Z_{red} for all $Z \in \Gamma$.

Example 3.5. Fix $P \in \mathbf{P}^2$, a line $L \subset \mathbf{P}^2$ and an integer $x \geq 2$. Let $Z \subset L$ be the infinitesimal neighborhood of order $x - 1$ of P in L. Thus Z is a collinear and connected length x subscheme of $\mathbf{P}^2$. The postulation of a general union of general collinear suschemes is known ([11]). The associated sequence of Z with respect to L is $(x, 0)$, while for any line D with $P \in D$ and $D \neq L$ the associated sequence of Z with respect to D is $(1, \ldots, 1)$. A connected lenght x scheme $W \subset \mathbf{P}^2$ has associated sequence $(1, \ldots, 1)$ with respect to some line D with $W_{red} \in D$ if and only if it is collinear and this is true for all lines D with $W_{red} \in D$, except exactly one: the Zariski tangent space to W at W_{red}.

Example 3.6. Fix $P \in \mathbf{P}^2$, a line $L \subset \mathbf{P}^2$ and integers $x \geq y > 0$. Let Z be the closed subscheme of $\mathbf{P}^2$ with $(\mathcal{I}_{\{P\}})^x \cap (\mathcal{I}_L)^y$ as ideal sheaf and $\Gamma_{x,y}$ the invariant class formed by all schemes $h(Z)$ with $h \in \mathrm{Aut}(\mathbf{P}^2)$. If $x = y$, then $Z = xP$. Now assume $x > y$. The sequence of Z with respect to L is $(x, x-1, x-2, \ldots, x-y+1, 0)$. The sequence of Z with respect to any line $D \neq L$ with $P \in D$ is $(y, \ldots, y, y-1, \ldots, 2, 1)$. If $x = 3$ and $y = 2$, then Z is a $(2, 3)$-scheme in the sense of [5], §2.

Example 3.7. Fix $P \in \mathbf{P}^2$, a line $L \subset \mathbf{P}^2$ and an integer $x \geq 2$. Let Z be the closed subscheme of $\mathbf{P}^2$ which is the union of $(x-1)P$ and the scheme Z of Example 3.6 with respect to the integer x and the integer $y = 2$. Let $\Psi_{x,n}$ be the invariant class formed by all schemes $h(Z)$ with $h \in \mathrm{Aut}(\mathbf{P}^2)$. The sequence of Z associated to L is $(x, x-1, x-3, x-4, \dots, 2, 1)$. For any line $D \subset \mathbf{P}^2$ with $P \in D$ and $D \neq L$ the sequence of Z associated to D is $(2, 2, \dots, 2, 1)$.

Example 3.8. Fix a line $L \subset \mathbf{P}^2$ and $P \in L$. As in the statement of theorem 1.4, let A be the degree 8 subscheme of $\mathbf{P}^2$ with ideal sheaf $\mathcal{I}_A = ((\mathcal{I}_{\{P\}})^4 + (\mathcal{I}_L)^2) \cap (\mathcal{I}_{\{P\}})^3$. Then A has type $(4, 3, 1)$ with respect to L. For any line D such that $P \in D$ and $D \neq L$, we have length$(D \cap A) = 3$ and A has type $(3, 2, 2, 1)$ with respect to D.

