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On a family of vector space categories

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Abstract: In continuation of our earlier work [2] we describe the indecomposable representations and the Auslander–Reiten quivers of a family of vector space categories playing an important role in the study of domestic finite dimensional algebras over an algebraically closed field. The main results of the paper are applied in our paper [3] where we exhibit a wide class of almost sincere domestic simply connected algebras of arbitrary large finite global dimensions and describe their Auslander–Reiten quivers.
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Introduction

Throughout the paper K will denote a fixed algebraically closed field. By an algebra we mean an associative finite dimensional K-algebra with an identity, and by a module we mean a finite dimensional left module.

Linear representations of vector space categories have for a long time been recognized as an important tool in the study of indecomposable representations of groups, algebras, lattices over orders, and Cohen–Macaulay modules. In particular, many of the classification results and representation type criteria for algebras depend essentially on the representation theory of vector space categories (see [4], [5], [6] for more details). The Auslander–Reiten quiver is another important combinatorial and homological invariant of the category of finite dimensional representations of a vector space category or an algebra.

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One of the main open problems of the representation theory of algebras is to describe the module categories of finite dimensional modules over the tame algebras, that is algebras for which the indecomposable modules occur in each dimension in a finite number of discrete and a finite number of one-parameter families. An important class of tame algebras is formed by the domestic algebras for which there is a common bound for the numbers of one-parametric families of indecomposable modules of any fixed dimension. Classical examples of domestic algebras are provided by the path algebras of Dynkin and the Euclidean quivers.

Frequently, applying covering techniques, geometric deformations and vector space category methods, we may reduce the representation theory of a given algebra to that of the known tame algebras of finite global dimension.

The main aim of this paper is to give a complete classification of indecomposable finite dimensional representations of a family of domestic vector space categories and describe the structure of all connected components in the associated Auslander–Reiten quivers.

The interest in these vector space categories is motivated by the fact that they seem to play a crucial role in the study of domestic simply connected algebras. In fact, applying the main result of this paper, we exhibit in our paper [3] a new wide class of sincere (all simple modules occur as composition factors of an indecomposable module) domestic algebras of arbitrary large global dimension, which gives us a new view of the representation theory of domestic algebras.

The paper is organized as follows. In Section 1 we present the family of vector space categories we are interested in and state our main result on the shape of the Auslander–Reiten quiver of the associated subspace category. Section 2 contains some notation and preliminary facts on the indecomposable objects of the subspace categories of vector space categories. In Sections 3 and 4 we prove our main result, presented in the first section.

1 A family of vector space categories

Following [6] (see also [5]), a vector space category is defined as a pair $(\mathbb{K}, |-|)$, where $\mathbb{K}$ is a category having the finite unique decomposition property and $|-| : \mathbb{K} \to \operatorname{mod} K$ is an additive faithful functor.

Given a vector space category $(\mathbb{K}, |-|)$ we consider the subspace category $\mathcal{U}(\mathbb{K}, |-|)$. The objects of $\mathcal{U}(\mathbb{K}, |-|)$ are triples $V = (V_0, V_\omega, \gamma_V)$ with $V_0 \in \mathbb{K}$, $V_\omega \in \operatorname{mod} K$ and $\gamma_V : V_\omega \to |V_0|$ a K-linear map. If $V = (V_0, V_\omega, \gamma_V)$ and $W = (W_0, W_\omega, \gamma_W)$ are two objects of $\mathcal{U}(\mathbb{K}, |-|)$ then a morphism $f : V \to W$ in $\mathcal{U}(\mathbb{K}, |-|)$ is a pair $f = (f_0, f_\omega)$, where $f_0 : V_0 \to W_0$ is a morphism in $\mathbb{K}$, $f_\omega : V_\omega \to W_\omega$ is a K-linear map and the condition $|f_0|\gamma_V = \gamma_W f_\omega$ is satisfied.

Let $m \geq 1$ and $n \geq 0$ be fixed integers. Denote

$$\alpha(i) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i = 1, \dots, m, \end{cases}$$

and

$$\beta(i) = \begin{cases} 2m+n-2i-1 & \text{if } i = 0, \dots, m-1, \\ n & \text{if } i = m. \end{cases}$$

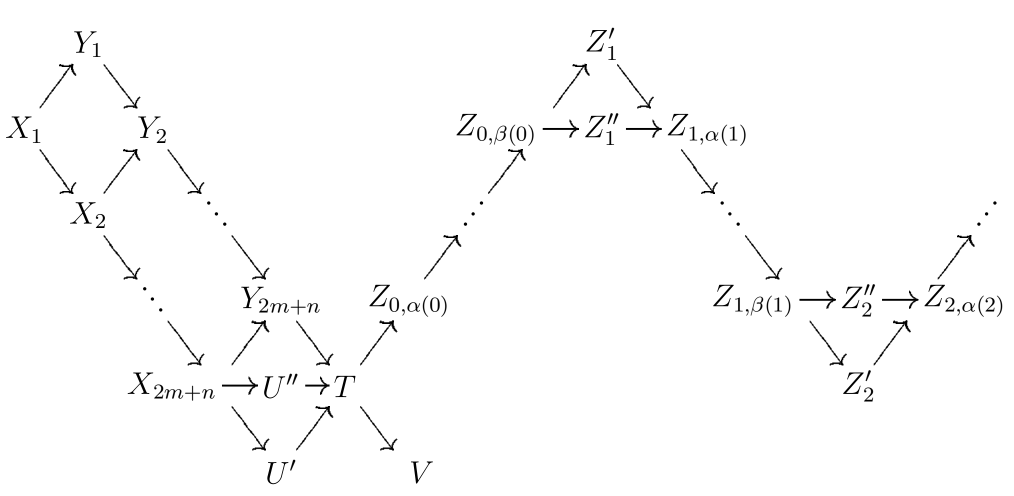
The main goal of this paper is to describe the Auslander–Reiten quiver of the subspace category $\mathcal{U}(\mathbb{K}_{m,n}, |-|)$, where $(\mathbb{K}_{m,n}, |-|)$ is the following vector space category. The representatives of isomorphism classes of indecomposable objects in $\mathbb{K}_{m,n}$ are X_s, $s = 1, \dots, 2m+n$, Y_s, $s = 1, \dots, 2m+n$, U', U'', T, V, $Z_{i,s}$, $s = \alpha(i), \dots, \beta(i)$, $i = 0, \dots, m$, Z'_i, $i = 1, \dots, m$, Z''_i, $i = 1, \dots, m$. For each indecomposable object X of $\mathbb{K}_{m,n}$, we have

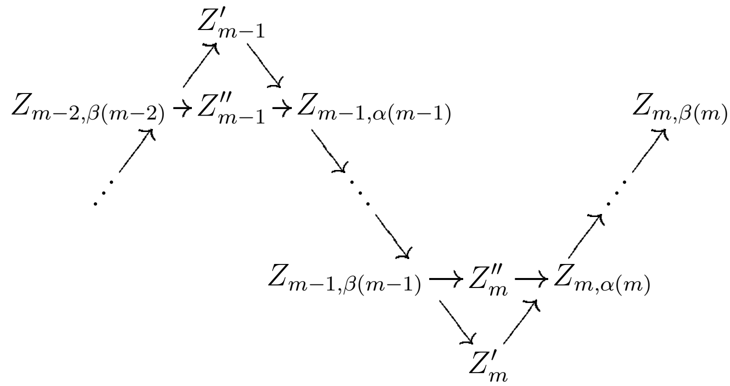
$$\dim_K |X| = \begin{cases} 1 & \text{if } X \not\simeq T, \\ 2 & \text{if } X \simeq T. \end{cases}$$

Further, $\dim_K \operatorname{Hom}_{\mathbb{K}}(X,Y) \leq 2$ for indecomposable objects X and Y of $\mathbb{K}_{m,n}$. Moreover, $\operatorname{Hom}_{\mathbb{K}}(X,Y) \neq 0$ if and only if one of the following conditions is satisfied:

