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Decompositions of the category of noncommutative sets and Hochschild and cyclic homology

Jolanta Słomińska*

Faculty of Mathematics and Information Sciences,
Technical University of Warsaw,
Plac Politechniki 1, 00-661 Warszawa, Poland

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Abstract: In this note we show that the main results of the paper [PR] can be obtained as consequences of more general results concerning categories whose morphisms can be uniquely presented as compositions of morphisms of their two subcategories with the same objects. First we will prove these general results and then we will apply it to the case of finite noncommutative sets.

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1 General results

Let $\mathcal{C}$ be a small category. Let $\mathcal{A}$ and $\mathcal{B}$ be subcategories of $\mathcal{C}$ with the same objects as $\mathcal{C}$. We will say that $\mathcal{C}$ is a composition of $\mathcal{A}$ and $\mathcal{B}$ if every morphism f of $\mathcal{C}$ can be uniquely presented as a composition $f_1 f_2$ where f_1 is a morphism of $\mathcal{A}$ and where f_2 is a morphism of $\mathcal{B}$. We will use the notation $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$. The natural inclusion $\mathcal{A} \subset \mathcal{C}$ will be denoted by $i_{\mathcal{A}}$.

We will assume that K is a commutative ring. The category of K modules will be denoted by $K-Mod$. A free K module with a base X will be denoted by $K(X)$. Constant functors defined by K will be denoted by the same letter.

For every two functors $M : \mathcal{C} \to K - Mod$ and $N : \mathcal{C}^{op} \to K - Mod$, a tensor product $N \otimes_{\mathcal{C}} M$ is a quotient of $\bigoplus_{c \in \mathcal{C}} N(c) \otimes_K M(c)$ modulo the relation $N(f)x \otimes y = x \otimes M(f)y$

* E-mail: jolslom@impan.gov.pl

where $f : c \to c', x \in N(c'), y \in M(c)$.

Let $C_*(\mathcal{C}, K) = C_*(\mathcal{C}, Z) \otimes_Z K$ denote the chain complex with coefficients in K of the simplicial nerve $B\mathcal{C}$ of $\mathcal{C}$. The groups $Tor^{\mathcal{C}}_*(K, M)$ are isomorphic to the homology groups of the complex $C_*(-\backslash\mathcal{C}, K) \otimes_{\mathcal{C}} M$, where $c\backslash\mathcal{C}$ is the category whose objects are morphisms $c \to y$ of $\mathcal{C}$ and whose morphisms are morphisms $y \to y'$ in $\mathcal{C}$ which give commutative triangles.

Proposition 1.1. Let $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$. Then there exists a functor $K_{\mathcal{B},\mathcal{C}} : \mathcal{C}^{op} \to K - Mod$ such that, for every functor $M : \mathcal{C} \to K - Mod$,

$$Tor^{\mathcal{C}}_*(K_{\mathcal{B},\mathcal{C}}, M) = Tor^{\mathcal{A}}_*(K, Mi_{\mathcal{A}})$$

and for every object c of $\mathcal{C}$,

$$K_{\mathcal{B},\mathcal{C}}(c) = K(\bigsqcup_{y\in\mathcal{C}} Mor_{\mathcal{B}}(c, y)).$$

Proof. If $\mathcal{A}$ is a subcategory of $\mathcal{C}$, then the groups $Tor^{\mathcal{A}}_*(K, Mi_{\mathcal{A}})$ are isomorphic to the homology groups of the complex $C_*(-\backslash i_{\mathcal{A}}, K) \otimes_{\mathcal{C}} M$ where $c\backslash i_{\mathcal{A}}$ is the category whose objects are morphisms $c \to y$ of $\mathcal{C}$ and whose morphisms are morphisms $y \to y'$ in $\mathcal{A}$ which give commutative triangles. This follows from the fact that