Remark 3.9. Let Γ be an invariant class of length 5 subschemes of type $(4, 1)$ on $\mathbf{P}^2$. Let $Z_i \in \Gamma$, $i = 1, 2$, be schemes such that $(Z_2)_{red} \neq (Z_2)_{red}$. Let $L_i \subset \mathbf{P}^2$ be the line such that length$(L_i \cap Z_i) = 4$. If $L_1 = L_2$, then $Z_1 \cup Z_2$ is contained in the double line $2L_1$ and hence it has not maximal rank. Now assume $L_1 \neq L_2$. Since $h^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) = 0$, the restriction map $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3)) \to H^0(L_1 \cup L_2, \mathcal{O}_{L_1 \cup L_2}(3))$ is surjective. Since length$((L_1 \cup L_2) \cap (Z_1 \cup Z_2)) \geq 8 > h^0(L_1 \cup L_2, \mathcal{O}_{L_1 \cup L_2}(3))$, the restriction map $\rho_{Z_1 \cup Z_2, 3}$ is not surjective. Thus $Z_1 \cup Z_2$ has not maximal rank. Alternatively, it is easy to see that $Z_1 \cup Z_2$ is contained in the reducible cubic $L_1 \cup L_2 \cup R$, where R is the line containing $\{(Z_1)_{red}, (Z_2)_{red}\}$. Fix integers $b \geq 5$ and a such that $1 \leq a \leq b - 3$. Let Φ be an irreducible invariant class of type $(b, a, 0)$ with respect to a line. Let W_i, $i = 1, 2$, be two general elements of Φ, L_i, $i = 1, 2$, the line such that length$(L_i \cap W_i) = b$ and R the line containing $\{(Z_1)_{red}, (W_2)_{red}\}$. Thus length$((L_1 \cup L_2) \cap (W_1 \cup W_2)) \geq 2b > h^0(L_1 \cup L_2, \mathcal{O}_{L_1 \cup L_2}(b-1))$ and $W_1 \cup W_2$ is contained in any reducible degree $b-1$ curve containing L_1, L_2, and R with multiplicity at least a. Hence $W_1 \cup W_2$ has not maximal rank. Alternatively, $\rho_{W_1 \cup W_2, b-1}$ is not surjective because length$((W_1 \cup W_2) \cap (L_1 \cup L_2)) \geq 2b$, hence the restriction map $H^0(L_1 \cup L_2, \mathcal{O}_{L_1 \cup L_2}(b-1)) \to H^0(W_1 \cup W_2, \mathcal{O}_{W_1 \cup W_2}(b-1))$ is not surjective.

Remark 3.10. Fix a line $L \subset \mathbf{P}^2$ and $P \in L$. Let A be a zero-dimensional scheme such that $A_{red} = \{P\}$ and A has type $(a_1, \dots, a_s)$, $a_s > 0$, $s \geq 2$, with respect to L. Fix an integer $d > 0$, an integer j with $2 \leq j \leq s$, and a zero-dimensional scheme $W \subset \mathbf{P}^2$ such that $P \notin W_{red}$. Let Γ be the invariant class formed by all zero-dimensional schemes $h(A)$ for some $h \in \mathrm{Aut}(\mathbf{P}^2)$. Assume $a_j + \text{length}(W \cap L) \leq d + 1$. There is a zero-dimensional scheme B with $B_{red} = \{P\}$ and type $(b_1, \dots, b_{s-1})$ with respect to L, $b_i = a_i$ for $i < j$, $b_i = a_{i+1}$ for $j \leq i \leq s-1$, such that to show that $h^1(\mathbf{P}^2, \mathcal{I}_{E \cup W}(d)) = 0$ for a general $E \in \Gamma$ it is sufficient to prove $h^1(\mathbf{P}^2, \mathcal{I}_{B \cup \mathrm{Res}_L(W)}(d-1)) = 0$ ([3], Fig. 1 at p. 308). Similarly, if $a_j + \text{length}(W \cap L) \geq d + 1$ to check that $h^0(\mathbf{P}^2, \mathcal{I}_{E \cup W}(d)) = 0$ for a general $E \in \Gamma$ it is sufficient to prove $h^0(\mathbf{P}^2, \mathcal{I}_{B \cup \mathrm{Res}_L(W)}(d-1)) = 0$. We will call this observation the $(a_j, b_1, \dots, b_{s-1})$-trick with respect to the pair (L, P).

For every integer $d \geq 4$ define the integers z_d and w_d by the relations:

$$8z_d + w_d = (d+2)(d+1)/2, 0 \leq w_d \leq 7 \tag{2}$$

For all integers $d \geq 4$ define the following assertions $C(d)$ and $D(d)$:

$C(d)$: Let Γ be an invariant irreducible class of zero-dimensional subschemes of type $(4,3,1)$ of $\mathbf{P}^2$. Then for a general union Z of z_d elements in Γ we have $h^1(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$, i.e. $h^0(\mathbf{P}^2, \mathcal{I}_Z(d)) = w_d$.

$D(d)$: Let Γ be an invariant irreducible class of zero-dimensional subschemes of type $(4,3,1)$ of $\mathbf{P}^2$. Then for a general union Z of z_d+1 elements in Γ we have $h^0(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$, i.e. $h^1(\mathbf{P}^2, \mathcal{I}_Z(d)) = 8 - w_d$.