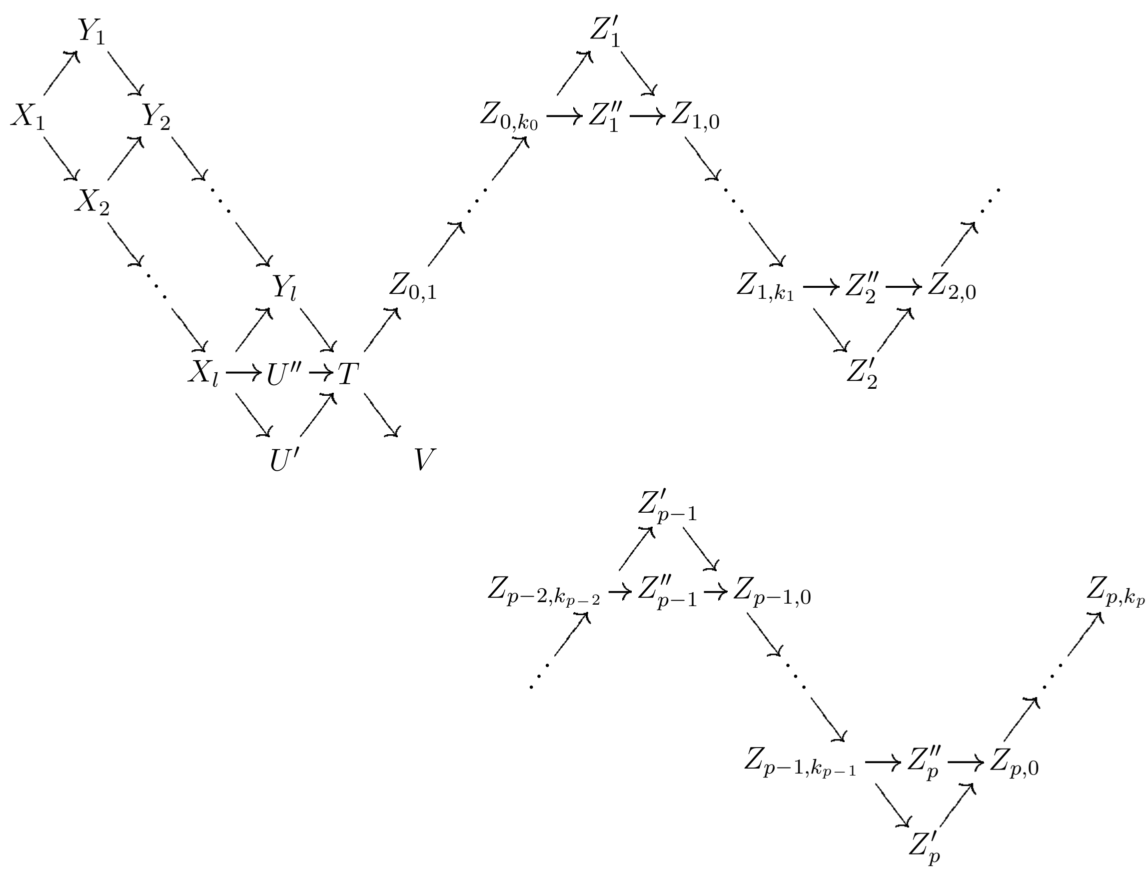
- $X \simeq X_s$, $Y \simeq X_t, Y_t, U', U'', T, V, Z_{i,r}, Z'_j, Z''_j$, $s \leq t$,
- $X \simeq Y_s$, $Y \simeq Y_t, T, V$, $s \leq t$,
- $X \simeq U'$, $Y \simeq U', T, Z_{i,s}, Z'_j, Z''_j$,
- $X \simeq U''$, $Y \simeq U'', T, V, Z_{i,s}, Z'_j, Z''_j$,
- $X \simeq T$, $Y \simeq T, V, Z_{i,s}, Z'_j, Z''_j$,
- $X \simeq V$, $Y \simeq V$,
- $X \simeq Z_{i,s}$, $Y \simeq Z_{j,t}, Z'_k, Z''_k$, $i < j$ or $i = j$ and $s \leq t$, $i < k$,
- $X \simeq Z'_i$, $Y \simeq Z_{j,s}, Z'_j, Z''_k$, $i \leq j$, $i < k$,
- $X \simeq Z''_i$, $Y \simeq Z_{j,s}, Z'_k, Z''_j$, $i \leq j$, $i < k$.

Finally, $\dim_K \operatorname{Hom}_{\mathbb{K}}(X,Y) = 2$ if and only if $X \simeq X_i$ and $Y \simeq T$. The vector space category $(\mathbb{K}_{m,n}, |-|)$ may be represented as the following quiver whose vertices are representatives of isomorphism classes of the indecomposable objects in $\mathbb{K}_{m,n}$ and arrows correspond to irreducible maps.





The subspace categories $\mathcal{U}(\mathbb{K}_{m,n}, |-|)$ play an important role in the description of indecomposable modules over a wide class of domestic algebras (see [3]). Note also that knowledge of the categories $\mathcal{U}(\mathbb{K}_{m,n}, |-|)$ for all possible m and n immediately implies knowledge of the subspace category of the following vector space category



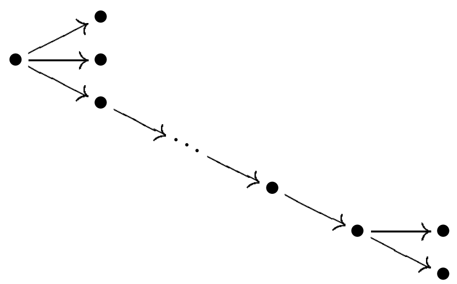
where $l \geq 1$, $p \geq 0$ and $k_0, \ldots, k_p \geq 1$ are arbitrary.

Recall that following [2] a translation quiver is called a translation quiver of the 1st type if its stable part is of the form $\mathbb{Z}\mathbb{A}_\infty$, its left stable part is of the form $(-\mathbb{N})\mathbb{D}_\infty$, and its right stable part is of the form $\mathbb{N}\mathbb{D}_\infty$. See [1], [5], [6] for basic background on the translation quivers and Auslander–Reiten theory.

The following theorem is the main result of the paper.

Main Theorem. The Auslander–Reiten quiver of the subspace category $\mathcal{U}(\mathbb{K}_{m,n}, |-|)$ consists of the following components:

- a preprojective component of the form $\mathbb{N}\Delta$, where Δ is the following quiver of type $\tilde{\mathbb{D}}_{2m+n+3}$



- a coray tube of rank 2, a stable tube of rank 2 and a family of homogeneous stable tubes indexed by the elements of $K \setminus \{0, 1\}$;
- m components of 1st type.

In fact, in Section 3 we not only prove the above theorem, but we also give, in Lemmas 3.2, 3.3 and 3.4, the precise description of indecomposable objects and Auslander–Reiten sequences.

2 Preliminaries on subspace categories

In this section we recall some known facts on vector space categories and introduce notation for the objects of the associated subspace categories.

With a vector space category $(\mathbb{K}, |-|)$ we may associate, in addition to the subspace category $\mathcal{U}(\mathbb{K}, |-|)$, the factor space category $\mathcal{V}(\mathbb{K}, |-|)$. Its objects are triples $V = (V_0, V_\omega, \gamma_V)$ with $V_0 \in \mathbb{K}$, $V_\omega \in \operatorname{mod} K$ and $\gamma_V : |V_0| \to V_\omega$ a K-linear map. The morphisms in $\mathcal{V}(\mathbb{K}, |-|)$ are defined in a similar way as the morphisms in $\mathcal{U}(\mathbb{K}, |-|)$.

Let $\mathcal{A}$ be a category and let $\mathcal{F} : \mathcal{A} \to \operatorname{mod} K$ be a covariant functor. We denote by $\mathcal{F}(\mathcal{A})$ the vector space category defined as the pair $(\mathcal{A}/\operatorname{Ker}\mathcal{F}, |-|)$, where the functor $|-| : \mathcal{A}/\operatorname{Ker}\mathcal{F} \to \operatorname{mod} K$ is induced by $\mathcal{F}$. The vector space categories of the above type arise while considering one-point extensions and one-point coextensions of algebras.

If A is an algebra and M is an A-module then the algebra given in the matrix form as

$$\begin{bmatrix} A & M \\ 0 & K \end{bmatrix}$$

is called the one-point extension of A by M and is denoted by $A[M]$. It is known that the category $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} A))$ is representation equivalent to the full subcategory of $\operatorname{mod} A[M]$ consisting of $A[M]$-modules without indecomposable direct summands X such that $X \in \operatorname{mod} A$ and $\operatorname{Hom}_A(M, X) = 0$. We will identify the objects of $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} A))$ with the corresponding $A[M]$-modules. The Auslander–Reiten quiver $\Gamma(\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} A)))$ of $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} A))$ is a full subquiver of the Auslander–

Reiten quiver $\Gamma(\operatorname{mod} A[M])$ of $\operatorname{mod} A[M]$. Moreover, using so called lifted sequences one can construct $\Gamma(\operatorname{mod} A[M])$ if $\Gamma(\operatorname{mod} A)$ and $\Gamma(\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} A)))$ are known (see [5, 2.5,(6)]). In particular, it is known that a component $\mathcal{C}$ of $\Gamma(\operatorname{mod} A)$ is a component of $\Gamma(\operatorname{mod} A[M])$ provided $\operatorname{Hom}_A(M, X) = 0$ for all $X \in \mathcal{C}$. Finally, $\operatorname{Hom}(N, \operatorname{mod} A[M]) = \operatorname{Hom}(N, \operatorname{mod} A)$, provided the vector space categories $\operatorname{Hom}(M, \operatorname{mod} A)$ and $\operatorname{Hom}(N, \operatorname{mod} A)$ have no nonzero common objects.