$$C_*(c\backslash i_{\mathcal{A}}, K) = C_*(-\backslash\mathcal{A}, K) \otimes_{\mathcal{A}} K(Mor_{\mathcal{C}}(c, i_{\mathcal{A}}(-)))$$

and $K(Mor_{\mathcal{C}}(-, i_{\mathcal{A}}(-))) \otimes_{\mathcal{C}} M = Mi_{\mathcal{A}}$. The homology groups of the complex $C_*(c\backslash i_{\mathcal{A}}, K)$ are equal to

$$Tor^{\mathcal{A}}_*(K, K(Mor_{\mathcal{C}}(c, i_{\mathcal{A}}(-)))).$$

It follows from our assumptions that after restriction to $\mathcal{A}$

$$K(Mor_{\mathcal{C}}(c, i_{\mathcal{A}}(-))) = K(\bigsqcup_{y\in\mathcal{C}} \bigsqcup_{Mor_{\mathcal{B}}(c,y)} Mor_{\mathcal{A}}(y, -)).$$

Hence the functors $K(Mor_{\mathcal{C}}(c, i_{\mathcal{A}}(-)))$ are projective and all homology groups of complexes $C_*(c\backslash i_{\mathcal{A}}, K)$ vanish except

$$H_0(C_*(c\backslash i_{\mathcal{A}}, K)) = K(\bigsqcup_{y\in\mathcal{C}} Mor_{\mathcal{B}}(c, y)).$$

Let

$$K_{\mathcal{B},\mathcal{C}} = H_0(C_*(-\backslash i_{\mathcal{A}}, K)).$$

The functors $C_n(-\backslash i_{\mathcal{A}}, K)$ are direct sums of functors of the form $K(Mor_{\mathcal{C}}(-, y))$ where y is an object of $\mathcal{C}$. Hence $C_*(-\backslash i_{\mathcal{A}}, K)$ is a projective resolution of $K_{\mathcal{B},\mathcal{C}}$ and this implies the result.

The following properties of functors $K_{\mathcal{B},\mathcal{C}}$ are immediate consequences of the definitions.

Corollary 1.2. Suppose that $K : (\mathcal{A})^{op} \to K - Mod$ is equal to $Coker\ \alpha$ where

$$\alpha = \sum_f k_f f_{\mathcal{A}} : K(Mor_{\mathcal{A}}(-, c)) \to K(Mor_{\mathcal{A}}(-, c'))$$

is a linear combination such that, $f \in Mor_{\mathcal{A}}(c, c'), k_f \in K$ and $f_{\mathcal{A}}$ is a natural transformation induced by f. Then $K_{\mathcal{B},\mathcal{C}}$ is equal to $Coker\ \alpha'$ where

$$\alpha' = \sum_f k_f f_{\mathcal{C}} : K(Mor_{\mathcal{C}}(-, c)) \to K(Mor_{\mathcal{C}}(-, c')).$$

Corollary 1.3. If $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ and $\mathcal{A}'$, $\mathcal{B}'$ are subcategories of $\mathcal{A}$ and $\mathcal{B}$ with the same object sets and such that $\mathcal{C}' = \mathcal{A}' \circ \mathcal{B}'$, then, after restriction of $K_{\mathcal{B},\mathcal{C}}$ to $\mathcal{C}'$, there is a natural transformation of functors $K_{\mathcal{B}',\mathcal{C}'} \to K_{\mathcal{B},\mathcal{C}}$. In the case where $\mathcal{B}' = \mathcal{B}$, $K_{\mathcal{B},\mathcal{C}'}$ is equal to $K_{\mathcal{B},\mathcal{C}}$.