Remark 3.11. Obviously, $C(d)$ implies $D(d)$ if $w_d = 0$. Since a general point imposes a non-trivial condition to any non-empty linear system, $C(d)$ implies $D(d)$ if $w_d = 1$. We have $w_d = 1$ if $d \equiv 0, 13 \pmod{16}$, $w_d = 3$ if $d \equiv 1, 12 \pmod{16}$, $w_d = 6$ if $d \equiv 2, 11 \pmod{16}$, $w_d = 4$ if $d \equiv 5, 6 \pmod{16}$, $w_d = 2$ if $d \equiv 3, 10 \pmod{16}$, $w_d = 7$ if $d \equiv 4, 9 \pmod{16}$, $w_d = 5$ if $d \equiv 5, 8 \pmod{16}$, and $w_d = 0$ if $d \equiv 14, 15 \pmod{16}$.

In order to show in the case of low degree curves the subtilities of the postulation of general unions $Z \subset \mathbf{P}^2$ of some elements of an invariant class Γ, we give the following three examples.

Example 3.12. Let Γ be an invariant irreducible class of zero-dimensional subschemes of type $(4,3)$ of $\mathbf{P}^2$. Here we will check that the union Z of two general elements of Γ is contained in exactly one degree 4 curve, which is a line counted with multiplicity 4. Take $P_1, P_2 \in \mathbf{P}^2$ with $P_1 \neq P_2$ and let L be the line spanned by P_1 and P_2. There is $Z_i \in \Gamma$ such that $(Z_i)_{red} = \{P_i\}$. $Z \subset 4L$ because $\mathrm{Res}^3_L(P_i^{(3)}) = \emptyset$ and $Z \subset P_1^{(3)} \cup P_2^{(3)}$. Using Horace's Lemma three times with respect to L we obtain $h^0(\mathbf{P}^2, \mathcal{I}_Z(4)) \leq 1$.

The same proof gives the following result.

Example 3.13. Fix a line $D \subset \mathbf{P}^2$, $P \in D$ and a scheme $A \subset \mathbf{P}^2$ such that A has type $(3,3,2)$ with respect to D. Let Γ be the irreducible invariant class formed by all schemes $h(A)$ with $h \in \mathrm{Aut}(\mathbf{P}^2)$. The union, Z, of two general elements satisfies $h^0(\mathcal{I}_Z(4)) \neq 0$ if and only if for a general line R through P the scheme A is contained in the quadruple line $4R$; if this is the case, then $h^0(\mathcal{I}_Z(4)) = 1$ and the unique degree 4 plane curve containing Z is the quadruple line $4L$, where L is the line spanned by Z_{red}.

Example 3.14. Fix a line $D \subset \mathbf{P}^2$, $P \in D$ and a scheme $A \subset \mathbf{P}^2$ such that A has type $(3,3,2)$ or type $(4,3,1)$ with respect to D. Let Γ be the irreducible invariant class formed by all schemes $h(A)$ with $h \in \mathrm{Aut}(\mathbf{P}^2)$. Let $W \subset \mathbf{P}^2$ be a general union of 3 elements of Γ. Here we check that $h^0(\mathcal{I}_W(4)) = 0$. Let $Z \subset \mathbf{P}^2$ be a general union of 2 elements of Γ. By Example 3.13 we have $h^0(\mathcal{I}_Z(4)) \leq 1$, hence we eliminate this cohomology group just by choosing another connected component.

Remark 3.15. By Example 3.14, theorem 1.4 (hence theorem 1.3) is true for the integers $d = 4$ and $x = 3$.

Lemma 3.16. $C(2)$ and $D(2)$ are true.

Proof 3.17. Notice that C(2) is the empty statement, because $z_2 = 0$. Turning to D(2), take $W \in \Gamma$ of type $(4,3,1)$ with respect to (L,P); we need to check that $h^0(\mathbf{P}^2, \mathcal{I}_W(2)) = 0$. Any conic containing W must be singular at P. Since $\mathrm{Res}_L^k(W) = \{P\} \neq \emptyset$, we have $W \not\subset 2L$. If $D \neq L$ is a line with $D \neq L$, then length$(2D \cap W) \leq 6$ and $2D$ does not contain W. If $D_1 \cup D_2$ is a reducible conic with $D_1 \neq D_2$, it is easy to see that length$((D_1 \cup D_2) \cap W) \leq 7$, hence we deduce $W \not\subset D_1 \cup D_2$.