Dually, we define the one-point coextension $[M]A$ of A by M as

$$[M]A = \begin{bmatrix} K & D(M) \\ 0 & A \end{bmatrix},$$

where $D = \operatorname{Hom}_K(-, K)$ is the standard duality on $\operatorname{mod} A$. We have the corresponding vector space category $D(M) \otimes \operatorname{mod} A \simeq D \operatorname{Hom}(\operatorname{mod} A, M)$, and similar results as above hold for $\mathcal{V}(D \operatorname{Hom}(\operatorname{mod} A, M))$.

We will now introduce some notation for objects of subspace categories. They will be defined in terms of chosen elements, but it can be shown that the isomorphism classes of defined objects do not depend on the choice. Since it is done by standard calculations we will not present the proofs. On the other hand, in order to give the reader an overview of the method, we present the calculations for one of the most complicated cases in Section 4.

Let $(\mathbb{K}, |-|)$ be a vector space category and $X_1, \ldots, X_t$, $t \geq 1$, be pairwise nonisomorphic indecomposable objects of $\mathbb{K}$ such that $\dim_K |X_i| = 1$. For each i let $\mathfrak{x}_i$ be a nonzero element of X_i. We will denote by $\overline{X_1 \cdots X_t}$ the object $(X_1 \oplus \cdots \oplus X_t, K, f)$ of the subspace category $\mathcal{U}(\mathbb{K}, |-|)$, where $f : K \to |X_1 \oplus \cdots \oplus X_t|$ is given by

$$f(1) = \mathfrak{x}_1 + \cdots + \mathfrak{x}_t.$$

For the rest of this section we fix a vector space category $(\mathbb{K}, |-|)$ and an indecomposable object T of $\mathbb{K}$ such that $\dim_K |T| = 2$. If $\mathfrak{t}$ is an element of $|T|$, then we denote by $\mathfrak{t}^{(1)}, \ldots, \mathfrak{t}^{(p)}$ the induced elements of $|T^p|$. Moreover, we always denote by $\mathfrak{e}_1, \ldots, \mathfrak{e}_p$ the standard basis of K^p.

Let $p \geq 1$. By $\overline{T^p}^{p-1}$ we denote the triple (T^p, K^{p-1}, f), where $f : K^{p-1} \to |T^p|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i)} + \mathfrak{t}_1^{(i+1)}, \ i = 1, \ldots, p-1,$$

for some basis $\mathfrak{t}_1$, $\mathfrak{t}_2$ of $|T|$. Note that $\overline{T^1}^0 = T$.

Let $p \geq 0$ and U be an indecomposable object of $\mathbb{K}$ such that $\dim_K |U| = 1$ and $\dim_K \operatorname{Hom}_{\mathbb{K}}(U, T) = 1$. Let $\mathfrak{u}$ be a nonzero element of $|U|$ and $\varphi : U \to T$ be a nonzero map. Put $\mathfrak{t}_1 = |\varphi|(\mathfrak{u})$ and choose $\mathfrak{t}_2 \notin \operatorname{Im} |\varphi|$. By $\overline{UT^p}^p$ we denote the triple $(U \oplus T^p, K^p, f)$, where $f : K^p \to |U \oplus T^p|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i)} + \mathfrak{t}_1^{(i+1)}, \ i = 1, \ldots, p,$$

with $\mathfrak{t}_1^{(p+1)} = \mathfrak{u}$. Note that $\overline{UT^0}^0 = U$.

Let $p \geq 0$ and U', U'' be indecomposable objects of $\mathbb{K}$ with $\dim_K |U'| = 1$, $\dim_K |U''| = 1$, $\dim_K \operatorname{Hom}_{\mathbb{K}}(U', T) = 1$ and $\dim_K \operatorname{Hom}_{\mathbb{K}}(U'', T) = 1$. Let $\varphi' : U' \to T$ and $\varphi'' : U'' \to T$ be nonzero maps. We assume in addition that $\operatorname{Im} |\varphi'| \cap \operatorname{Im} |\varphi''| = 0$. Choose nonzero elements $\mathfrak{u}' \in |U'|$ and $\mathfrak{u}'' \in |U''|$, and let $\mathfrak{t}_1 = |\varphi'|(\mathfrak{u}')$ and $\mathfrak{t}_2 = |\varphi''|(\mathfrak{u}'')$. By $\overline{U'U''T^p}^{p+1}$ we denote the triple $(U' \oplus U'' \oplus T^p, K^{p+1}, f)$, where $f : K^{p+1} \to |U' \oplus U'' \oplus T^p|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = 1, \ldots, p+1,$$

with $\mathfrak{t}_2^{(0)} = \mathfrak{u}''$ and $\mathfrak{t}_1^{(p+1)} = \mathfrak{u}'$. Note that $\overline{U'U''T^0}^1 = \overline{U''U'}$.

Let $p \geq 0$ and Y', U', U'' be indecomposable objects of $\mathbb{K}$ such that the vector spaces $|Y'|$, $|U'|$, $|U''|$, $\operatorname{Hom}_{\mathbb{K}}(Y', T)$, $\operatorname{Hom}_{\mathbb{K}}(U', T)$ and $\operatorname{Hom}_{\mathbb{K}}(U'', T)$ are one-dimensional. Let $\psi' : Y' \to T$, $\varphi' : U' \to T$ and $\varphi'' : U'' \to T$ be nonzero maps. We assume in addition that $\operatorname{Im} |\psi'| \cap \operatorname{Im} |\varphi'| = 0$, $\operatorname{Im} |\psi'| \cap \operatorname{Im} |\varphi''| = 0$ and $\operatorname{Im} |\varphi'| \cap \operatorname{Im} |\varphi''| = 0$. Let $\mathfrak{y}' \in |Y'|$, $\mathfrak{u}' \in |U'|$ and $\mathfrak{u}'' \in |U''|$ be nonzero elements, and put $\mathfrak{t}_0 = |\psi'|(\mathfrak{y}')$, $\mathfrak{t}_1 = |\varphi'|(\mathfrak{u}')$ and $\mathfrak{t}_2 = |\varphi''|(\mathfrak{u}'')$. By $\overline{Y'U'U''T^p}^{p+2}$ we denote the triple $(Y' \oplus U' \oplus U'' \oplus T^p, K^{p+2}, f)$, where $f : K^{p+2} \to |Y' \oplus U' \oplus U'' \oplus T^p|$ is given by

$$\begin{aligned} f(\mathfrak{e}_1) &= \mathfrak{y}' + \mathfrak{u}'', \\ f(\mathfrak{e}_i) &= \mathfrak{t}_2^{(i-2)} + \mathfrak{t}_1^{(i-1)},\ i = 2, \ldots, p+2, \end{aligned}$$

with $\mathfrak{t}_2^{(0)} = \mathfrak{u}''$ and $\mathfrak{t}_1^{(p+1)} = \mathfrak{u}'$.