2 Noncommutative sets

In [PR] there was introduced a category $F(ass)$ of noncommutative sets whose objects are finite sets $[n] = \{0, 1, .., n\}$ and whose morphisms are maps $f : [n] \to [m]$ together with a total ordering of preimages $f^{-1}(j)$ for all $j \in [m]$. It was proved there that $F(ass)$ is isomorphic to the category ΔS from [FL, L]. This category has two subcategories with the same object sets: the category Δ of linearly ordered sets and the isomorphism subcategory Σ. It follows from the definition that $\Delta S = \Delta \circ \Sigma$. The subcategory of $F(ass)$ consisting of all morphisms such that $f(0) = 0$ was denoted in [PR] by $\Gamma(ass)$. This category corresponds to the subcategory of ΔS denoted in E.6.4.1 of [L] by $\Delta^{op} S'$.

Let $\Sigma_{n+1} = Mor_{\Sigma}([n], [n])$ for all natural numbers n. Let Σ' be the isomorphism subcategory of $\Gamma(ass)$. It consists of all permutations $\sigma \in \Sigma_{n+1}$ such that $\sigma(0) = 0$. Hence Σ'_{n+1} is isomorphic to Σ_n.

Let ΔC be the category of cyclic sets constructed by A. Connes in [C] and let C be the isomorphism subcategory of ΔC. Then C is a subcategory of Σ and $\Delta C = \Delta \circ C$ is a subcategory of ΔS. For every n, C_{n+1} is a cyclic subgroup of Σ_{n+1} generated by the permutation $t : [n] \to [n]$ such that $t(k) = k + 1 \mod n + 1$.

Proposition 2.1. There are the following decomposition

$$F(ass) = (\Delta C)^{op} \circ \Sigma', \quad \Gamma(ass) \cong \Delta^{op} \circ \Sigma', \quad F(ass) = C^{op} \circ \Gamma(ass).$$

Proof. Let f be a morphism of $F(ass)$ such that $f(0) = k$. Then $t^{-k} f$ belongs to $\Gamma(ass)$ and gives us a decomposition $F(ass) = C^{op} \circ \Gamma(ass)$. Similarly let $\sigma \in \Sigma_{n+1}$ and let $\sigma(0) = k$. Then $\sigma' = t^{-k}\sigma$ belongs to Σ'_{n+1} and this gives us decompositions

$$\Sigma = C \circ \Sigma', \quad \Delta S = \Delta C \circ \Sigma'.$$

The fact that Σ' is a subcategory of $\Gamma(ass)$ implies that we have a decomposition

$$\Gamma(ass) = (\Delta C \cap \Gamma(ass)) \circ \Sigma'.$$

There is a natural isomorphism of categories $i : (\Delta C)^{op} = C^{op} \circ \Delta^{op} \to \Delta C$ ([L]) which gives us inclusions of categories $i_{(\Delta C)^{op}} : (\Delta C)^{op} \to F(ass)$, $i_{\Delta^{op}} : \Delta^{op} \to F(ass)$ and decompositions

$$\Delta S = (\Delta C)^{op} \circ \Sigma' = C^{op} \circ \Delta^{op} \circ \Sigma'.$$

From the fact that $i_{\Delta^{op}}(\Delta^{op})$ is a subcategory of $\Gamma(ass)$, it follows that $i_{\Delta^{op}}$ induces an inclusion of categories $\hat{C} : \Delta^{op} \to \Gamma(ass)$ (E.6.4.1 of [L], [PR]). The category $\hat{C}(\Delta^{op})$ is isomorphic to Δ^{op} and

$$\Delta C \cap \Gamma(ass) = (\Delta C)^{op} \cap \Gamma(ass) = \hat{C}(\Delta^{op}).$$

Indeed, if f is a morphism of $(\Delta C)^{op} = C^{op} \circ \Delta^{op}$ then $f = t^k f'$ where $f' \in \hat{C}(\Delta^{op}) \subseteq \Gamma(ass)$ and $k = f(0)$. This implies that $f \in \Gamma(ass)$ if and only if $k = 0$ and $t^k = id$ and that

$$\Gamma(ass) = \hat{C}(\Delta^{op}) \circ \Sigma'.$$

In [PR], for every

$$M : F(ass) \to K - Mod \quad \text{and} \quad N : \Gamma(ass) \to K - Mod,$$

the cyclic homology $HC_*(M)$ were defined as cyclic homology $HC_*(Mi_{(\Delta C)^{op}})$ of the cyclic module $Mi_{(\Delta C)^{op}}$ and the Hochschild homology $H_*(N)$ as the homotopy $\pi_*(N\hat{C})$ of the simplicial module $N\hat{C}$. As immediate consequence of previous results we obtain the following results of [PR].