Lemma 3.18.
(i) $C(3)$ is true.
(ii) $D(3)$ is true.
(iii) Theorems 1.3 and 1.4 are true for $d = 3$.

Proof 3.19. We have $z_3 = 1$. Let A be a scheme of type $(4,3,1)$ with respect to a line L. If (as in the case needed for theorem 1.3) $A \subset P^{(3)}$, where $\{P\} := A_{red}$, we have $h^1(\mathbf{P}^2, \mathcal{I}_A(3)) = 0$ because $h^1(\mathbf{P}^2, \mathcal{I}_{P^{(3)}}(3)) = 0$. In the remaining cases, use Horace's Lemma with respect to a line D with length$(A \cap D) \geq 3$ to obtain part (i). To check part (ii) we use Horace's Lemma three times in the following way. Fix $P_1, P_2 \in \mathbf{P}^2$ such that $P_1 \neq P_2$ and call D the line spanned by P_1 and P_2. Let $Z_i \in \Gamma$, $i = 1, 2$, be general with the restriction $(Z_i)_{red} = \{P_i\}$. Set $W_1 := Z_1 \cup Z_2$, $W_2 := \mathrm{Res}_D(W_1)$, $W_3 := \mathrm{Res}_D(W_2)$ and $W_4 := \mathrm{Res}_D(W_3)$. We show that W_1 solves $D(3)$. First assume $Z_i \subset P_i^{(3)}$. By the generality of Z_i, the scheme Z_i has type $(3,2,2,1)$ with respect to D (Example 3.7), hence lenght$(W_1 \cap D) = 6$, lenght$(W_2 \cap D) =$ lenght$(W_3 \cap D) = 4$ and lenght$(W_4 \cap D) = 2$. Thus $h^0(\mathbf{P}^2, \mathcal{I}_{W_1}(3)) \leq h^0(\mathbf{P}^2, \mathcal{I}_{W_2}(2)) \leq h^0(\mathbf{P}^2, \mathcal{I}_{W_3}(1)) \leq h^0(\mathbf{P}^2, \mathcal{I}_{W_4}) = 0$. In the general case, we have $4 \leq$ lenght$(W_1 \cap D) \leq 6$, hence we still have length$(W_i) \geq 5 - i$. Therefore Horace's Lemma proves $D(3)$ even for an arbitrary Γ. Part (iii) follows from (i) and (ii).

Lemma 3.20. $C(5)$ and $D(5)$ are true.

Proof 3.21. We have $y_5 = 2$ and $w_5 = 4$. Take a line $D \subset \mathbf{P}^2$, $P_1, P_2 \in D$ with $P_1 \neq P_2$ and general $Z_i \in \Gamma$, $i = 1, 2$ such that $(Z_i)_{red} = \{P\}$. Set $A := \mathrm{Res}_L(Z_1 \cup Z_2)$ and $B := \mathrm{Res}_L(A)$. By the generality of Z_i we have length$((Z_1 \cup Z_2) \cap L) = 6$ and length$(A \cap L) = 4$. Applying Horace's Lemma twice with respect to L we obtain $h^1(\mathbf{P}^2, \mathcal{I}_{Z_1 \cup Z_2}(5)) = h^1(\mathbf{P}^2, \mathcal{I}_A(4)) = h^1(\mathbf{P}^2, \mathcal{I}_B(3))$ and $h^0(\mathbf{P}^2, \mathcal{I}_{Z_1 \cup Z_2}(5)) = h^0(\mathbf{P}^2, \mathcal{I}_A(4)) \leq h^0(\mathbf{P}^2, \mathcal{I}_B(3)) + 1$. Since both y_5 and $8 - y_5$ are large, we conclude by an analysis of cubic curves containing B and (for $C(5)$) 4 collinear points of the plane.

Lemma 3.22. $C(6)$ and $D(6)$ are true.

Proof 3.23. We have $y_5 = 2$ and $w_5 = 5$. Take a line $D \subset \mathbf{P}^2$, $P_1, P_2 \in D$ with $P_1 \neq P_2$ and $Z_i \in \Gamma$, $i = 1, 2$ such that $(Z_i)_{red} = \{P\}$, lenght$(Z_1 \cap L) = 4$ and lenght$(Z_2 \cap L) = 3$.