In the same situation as above, we denote by $\overline{Y'U'U''T^p}^{p+1}$ the triple $(Y' \oplus U' \oplus U'' \oplus T^p, K^{p+1}, f)$, where $f : K^{p+1} \to |Y' \oplus U' \oplus U'' \oplus T^p|$ is defined as follows. If $p = 2q$, $q \geq 0$, then f is given by

$$\begin{aligned} f(\mathfrak{e}_i) &= \mathfrak{t}_0^{(i)} + \mathfrak{t}_1^{(i+1)},\ i = 1, \ldots, q-1, \\ f(\mathfrak{e}_q) &= \mathfrak{t}_0^{(q)} + \mathfrak{u}', \\ f(\mathfrak{e}_{q+1}) &= \mathfrak{u}' + \mathfrak{u}'' + \mathfrak{t}_0^{(q+1)}, \\ f(\mathfrak{e}_i) &= \mathfrak{t}_1^{(i-1)} + \mathfrak{t}_0^{(i)},\ i = q+2, \ldots, p+1, \end{aligned}$$

with $\mathfrak{t}_0^{(p+1)} = \mathfrak{y}'$. If $p = 2q+1$, $q \geq 0$, then f is given by

$$\begin{aligned} f(\mathfrak{e}_i) &= \mathfrak{t}_0^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = 1, \ldots, q, \\ f(\mathfrak{e}_{q+1}) &= \mathfrak{t}_0^{(q)} + \mathfrak{u}', \\ f(\mathfrak{e}_{q+2}) &= \mathfrak{u}' + \mathfrak{u}'' + \mathfrak{t}_0^{(q+1)}, \\ f(\mathfrak{e}_i) &= \mathfrak{t}_1^{(i-2)} + \mathfrak{t}_0^{(i-1)},\ i = q+3, \ldots, p+1, \end{aligned}$$

with $\mathfrak{t}_0^{(0)} = \mathfrak{y}'$. Note that $\overline{Y'U'U''T^0}^1 = \overline{Y'U'U''}$.

Now we assume in addition to the previous case that Y'' is an indecomposable object of $\mathbb{K}$ such that $\dim_K |Y''| = 1$ and $\dim_K \operatorname{Hom}_{\mathbb{K}}(Y'', T) = 1$. Let $\psi'' : Y'' \to T$ be a nonzero map. We assume also that $\operatorname{Im} |\psi''| = \operatorname{Im} |\psi'|$. Let $\mathfrak{y}''$ be a nonzero element of $|Y''|$ such that $|\psi''|(\mathfrak{y}'') = \mathfrak{t}_0$. By $\overline{Y'Y''U'U''T^p}^{p+2}$ we denote the triple $(Y' \oplus Y'' \oplus U' \oplus U'' \oplus T^p, K^{p+2}, f)$, where $f : K^{p+2} \to |Y' \oplus Y'' \oplus U' \oplus U'' \oplus T^p|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_0^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = 1, \ldots, q,$$

$$f(\mathfrak{e}_{q+1}) = \mathfrak{t}_0^{(q)} + \mathfrak{u}',$$
$$f(\mathfrak{e}_{q+2}) = \mathfrak{u}' + \mathfrak{u}'' + \mathfrak{t}_0^{(q+1)},$$
$$f(\mathfrak{e}_i) = \mathfrak{t}_1^{(i-2)} + \mathfrak{t}_0^{(i-1)},\ i = q+3, \dots, p+2,$$

with $q = \lfloor \frac{p}{2} \rfloor$, $\mathfrak{t}_0^{(0)} = \mathfrak{y}'$ and $\mathfrak{t}_0^{(p+1)} = \mathfrak{y}''$.

Let $p \geq 0$ and U be an indecomposable object of $\mathbb{K}$ such that $\dim_K |U| = 1$ and $\dim_K \operatorname{Hom}_{\mathbb{K}}(U, T) = 1$. Let $\mathfrak{u} \in |U|$ be a nonzero element and $\varphi : U \to T$ be a nonzero map. Put $\mathfrak{t}_1 = |\varphi|(\mathfrak{u})$ and choose $\mathfrak{t}_2 \notin \operatorname{Im} |\varphi|$. By $\overline{UT^p}^{p+1}$ we denote the triple $(U \oplus T^p, K^{p+1}, f)$, where $f : K^{p+1} \to |U \oplus T^p|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = 1, \dots, p+1,$$

with $\mathfrak{t}_2^{(0)} = 0$ and $\mathfrak{t}_1^{(p+1)} = \mathfrak{u}$. Note that $\overline{UT^0}^1 = \overline{U}$.

Let $p \geq 0$ and V be an indecomposable object of $\mathbb{K}$ such that $\dim_K |V| = 1$ and $\dim_K \operatorname{Hom}_{\mathbb{K}}(T, V) = 1$. Let $\psi : T \to V$ be a nonzero map and $\mathfrak{t}_1 \in \operatorname{Ker} |\psi|$ be a nonzero element. Choose $\mathfrak{t}_2 \notin \operatorname{Ker} |\psi|$ and put $\mathfrak{v} = |\psi|(\mathfrak{t}_2)$. By $\overline{T^pV}^p$ we denote the triple $(T^p \oplus V, K^p, f)$, where $f : K^p \to |T^p \oplus V|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = 1, \dots, p,$$

with $\mathfrak{t}_2^{(0)} = \mathfrak{v}$. Note that $\overline{T^0V}^0 = V$.

Let $p \geq 0$ and U, V be indecomposable objects of $\mathbb{K}$ with $\dim_K |U| = 1$, $\dim_K |V| = 1$, $\dim_K \operatorname{Hom}_{\mathbb{K}}(U, T) = 1$, $\dim_K \operatorname{Hom}_{\mathbb{K}}(T, V) = 1$, $\operatorname{Hom}_{\mathbb{K}}(U, V) = 0$. Let $\mathfrak{u} \in |U|$ be a nonzero element, and $\varphi : U \to T$ and $\psi : T \to V$ be nonzero maps. Put $\mathfrak{t}_1 = |\varphi|(\mathfrak{u})$, choose $\mathfrak{t}_2 \notin \operatorname{Im} |\varphi|$ and put $\mathfrak{v} = |\psi|(\mathfrak{t}_2)$. By $\overline{UT^pV}^{p+1}$ we denote the triple $(U \oplus T^p \oplus V, K^{p+1}, f)$, where $f : K^{p+1} \to |U \oplus T^p \oplus V|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = 1, \dots, p+1,$$

with $\mathfrak{t}_2^{(0)} = \mathfrak{v}$ and $\mathfrak{t}_1^{(p+1)} = \mathfrak{u}$. Note that $\overline{UT^0V} = \overline{VU}$.

Let $p \geq 0$. By $\overline{T^p}^{p+1}$ we denote the triple (T^p, K^{p+1}, f), where $f : K^{p+1} \to |T^p|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = 1, \dots, p+1,$$

for some basis $\mathfrak{t}_1$, $\mathfrak{t}_2$ of $|T|$, with $\mathfrak{t}_2^{(0)} = 0 = \mathfrak{t}_1^{(p+1)}$. In particular, $\overline{T^0}^1 = (0, K, 0)$ is the unique (up to isomorphism) simple injective object of $\mathcal{U}(\mathbb{K}, |-|)$.

Let $p \geq 0$ and V be an indecomposable object of $\mathbb{K}$ such that $\dim_K |V| = 1$ and $\dim_K \operatorname{Hom}_{\mathbb{K}}(T, V) = 1$. Let $\psi : T \to V$ be a nonzero map and $\mathfrak{t}_1 \in \operatorname{Ker} |\psi|$ be a nonzero element, choose $\mathfrak{t}_2 \notin \operatorname{Ker} |\psi|$ and put $\mathfrak{v} = |\psi|(\mathfrak{t}_2)$. By $\overline{T^pV}^{p+1}$ we denote the triple $(T^p \oplus V, K^{p+1}, f)$, where $f : K^{p+1} \to |T^p \oplus V|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = 1, \dots, p+1,$$

with $\mathfrak{t}_2^{(0)} = \mathfrak{v}$ and $\mathfrak{t}_1^{(p+1)} = 0$. Note that $\overline{T^0V}^1 = \overline{V}$.

Let $p \geq 0$ and V', V'' be indecomposable objects of $\mathbb{K}$ with $\dim_K |V'| = 1$, $\dim_K |V''| = 1$, $\dim_K \operatorname{Hom}_{\mathbb{K}}(T, V') = 1$ and $\dim_K \operatorname{Hom}_{\mathbb{K}}(T, V'') = 1$. Let $\psi' : T \to V'$ and $\psi'' : T \to V''$

be nonzero maps. We assume in addition that $\operatorname{Ker}|\psi'| \cap \operatorname{Ker}|\psi''| = 0$. Let $\mathfrak{t}_1 \in \operatorname{Ker}|\psi'|$ and $\mathfrak{t}_2 \in \operatorname{Ker}|\psi''|$ be nonzero elements, and put $\mathfrak{v}' = |\psi'|(\mathfrak{t}_2)$ and $\mathfrak{v}'' = |\psi''|(\mathfrak{t}_1)$. By $\overline{T^p V' V''}^{p+1}$ we denote the triple $(T^p \oplus V' \oplus V'', K^{p+1}, f)$, where $f : K^{p+1} \to |T^p \oplus V' \oplus V''|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = 1, \ldots, p+1,$$

with $\mathfrak{t}_2^{(0)} = \mathfrak{v}'$ and $\mathfrak{t}_1^{(p+1)} = \mathfrak{v}''$. Note that $\overline{T^0 V' V''}^1 = \overline{V' V''}$.