Corollary 2.2. There exist functors

$$K_{\Sigma',F(ass)} : F(ass)^{op} \to K - Mod, \quad K_{\Sigma',\Gamma(ass)} : \Gamma(ass)^{op} \to K - Mod$$

such that

(i) $K_{\Sigma',\Gamma(ass)}$ is a restriction of $K_{\Sigma',F(ass)}$,

(ii) for every object $[n]$ of $F(ass)$,

$$K_{\Sigma',F(ass)}([n]) = K_{\Sigma',\Gamma(ass)}([n]) = \Sigma_n.$$

(iii) for every $M : F(ass) \to K - Mod$, $N : \Gamma(ass) \to K - Mod$,

$$Tor_*^{F(ass)}(K_{\Sigma',F(ass)}, M) = HC_*(M),$$
$$Tor_*^{\Gamma(ass)}(K_{\Sigma',\Gamma(ass)}, N) = H_*(N).$$

Proof. It is well known ([C], [L]) that, for every

$$M' : (\Delta C)^{op} \to K - Mod, \quad N' : \Delta^{op} \to K - Mod,$$

the groups $Tor_*^{(\Delta C)^{op}}(K, M')$ are equal to the cyclic homology $HC_*(M')$ of M' and $Tor_*^{\Delta^{op}}(K, N')$ are equal to the homotopy groups $\pi_*(N')$ of the simplicial module N'. Hence

$$Tor_*^{(\Delta C)^{op}}(K, Mi_{(\Delta C)^{op}}) = HC_*(M), \quad Tor_*^{\Delta^{op}}(K, N\hat{C}) = H_*(N).$$

Now the result follows from 1.1 and 1.3.

Corollary 2.3. The functor $K_{(\Sigma',F(ass))}$ is equal to the $Coker\ d_F$, where

$$d_F = d_1 - d_0 : K(Mor_{F(ass)}(-,[1])) \to K(Mor_{F(ass)}(-,[0]))$$

and the functor $K_{(\Sigma',\Gamma(ass)}$ is equal to the $Coker\ d_\Gamma$, where

$$d_\Gamma = d_1 - d_0 : K(Mor_{\Gamma(ass)}(-,[1])) \to K(Mor_{\Gamma(ass)}(-,[0]))$$

and d_0, d_1 are induced by different elements of the set

$$Mor_{\Delta^{op}}([1],[0]) \subset Mor_{(\Delta C)^{op}}([1],[0]).$$

Proof. For every $[s]$, the functor $K(Mor_\Delta(-,[s]))$ is a simplicial resolution of K and the constant contravariant functor K on Δ^{op} is equal to the cokernel of the sequence

$$d_\Delta = d_1 - d_0 : K(Mor_{\Delta^{op}}(-,[1])) \to K(Mor_{\Delta^{op}}(-,[0])).$$

Similarly, it is proved in 7.1 of [L], that, for every $[s]$, the functor $Mor_{\Delta C}(-,[s])$, after restriction to Δ^{op}, is a simplicial sphere S^1. Hence the constant contravariant functor K on $(\Delta C)^{op}$ is equal to the cokernel of the sequence

$$d_{\Delta C} = d_1 - d_0 : K(Mor_{(\Delta C)^{op}}(-,[1])) \to K(Mor_{(\Delta C)^{op}}(-,[0])).$$

Now the result follows from 1.2.

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Remark. After this article was written, a preprint [Z], in which a similar approach was developed, appeared on the web page ArXiv.

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