Set $A := \text{Res}_L(Z_1 \cup Z_2)$ and $B := \text{Res}_L(A)$. We have lenght$(A \cap L) = 6$. By applying Horace's Lemma twice with respect to L we obtain $h^i(\mathbf{P}^2, \mathcal{I}_{Z_1 \cup Z_2}(6)) = h^i(\mathbf{P}^2, \mathcal{I}_A(5)) = h^i(\mathbf{P}^2, \mathcal{I}_B(4))$, $i = 1, 2$; then we continue as in the case $A(4)$, by using the inequality $h^0(\mathbf{P}^2, \mathcal{I}_E(4)) \leq 2$, where E is a general union of two elements of Γ.

Lemma 3.24. Assume $d \geq 10$ and $d \notin \{12, 13, 16, 20\}$. If $C(d-2)$ and $D(d-3)$ are true, then $C(d)$ is true.

Proof 3.25. Fix a line $L \subset \mathbf{P}^2$. Since 4 and 3 are coprime, there are uniquely determined integers a, b such that $4a+3b = d+1$ and $0 \leq b \leq 3$. Since $d \geq 10$ and $d \notin \{12, 13, 16, 20\}$, a direct check shows that $a > b$. Fix general $P_i \in L$, $1 \leq i \leq a+b$. Take $Z_j \in \Gamma$, $1 \leq j \leq a$, such that Z_j has type $(4, 3, 1)$ with respect to L. If $b = 0$ set $A = A' = \emptyset$. If $b > 0$ we apply the $(3, 4, 1)$-trick with respect to the pairs (L, P_i), $a+1 \leq i \leq a+b$. Let $A_i \subset L$ (resp. A'_i), $a+1 \leq i \leq a+b$, be the length 3 (resp. 4) subscheme of L with P_i as support and A''_i the length 5 scheme with $(A''_i)_{red} = \{P_i\}$ and type $(4, 1)$ with respect to L. Set $A := \cup_{i=a+1}^{a+b} A_i$, $A'' := \cup_{i=a+1}^{a+b} A''_i$ and $B := \cup_{1=1}^{a} \text{Res}_L(Z_i)$. Since $4a+3b = d+1 = h^0(L, \mathcal{O}_L(d))$, in order to get $C(d)$ it is sufficient to prove that $h^0(\mathbf{P}^2, \mathcal{I}_{A'' \cup B \cup T}(d-1)) = w_d$ (i.e. $h^1(\mathbf{P}^2, \mathcal{I}_{A'' \cup B \cup T}(d-1)) = 0$), where T is a general union of $z_d - a - b$ elements of Γ. We have $z_{d-3} < \text{card}(T_{red}) \leq z_{d-2}$. Hence by the generality of T, $D(d-3)$ and $C(d-2)$, we have $h^0(\mathbf{P}^2, \mathcal{I}_T(d-3)) = h^1(\mathbf{P}^2, \mathcal{I}_T(d-2)) = 0$. Let $\rho : H^0(\mathbf{P}^2, \mathcal{I}_T(d-1)) \to H^0(2L, \mathcal{O}_{2L}(d-1))$ be the restriction map, where $2L$ denotes the unreduced plane conic with L as support. Since $h^0(\mathbf{P}^2, \mathcal{I}_T(d-3)) = 0$, ρ is injective. Since $h^1(\mathbf{P}^2, \mathcal{I}_T(d-2)) = 0$, then $h^0(\mathbf{P}^2, \mathcal{I}_T(d-1)) = w_d + 4a + 5b$. Thus to prove $h^1(\mathbf{P}^2, \mathcal{I}_{A'' \cup B \cup T}(d-1)) = 0$, hence the lemma, it is sufficient to show that $B \cap A''$ gives $4a+5b$ independent conditions to the linear system $W := \text{Im}(\rho)$. Let $\eta : W \to H^0(L, \mathcal{O}_L(d-1))$ be the restriction map and $M := \text{Im}(\eta)$. Since $\cup_{i=1}^{a+b}\{P_i\} = \text{Res}_L(A'' \cup B)$ and $a+b \leq d-2$, we have $\dim(Ker)(\eta) = a+b$, i.e. $\dim(M) = 3a+4b$. Hence it is sufficient to prove that $B \cup A''$ gives $3a+4b$ independent conditions to the linear system M. Since $(A \cup B)_{red}$ is general in L (and general independently of the choice of T, hence general independently of M) and $\text{char}(K) = 0$, we may apply [8] (or, equivalently, the fact that every linear system on a smooth curve has only finitely many ramification points) to obtain that $B \cup A''$ imposes independent conditions to M, so concluding the proof.