For the rest of this section we fix indecomposable objects Y', Y'', Z', Z'', V' and V'' of $\mathbb{K}$ such that the vector spaces $|Y'|$, $|Y''|$, $|Z'|$, $|Z''|$, $|V'|$, $|V''|$, $\operatorname{Hom}_{\mathbb{K}}(Y', T)$, $\operatorname{Hom}_{\mathbb{K}}(Y'', T)$, $\operatorname{Hom}_{\mathbb{K}}(T, Z')$, $\operatorname{Hom}_{\mathbb{K}}(T, Z'')$, $\operatorname{Hom}_{\mathbb{K}}(T, V')$ and $\operatorname{Hom}_{\mathbb{K}}(T, V'')$ are one-dimensional. Let $\varphi' : Y' \to T$, $\varphi'' : Y'' \to T$, $\psi' : T \to Z'$, $\psi'' : T \to Z''$, $\varrho' : T \to V'$ and $\varrho'' : T \to V''$ be nonzero maps. We assume that $\operatorname{Im}|\varphi'| = \operatorname{Im}|\varphi''| = \operatorname{Ker}|\psi'| = \operatorname{Ker}|\psi''|$, $\operatorname{Ker}|\varrho'| = \operatorname{Ker}|\varrho''|$ and $\operatorname{Im}|\varphi'| \cap \operatorname{Ker}|\varrho'| = 0$. Let $\mathfrak{y}' \in |Y'|$, $\mathfrak{y}'' \in |Y''|$ and $\mathfrak{t}_2 \in \operatorname{Ker}|\varrho'|$ be nonzero elements, and put $\mathfrak{t}_1 = |\varphi'|(\mathfrak{y}')$, $\mathfrak{z}' = |\psi'|(\mathfrak{t}_2)$, $\mathfrak{z}'' = |\psi''|(\mathfrak{t}_2)$, $\mathfrak{v}' = |\varrho'|(\mathfrak{t}_1)$ and $\mathfrak{v}'' = |\varrho''|(\mathfrak{t}_1)$.

Let $p > q \geq 0$. By $\overline{T^q Z' Z'' T^p}^{p+q}$ we denote the triple $(T^{p+q} \oplus Z' \oplus Z'', K^{p+q}, f)$, where $f : K^{p+q} \to |T^{p+q} \oplus Z' \oplus Z''|$ is given by

$$\begin{aligned} f(\mathfrak{e}_i) &= \mathfrak{t}_1^{(i)} + \mathfrak{t}_2^{(i+1)},\ i = 1, \ldots, q-1, \\ f(\mathfrak{e}_q) &= \mathfrak{t}_1^{(q)} + \mathfrak{z}', \\ f(\mathfrak{e}_{q+1}) &= \mathfrak{z}' + \mathfrak{z}'' + \mathfrak{t}_1^{(q+1)}, \\ f(\mathfrak{e}_i) &= \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = q+2, \ldots, p+q. \end{aligned}$$

Let $p > q \geq 0$. By $\overline{Y' T^q Z' Z'' T^p}^{p+q+1}$ we denote the triple $(Y' \oplus T^{p+q} \oplus Z' \oplus Z'', K^{p+q+1}, f)$, where $f : K^{p+q+1} \to |Y' \oplus T^{p+q} \oplus Z' \oplus Z''|$ is given by

$$\begin{aligned} f(\mathfrak{e}_i) &= \mathfrak{t}_1^{(i-1)} + \mathfrak{t}_2^{(i)},\ i = 1, \ldots, q, \\ f(\mathfrak{e}_{q+1}) &= \mathfrak{t}_1^{(q)} + \mathfrak{z}', \\ f(\mathfrak{e}_{q+2}) &= \mathfrak{z}' + \mathfrak{z}'' + \mathfrak{t}_1^{(q+1)}, \\ f(\mathfrak{e}_i) &= \mathfrak{t}_2^{(i-2)} + \mathfrak{t}_1^{(i-1)},\ i = q+3, \ldots, p+q+1, \end{aligned}$$

with $\mathfrak{t}_1^{(0)} = \mathfrak{y}'$.

Let $p \geq q \geq 0$. By $\overline{T^q Z' Z'' T^p Y''}^{p+q+1}$ we denote the triple $(Y'' \oplus T^{p+q} \oplus Z' \oplus Z'', K^{p+q+1}, f)$, where $f : K^{p+q+1} \to |Y'' \oplus T^{p+q} \oplus Z' \oplus Z''|$ is given by

$$\begin{aligned} f(\mathfrak{e}_i) &= \mathfrak{t}_1^{(i)} + \mathfrak{t}_2^{(i+1)},\ i = 1, \ldots, q-1, \\ f(\mathfrak{e}_q) &= \mathfrak{t}_1^{(q)} + \mathfrak{z}', \\ f(\mathfrak{e}_{q+1}) &= \mathfrak{z}' + \mathfrak{z}'' + \mathfrak{t}_1^{(q+1)}, \\ f(\mathfrak{e}_i) &= \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = q+2, \ldots, p+q+1, \end{aligned}$$

with $\mathfrak{t}_1^{(p+q+1)} = \mathfrak{y}''$. Note that $\overline{T^0 Z' Z'' T^0 Y''}^1 = \overline{Z' Z'' Y''}$.

Let $p \geq q \geq 0$. By $\overline{Y' T^q Z' Z'' T^p Y''}^{p+q+2}$ we denote the triple $(Y' \oplus Y'' \oplus T^{p+q} \oplus Z' \oplus Z'', K^{p+q+2}, f)$, where $f : K^{p+q+2} \to |Y' \oplus Y'' \oplus T^{p+q} \oplus Z' \oplus Z''|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_1^{(i-1)} + \mathfrak{t}_2^{(i)},\ i = 1, \ldots, q,$$

$$f(\mathfrak{e}_{q+1}) = \mathfrak{t}_1^{(q)} + \mathfrak{z}',$$
$$f(\mathfrak{e}_{q+2}) = \mathfrak{z}' + \mathfrak{z}'' + \mathfrak{t}_1^{(q+1)},$$
$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-2)} + \mathfrak{t}_1^{(i-1)},\ i = q+3, \dots, p+q+2,$$

with $\mathfrak{t}_1^{(0)} = \mathfrak{y}'$ and $\mathfrak{t}_1^{(p+q+1)} = \mathfrak{y}''$.

Let $p, q \geq 0$. By $\overline{T^q Z' Z'' T^p}^{p+q+1}$ we denote the triple $(T^{p+q} \oplus Z' \oplus Z'', K^{p+q+1}, f)$, where $f : K^{p+q+1} \to |T^{p+q} \oplus Z' \oplus Z''|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_1^{(i)} + \mathfrak{t}_2^{(i+1)},\ i = 1, \dots, q-1,$$
$$f(\mathfrak{e}_q) = \mathfrak{t}_1^{(q)} + \mathfrak{z}',$$
$$f(\mathfrak{e}_{q+1}) = \mathfrak{z}' + \mathfrak{z}'' + \mathfrak{t}_1^{(q+1)},$$
$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = q+2, \dots, p+q+1,$$

with $\mathfrak{t}_1^{(p+q+1)} = 0$. Note that $\overline{T^0 Z' Z'' T^0}^1 = \overline{Z' Z''}$.

Let $p, q \geq 0$. The triple $(Y' \oplus T^{p+q} \oplus Z' \oplus Z'', K^{p+q+2}, f)$, where $f : K^{p+q+2} \to |Y' \oplus T^{p+q} \oplus Z' \oplus Z''|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_1^{(i-1)} + \mathfrak{t}_2^{(i)},\ i = 1, \dots, q,$$
$$f(\mathfrak{e}_{q+1}) = \mathfrak{t}_1^{(q)} + \mathfrak{z}',$$
$$f(\mathfrak{e}_{q+2}) = \mathfrak{z}' + \mathfrak{z}'' + \mathfrak{t}_1^{(q+1)},$$
$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-2)} + \mathfrak{t}_1^{(i-1)},\ i = q+3, \dots, p+q+2,$$

with $\mathfrak{t}_1^{(0)} = \mathfrak{y}'$ and $\mathfrak{t}_1^{(p+q+1)} = 0$, will be denoted by $\overline{Y' T^q Z' Z'' T^p}^{p+q+2}$.

Let $p, q \geq 0$. The triple $(T^{p+q} \oplus Z' \oplus Z'' \oplus V'', K^{p+q+1}, f)$, where $f : K^{p+q+1} \to |T^{p+q} \oplus Z' \oplus Z'' \oplus V''|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_1^{(i)} + \mathfrak{t}_2^{(i+1)},\ i = 1, \dots, q-1,$$
$$f(\mathfrak{e}_q) = \mathfrak{t}_1^{(q)} + \mathfrak{z}',$$
$$f(\mathfrak{e}_{q+1}) = \mathfrak{z}' + \mathfrak{z}'' + \mathfrak{t}_1^{(q+1)},$$
$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = q+2, \dots, p+q+1,$$

with $\mathfrak{t}_1^{(p+q+1)} = \mathfrak{v}''$, will be denoted by $\overline{T^q Z' Z'' T^p V''}^{p+q+1}$. Note that $\overline{T^0 Z' Z'' T^0 V''}^1 = \overline{Z' Z'' V''}$,

Let $p, q \geq 0$. By $\overline{Y' T^q Z' Z'' T^p V''}^{p+q+2}$ we denote the triple $(Y' \oplus T^{p+q} \oplus Z' \oplus Z'' \oplus V'', K^{p+q+2}, f)$, where $f : K^{p+q+2} \to |Y' \oplus T^{p+q} \oplus Z' \oplus Z'' \oplus V''|$ is given by

$$\begin{aligned} f(\mathfrak{e}_i) &= \mathfrak{t}_1^{(i-1)} + \mathfrak{t}_2^{(i)},\ i = 1, \dots, q, \\ f(\mathfrak{e}_{q+1}) &= \mathfrak{t}_1^{(q)} + \mathfrak{z}', \\ f(\mathfrak{e}_{q+2}) &= \mathfrak{z}' + \mathfrak{z}'' + \mathfrak{t}_1^{(q+1)}, \\ f(\mathfrak{e}_i) &= \mathfrak{t}_2^{(i-2)} + \mathfrak{t}_1^{(i-1)},\ i = q+3, \dots, p+q+2, \end{aligned} \tag{$*$}$$

with $\mathfrak{t}_1^{(0)} = \mathfrak{y}'$ and $\mathfrak{t}_1^{(p+q+1)} = \mathfrak{v}''$.

Let $q > p \geq 0$. By $\overline{T^q Z' Z'' T^p}^{p+q+2}$ we denote the triple $(T^{p+q} \oplus Z' \oplus Z'', K^{p+q+2}, f)$, where $f : K^{p+q+2} \to |T^{p+q} \oplus Z' \oplus Z''|$ is given by the formulas $(*)$ with $\mathfrak{t}_1^{(0)} = 0 = \mathfrak{t}_1^{(p+q+1)}$.

Let $q \geq p \geq 0$. By $\overline{V' T^q Z' Z'' T^p}^{p+q+2}$ we denote the triple $(T^{p+q} \oplus Z' \oplus Z'' \oplus V', K^{p+q+2}, f)$, where $f : K^{p+q+2} \to |T^{p+q} \oplus Z' \oplus Z'' \oplus V'|$ is given by the formulas $(*)$ with $\mathfrak{t}_1^{(0)} = \mathfrak{v}'$ and $\mathfrak{t}_1^{(p+q+1)} = 0$.

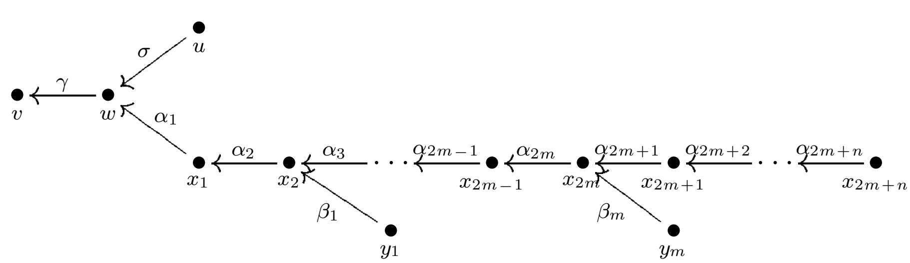
Let $q > p \geq 0$. By $\overline{T^q Z' Z'' T^p V''}^{p+q+2}$ we denote the triple $(T^{p+q} \oplus Z' \oplus Z'' \oplus V'', K^{p+q+2}, f)$, where $f : K^{p+q+2} \to |T^{p+q} \oplus Z' \oplus Z'' \oplus V''|$ is given by the formulas $(*)$ with $\mathfrak{t}_1^{(0)} = 0$ and $\mathfrak{t}_1^{(p+q+1)} = \mathfrak{v}''$.

Let $q > p \geq 0$. By $\overline{V' T^q Z' Z'' T^p V''}^{p+q+2}$ we denote the triple $(T^{p+q} \oplus Z' \oplus Z'' \oplus V' \oplus V'', K^{p+q+2}, f)$, where $f : K^{p+q+2} \to |T^{p+q} \oplus Z' \oplus Z'' \oplus V' \oplus V''|$ is given by the formulas $(*)$ with $\mathfrak{t}_1^{(0)} = \mathfrak{v}'$ and $\mathfrak{t}_1^{(p+q+1)} = \mathfrak{v}''$.

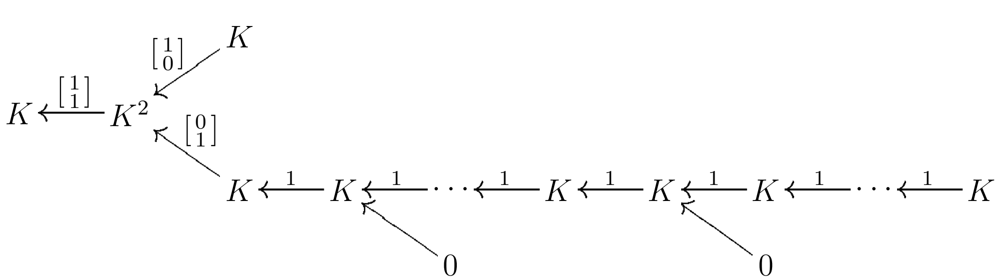
3 Proof of Main Theorem

We start with the following crucial observation.

Lemma 3.1. The vector space category $(\mathbb{K}_{m,n}, |-|)$ is isomorphic to the vector space category $\mathrm{Hom}(M, \mathrm{mod}\, A)$, where A is the path algebra of the quiver



bounded by the relations $\alpha_{2i-1}\alpha_{2i}\beta_i = 0$, $i = 1, \ldots, m$, and M is the A-module corresponding to the indecomposable representation



Proof. Let A_0 be the full subcategory of A given by the objects of u, v, w, x_s, $s = 1, \ldots, 2m+n$. Note that M is an A_0-module. Since A_0 is a hereditary algebra of type $\mathbb{D}_{2m+n+3}$, one may easily calculate the vector space category $\mathrm{Hom}(M, \mathrm{mod}\, A_0)$. As a result we obtain that $\mathrm{Hom}(M, \mathrm{mod}\, A_0)$ may be identified with the full vector space subcategory of $\mathbb{K}_{m,n}$ formed by the objects X_s, $s = 1, \ldots, 2m+n$, Y_s, $s = 1, \ldots, 2m+n$, U', U'', T, V, $Z_{i,1}$, $i = 0, \ldots, m-1$, Z'_i, $i = 1, \ldots, m$, $Z_{m,s}$, $s = 1, \ldots, n$. Moreover, $A = A_0[M_1] \cdots [M_m]$, where M_i, $i = 1, \dots, m$, is given by the representation $V_i = (V_{i,x}, V_{i,\alpha})$ with

$$V_{i,x} = \begin{cases} K & \text{if } x = x_{2i-1}, x_{2i}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$V_{i,\alpha} = \begin{cases} 1 & \text{if } \alpha = \alpha_{2i}, \\ 0 & \text{otherwise.} \end{cases}$$

We have $M_i = \tau_{A_0}^{-2i} M$, and $\tau_{A_0}^{-2i}$ induces an equivalence between the full vector space subcategory of $\operatorname{Hom}(M, \operatorname{mod} A_0)$ formed by the objects X_s, $s = 1, \dots, \beta(i-1)$, Y_s, $s = 1, \dots, \beta(i-1)$ and $\operatorname{Hom}(M_i, \operatorname{mod} A_0)$. Note that $\tau_{A_0}^{-2i} X_{\beta(i-1)} = Z_{i-1,1}$ and $\tau_{A_0}^{-2i} Y_{\beta(i-1)} = Z'_i$.