The same proof gives the following result.

Lemma 3.26. Assume $d \geq 10$ and $d \notin \{12, 13, 16, 20\}$. If $C(d-2)$ and $D(d-3)$ are true, then $D(d)$ is true.

Lemma 3.27. $C(9)$ and $D(9)$ are true.

Proof 3.28. Fix a line $L \subset \mathbf{P}^2$ and take distinct points $P_i \in L$, $i > 0$. Take $Z_i \in \Gamma$, $1 \leq i \leq 3$, such that $(Z_i)_{red} = \{P_i\}$, length$(Z_j \cap L) = 4$, for $j = 1, 2$ and length$(Z_3 \cap L) = 2$. Hence length$((Z_1 \cup Z_2 \cup Z_3) \cap L) = 10$. By Horace's Lemma we have $h^i(\mathbf{P}^2, \mathcal{I}_{Z_1 \cup Z_2 \cup Z_3}(9)) =$

$h^i(\mathbf{P}^2, \mathcal{I}_{A_1 \cup A_2 \cup A_3}(8))$, $i = 0, 1$. Then we use the $(1, 4, 3)$-trick with respect to the pair (L, P_4) and control in this way the postulation of $\mathcal{O}_L(8)$. Now on L we have the effective Cartier divisor $P_1 + P_2 + 2P_3 + 4P_4$ which controls the postulation of $\mathcal{O}_L(7)$ and allows us to apply Horace's Lemma with respect to L because $h^0(L, \mathcal{I}_{P_1+P_2+2P_3+4P_4}(7)) = h^1(L, \mathcal{I}_{P_1+P_2+2P_3+4P_4}(7)) = 0$.

Lemma 3.29. $C(8)$ and $D(8)$ are true.

Proof 3.30. Let $C \subset \mathbf{P}^2$ be a smooth cubic, P_1 and P_2 flexes of C and P_3, P_4, P_5 general points of C. Let L_i, $1 \leq i \leq 4$, be the tangent line to C at P_i and $Z_i \in \Gamma$ with P_i as support and type $(4, 3, 1)$ with respect to L_i. Set $W := Z_1 \cup Z_2 \cup Z_3 \cup Z_4$ and $A := \mathrm{Res}_C(W)$. Since L_1 and L_2 are flexes of C, while L_3, L_4 and L_3 are not flexes, the scheme $W \cap C$ is the degree 24 divisor $6P_1 + 6P_2 + 4P_3 + 4P_4 + 4P_5$ of C. By the generality of P_5 this divisor is not associated to $\mathcal{O}_C(8)$. Hence $W \cap C$ is not the complete intersection of C with a degree 8 plane curve. Thus we may apply Horace's Lemma with respect to C and reduce $A(8)$ to prove $h^1(\mathbf{P}^2, \mathcal{I}_{A \cup T}(5)) = 0$, where T is a general element of Γ and $A_{red} = \{P_1, \ldots, P_5\}$; the scheme A has a connected component of length 2 at P_1 and P_2 and of length 4 at P_3, P_4 and P_5. For general C the points P_1, P_2, P_3, P_4 and P_5 may be considered as general points of $\mathbf{P}^2$, hence we easily reduce to prove $h^1(\mathbf{P}^2, \mathcal{I}_T(5)) = 0$ (for instance by [7], if the result is false then the corresponding linear system has a base component occuring with multiplicity at least two and containing P_3, P_4 and P_5, hence a base divisor of degree at least 4). Since this last fact is trivially true, the proof is over.

Lemma 3.31. $C(7)$ and $D(7)$ are true.