The vector space categories of the above type are well-understood. In particular, we know the description of the Auslander–Reiten quiver of the category $\mathcal{U}(\operatorname{Hom}(M_i, \operatorname{mod} A_0))$ (see for example [2]). If we identify the objects of $\mathcal{U}(\operatorname{Hom}(M_i, \operatorname{mod} A_0))$ with the corresponding A-modules and use the above mentioned description of $\Gamma(\mathcal{U}(\operatorname{Hom}(M_i, \operatorname{mod} A_0)))$ we get that $\operatorname{Hom}(M, \mathcal{U}(\operatorname{Hom}(M_i, \operatorname{mod} A_0)))$ is equivalent to the full subcategory of $\mathbb{K}_{m,n}$ formed by the objects $Z_{i-1,s}$, $s = 1, \dots, \beta(i-1)$, Z'_i, Z''_i, $Z_{i,0}$, where an equivalence is given by the following assignments $Z_{i-1,s} \mapsto \overline{\tau_{A_0}^{-2i} X_{\beta(i-1)} \tau_{A_0}^{-2i} Y_{s-1}}$, $s = 2, \dots, \beta(i-1)$, $Z''_i \mapsto \overline{\tau_{A_0}^{-2i} X_{\beta(i-1)}}$ and $Z_{i,0} \mapsto \overline{\tau_{A_0}^{-2i} Y_{\beta(i-1)}}$. Finally, if we use the method of constructing $\Gamma(\operatorname{mod} A)$ from $\Gamma(\operatorname{mod} A_0)$ and $\Gamma(\mathcal{U}(M_i, \operatorname{mod} A_0))$, $i = 1, \dots, s$, we observe that the above equivalences extend to an equivalence of $\mathbb{K}_{m,n}$ and $\operatorname{Hom}(M, \operatorname{mod} A)$.

Throughout this section, we will use the notation introduced in the above lemma and its proof. From now on we will identify $\mathbb{K}_{m,n}$ with the category $\operatorname{Hom}(M, \operatorname{mod} A)$. It follows that, in order to describe $\mathcal{U}(\mathbb{K}_{m,n})$, it is enough to describe the category $\operatorname{mod} A[M]$.

Let A'_0 be the full subcategory of A_0 formed by all objects except v. We define $A' = A'_0[M']$, where M' is the restriction of M to A'. Then $A[M] = [N]A'[M_1] \cdots [M_m]$, where N is the A'-module corresponding to the representation $W = (W_{i,x}, W_{i,\alpha})$ with $W_{i,x} = K$ for each x and $W_{i,\alpha} = 1$ for each α. Since A' is a hereditary algebra of type $\tilde{\mathbb{A}}_{2m+n+2}$, the structure of $\Gamma(\operatorname{mod} A')$ is known. It consists of a preprojective component $\mathcal{P}$, a family of stable tubes $\mathcal{T}_\lambda$, $\lambda \in \mathbb{P}_1(K)$, and a preinjective component $\mathcal{I}$. We may assume that the tube $\mathcal{T}_0$ has rank $2m + n + 1$ and contains the modules $M_1, \dots, M_m$, the tube $\mathcal{T}_\infty$ has rank 2, the tube $\mathcal{T}_1$ is homogeneous and contains the module N, and all the remaining tubes $\mathcal{T}_\lambda$ are homogeneous. It is easy to see that the indecomposable objects of the vector space categories $\operatorname{Hom}(M_i, \operatorname{mod} A')$, $i = 1, \dots, m$, belong to $\mathcal{T}_0$ and $\mathcal{I}$, and the above categories have no nonzero common objects. Similarly, the indecomposable objects of the category $D \operatorname{Hom}(\operatorname{mod} A', N)$ belong to $\mathcal{P}$ and $\mathcal{T}_1$. Since the vector space categories $\operatorname{Hom}(M_i, \operatorname{mod} A')$, $i = 1, \dots, m$, and $D \operatorname{Hom}(\operatorname{mod} A', N)$ have no nonzero common objects, we obtain the following description of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A[M])$. It consists of the stable tubes $\mathcal{T}_\lambda$, $\lambda \neq 0, 1$, the components containing indecomposable modules from $\mathcal{P}$, $\mathcal{T}_1$ and $\mathcal{V}(D \operatorname{Hom}(\operatorname{mod} A', N))$, and the components containing indecomposable modules from $\mathcal{T}_0$, $\mathcal{I}$ and $\mathcal{U}(\operatorname{Hom}(M_i, \operatorname{mod} A'))$, $i = 1, \dots, m$. Moreover, in order to describe the components containing modules from $\mathcal{P}$, $\mathcal{T}_1$ and $\mathcal{V}(D\operatorname{Hom}(\operatorname{mod} A', N))$, it is enough to consider the coextension $[N]A'$, while in order to describe the components containing modules from $\mathcal{T}_0$, $\mathcal{I}$ and $\mathcal{U}(\operatorname{Hom}(M_i, \operatorname{mod} A'))$, $i = 1, \dots, m$, we may consider the iterated extension $A'[M_1] \cdots [M_m]$. Finally, the tubes $\mathcal{T}_\lambda$, $\lambda \neq 0, 1$, consist of A'-modules.

We first describe the tubes $\mathcal{T}_\lambda$, $\lambda \neq 0, 1$. This description is made in terms of the subspace category $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} A))$. We need to introduce some additional notation. Let $\mathfrak{t}_1$, $\mathfrak{t}_2$ be a basis of $|T|$ such that $\mathfrak{t}_1$ belongs to the image of a nonzero map $|Y_{2m+n}| \to |T|$ and $\mathfrak{t}_2$ belongs to the image of a nonzero map $|U'| \to |T|$. For $\lambda \in K \cup \{\infty\}$ and $p \geq 1$, we denote by $\overline{T^p}^p(\lambda)$ the triple (T^p, K^p, f), where $f : K^p \to |T^p|$ is given by

$$f(\mathfrak{e}_i) = \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)} + \lambda \mathfrak{t}_2^{(i)},\ i = 1, \dots, p,$$

if $\lambda \neq \infty$, and

$$f(\mathfrak{e}_i) = \mathfrak{t}_1^{(i-1)} + \mathfrak{t}_2^{(i)},\ i = 1, \dots, p,$$

if $\lambda = \infty$, with $\mathfrak{t}_1^{(0)} = \mathfrak{t}_2^{(0)} = 0$.

Lemma 3.2. If $\lambda \neq 0, 1, \infty$, then the tube $\mathcal{T}_\lambda$ consists of the modules

$$T_\lambda(p) = \overline{T^p}^p(\lambda),\ p \geq 1,$$

and the Auslander–Reiten sequences in $\mathcal{T}_\lambda$ are of the form

$$0 \to T_\lambda(p) \to T_\lambda(p-1) \oplus T_\lambda(p+1) \to T_\lambda(p) \to 0,\ p \geq 1,$$

where $T_\lambda(0) = 0$.

The tube $\mathcal{T}_\infty$ consists of the modules

$$\begin{aligned} T_\infty(2p-1, 0) &= \overline{U'T^{p-1}}^p,\ p \geq 1, \\ T_\infty(2p-1, 1) &= \overline{T^{p-1}V}^{p-1},\ p \geq 1, \\ T_\infty(2p, 0) &= \overline{T^p}^p(\infty),\ p \geq 1, \\ T_\infty(2p, 1) &= \overline{U'T^{p-1}V}^p,\ p \geq 1, \end{aligned}$$

and the Auslander–Reiten sequences in $\mathcal{T}_\infty$ are of the form

$$\begin{aligned} 0 \to T_\infty(p, i) \to T_\infty(p-1, i+1) \oplus T_\infty(p+1, i) \to T_\infty(p, i+1) \to 0, \\ i = 0, 1,\ p \geq 1, \end{aligned}$$

where $T_\infty(0, i) = 0$, $i = 0, 1$, and the addition on the second coordinate is performed modulo 2.