Proof 3.32. Fix a line $L \subset \mathbf{P}^2$ and take distinct points $P_i \in L$, $1 \leq i \leq 3$. Take $Z_i \in \Gamma$ such that $(Z_i)_{red} = \{P_i\}$, $\mathrm{length}(Z_1 \cap L) = 4$ and $\mathrm{length}(Z_j \cap L) = 2$ for $j = 2, 3$. Thus $\mathrm{length}((Z_1 \cup Z_2 \cup Z_3) \cap L) = 8$. Set $A := \mathrm{Res}_L(Z_1 \cup Z_2 \cup Z_3)$ and $B := \mathrm{Res}_L(A)$. Hence $\mathrm{length}(A \cap L) = 7$. By Horace's Lemma we have $h^i(\mathbf{P}^2, \mathcal{I}_{Z_1 \cup Z_2 \cup Z_3}(7)) = h^i(\mathbf{P}^2, \mathcal{I}_A(6)) = h^i(\mathbf{P}^2, \mathcal{I}_B(5))$, $i = 0, 1$. The scheme B is the union of P_1, a length 4 scheme B_2 supported by P_2 and a length 4 scheme supported by P_3. We have $\mathrm{length}(B \cap L) = 5$. Let Z_4 be a general element of Γ. Set $E := \mathrm{Res}_L(B)$. Hence $\mathrm{length}(E) = 4$. By Horace's Lemma to check $C(7)$ it is sufficient to prove $h^1(\mathbf{P}^2, \mathcal{I}_{E \cup Z_4}(4)) = 0$; this is obvious. To check $D(7)$ it is sufficient to take a general $Z_5 \in \Gamma$ such that $(Z_5)_{red} \in L$ and notice that $h^0(\mathbf{P}^2, \mathcal{I}_{E \cup Z_4 \cup \mathrm{Res}_L(Z_5)}(4)) = 0$.

Lemma 3.33. $C(12)$, $D(12)$, $C(13)$, $D(13)$, $C(20)$, and $D(20)$ are true.

Proof 3.34. Fix a smooth cubic $C \subset \mathbf{P}^2$ and take distinct flexes $P_i \in C$, $1 \leq i \leq 4$, and general $P_5, P_6 \in C$. There is $Z_i \in \Gamma$, $1 \leq i \leq 6$, such that $(Z_i)_{red} = \{P_i\}$, $\mathrm{length}(C \cap Z_j) = 7$, $1 \leq j \leq 4$, and $\mathrm{length}(C \cap Z_j) = 4$, $j = 5, 6$. To prove $C(12)$, and $D(12)$ apply Horace's Lemma with respect to C. Then continue as in the proof of lemma 3.24, but taking a line containing a length two subscheme of $\mathrm{Res}_C(Z_1 \cup \cdots \cup Z_6)$. Take another flex $P_7 \in C$

and $Z_7 \in \Gamma$ such that $(Z_7)_{red} = \{P_7\}$, length$(C \cap Z_7) = 7$. To check $C(13)$ and $D(13)$ we start with $Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \cup Z_6$, apply Horace's Lemma with respect to C and then continue as usual. Let P_8 be another flex of C and P_9 a general point of C. Take $Z_i \in \Gamma$, $i = 8, 9$, such that $(Z_i)_{red} = \{P_i\}$, length$(C \cap Z_8) = 7$ and length$(C \cap Z_9) = 2$. To check $C(16)$ and $D(16)$ we use $Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \cup Z_7 \cup Z_8 \cup Z_9$ and apply Horace's Lemma with respect to C. To check $C(20)$ and $D(20)$ we cannot take 8 elements of Γ, each of them intersecting C in a length 7 scheme because the associated degree 63 divisor of C is a complete intersection with the seventh power of the Hessian cubic of C. We use 7 of these schemes, Z_5, Z_6, and Z_9, and then apply Horace's Lemma.

Proof (of theorem 1.4.). Take $d \neq 4$ and $x > 0$. First assume $x \leq z_d$. By $C(d)$ for a general union W of z_d elements of Γ we have $h^1(\mathbf{P}^2, \mathcal{I}_W(d)) = 0$ and $h^0(\mathbf{P}^2, \mathcal{I}_Z(d)) = w_d$. Hence for any union Z of x connected components of W we have $h^1(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$, so $h^0(\mathbf{P}^2, \mathcal{I}_Z(d)) = w_d + 8(z_d - x)$, i.e. $\rho_{Z,d}$ is surjective. Now assume $x > z_d$. Let M be the general union of $z_d + 1$ elements of Γ. By $D(d)$ we have $h^0(\mathbf{P}^2, \mathcal{I}_M(d)) = 0$. Thus for any disjoint union of x elements of Γ such that $M \subseteq Z$ we have $h^0(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$, i.e. $\rho_{Z,d}$ is injective.

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