Proof. We have already observed that the tubes $\mathcal{T}_\lambda$, $\lambda \neq 0, 1$, consist of A'-modules. Recall that $A' = A'_0[M']$. It follows from the definition that $M' = Y_1 \oplus U'$. Thus the category $\operatorname{Hom}(M', \operatorname{mod} A'_0)$ is a full vector space subcategory of the category $\operatorname{Hom}(M, \operatorname{mod} A_0)$

consisting of the objects Y_s, $s = 1, \ldots, 2m+n$, U', T, V, $Z_{i,1}$, $i = 0, \ldots, m-1$, Z'_i, $i = 1, \ldots, m$, $Z_{m,s}$, $s = 1, \ldots, n$. The claim is the consequence of the well-known procedure of constructing tubes for hereditary algebras of type $\tilde{\mathbb{A}}_{2m+n+2}$.

The second step is the following lemma.

Lemma 3.3. The modules from $\mathcal{P}$, $\mathcal{T}_1$ and $\mathcal{V}(D\operatorname{Hom}(\operatorname{mod} A', N))$ form a preprojective component $\mathcal{P}'$ and a nonstable tube $\mathcal{T}'$. The indecomposable objects of $\operatorname{Hom}(M, \operatorname{mod} A)$ belonging to $\mathcal{P}'$ are

$$
\begin{aligned}
& C'((2m+n+1)p+i) = \overline{Y_i T^p}^p,\ i = 1, \ldots, 2m+n,\ p \geq 0, \\
& C'((2m+n+1)p) = \overline{T^p}^{p-1},\ p \geq 1, \\
& C''(i) = \overline{X_i},\ i = 1, \ldots, 2m+n, \\
& C''((2m+n+1)p+i) = \overline{Y_i U' U'' T^{p-1}}^{p+1},\ i = 1, \ldots, 2m+n,\ p \geq 1, \\
& C''((2m+n+1)p) = \overline{U' U'' T^{p-1}}^p,\ p \geq 1, \\
& D(0,i) = X_i,\ i = 1, \ldots, 2m+n, \\
& D(j, i-j) = \overline{X_i Y_j},\ 1 \leq j < i \leq 2m+n, \\
& D((2m+n+1)p+i, 2m+n+1-i) = \overline{Y_i U' U'' T^{2p}}^{2p+1}, \\
& \qquad i = 1, \ldots, 2m+n,\ p \geq 0, \\
& D((2m+n+1)p+j, 2m+n+1+i-j) = \overline{Y_i Y_j U' U'' T^{2p}}^{2p+2}, \\
& \qquad 1 \leq i < j \leq 2m+n,\ p \geq 0, \\
& D((2m+n+1)p, i) = \overline{Y_i U' U'' T^{2p-1}}^{2p},\ i = 1, \ldots, 2m+n,\ p \geq 1, \\
& D((2m+n+1)p+i, j-i) = \overline{Y_j Y_i U' U'' T^{2p+1}}^{2p+3}, \\
& \qquad 1 \leq i < j \leq 2m+n,\ p \geq 0, \\
& E'((2m+n+1)p) = \overline{U' T^p}^p,\ p \geq 0, \\
& E'((2m+n+1)p+i) = \overline{Y_i U' T^p}^{p+1},\ i = 1, \ldots, 2m+n,\ p \geq 0, \\
& E''((2m+n+1)p) = \overline{U'' T^p}^p,\ p \geq 0, \\
& E''((2m+n+1)p+i) = \overline{Y_i U'' T^p}^{p+1},\ i = 1, \ldots, 2m+n,\ p \geq 0,
\end{aligned}
$$

and the Auslander–Reiten sequences in $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} A))$ starting at these objects are of the form

$$
\begin{aligned}
& 0 \to C'(p) \to D(p,1) \to C''(p+1) \to 0,\ p \geq 1, \\
& 0 \to C''(p) \to D(p,1) \to C'(p+1) \to 0,\ p \geq 1, \\
& 0 \to D(p,i) \to D(p+1, i-1) \oplus D(p, i+1) \to D(p+1, i) \to 0, \\
& \qquad i = 1, \ldots, 2m+n,\ p \geq 0, \\
& 0 \to E'(p) \to D(p+1, 2m+n) \to E''(p+1) \to 0,\ p \geq 0, \\
& 0 \to E''(p) \to D(p+1, 2m+n) \to E'(p+1) \to 0,\ p \geq 0,
\end{aligned}
$$

where $D(p,0) = C'(p) \oplus C''(p)$, $p \geq 1$, and $D(p, 2m+n+1) = E'(p) \oplus E''(p)$, $p \geq 0$. The tube $\mathcal{T}'$ consists of the modules

$$T_1(2p-1) = \overline{U''T^{p-1}}^p,\ p \geq 1,$$
$$T_1(2p) = \overline{T^p}^p(1),\ p \geq 1,$$

and the Auslander–Reiten sequences in $\mathcal{T}'$ are of the form

$$0 \to T_1(p) \to T_1(p-2) \oplus T_1(p+1) \to T_1(p-1) \to 0,\ p \geq 2,$$

where $T_1(0) = 0$.

Proof. We know that the components we are interested in consist of $[N]A'$-modules. Since $[N]A'$ is a tilted algebra of type $\tilde{\mathbb{D}}_{2m+n+2}$, these components are the preprojective component $\mathcal{P}'$ and the nonstable tube $\mathcal{T}'$. We also have that $[N]A' = A_0[M]$ and $\operatorname{Hom}(M, \operatorname{mod} A_0)$ is a full subcategory of the category $\operatorname{Hom}(M, \operatorname{mod} A)$ which has been calculated in the proof of Lemma 3.1. The description of indecomposable objects of $\operatorname{Hom}(M, \operatorname{mod} A)$ belonging to $\mathcal{T}'$ is the consequence of the method of constructing tubes. In order to check that $\mathcal{P}'$ is exactly of the form presented in the lemma, it is enough to verify that the dimension vectors of modules in $\mathcal{P}'$ are exactly the dimension vectors of modules from the lemma, and that the modules from the lemma are indecomposable. The former is done by inductive calculations of the preprojective component, and the latter can be done by standard calculations. Since they are sometimes long we omit them, however to help the reader to follow the proof, we present one of the more complicated cases in Section 4.

The final step of our proof is the following lemma.

Lemma 3.4. The modules from $\mathcal{U}(\operatorname{Hom}(M_i, \operatorname{mod} A'))$, $i = 1, \dots, m$, $\mathcal{T}_0$ and $\mathcal{I}$ form m components of 1st type. The indecomposable objects of $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} A))$ belonging to these components are

$$M(i, (2m+n+1)p, (2m+n+1)q) = \overline{T^q Z_i' Z_i'' T^p}^{p+q},$$
$$i = 1, \dots, m,\ p > q \geq 0,$$
$$M(i, (2m+n+1)p, (2m+n+1)q + s) = \overline{Y_s T^q Z_i' Z_i'' T^p}^{p+q+1},$$
$$i = 1, \dots, m,\ s = 1, \dots, 2m+n,\ p > q \geq 0,$$
$$M(i, (2m+n+1)p + r, (2m+n+1)q) = \overline{T^q Z_i' Z_i'' T^p Y_r}^{p+q+1},$$
$$i = 1, \dots, m,\ r = 1, \dots, 2m+n,\ p \geq q \geq 0,$$
$$M(i, (2m+n+1)p + r, (2m+n+1)q + s) = \overline{Y_s T^q Z_i' Z_i'' T^p Y_r}^{p+q+2},$$
$$i = 1, \dots, m,\ r, s = 1, \dots, 2m+n,\ p, q \geq 0,$$
$$(2m+n+1)p + r > (2m+n+1)q + s,$$
$$M'(i, (2m+n+1)p) = \overline{T^p Z_i'}^p,\ i = 1, \dots, m,\ p \geq 0,$$
$$M'(i, (2m+n+1)p + r) = \overline{Y_r T^p Z_i'}^{p+1},$$

$$
\begin{aligned}
&\quad i=1,\dots,m,\ r=1,\dots,2m+n,\ p\geq 0,\\
&M''(i,(2m+n+1)p)=\overline{T^pZ_i''}^{p},\ i=1,\dots,m,\ p\geq 0,\\
&M''(i,(2m+n+1)p+r)=\overline{Y_rT^pZ_i''}^{p+1},\\
&\quad i=1,\dots,m,\ r=1,\dots,2m+n,\ p\geq 0,\\
&N(i,(2m+n+1)p,s)=\overline{T^pZ_{i,s}}^{p},\ i=0,\dots,m,\ s=\alpha(i),\dots,\beta(i),\ p\geq 0,\\
&N(i,(2m+n+1)p+r,s)=\overline{Y_rT^pZ_{i,s}}^{p+1},\\
&\quad i=0,\dots,m,\ s=\alpha(i),\dots,\beta(i),\ r=1,\dots,2m+n,\ p\geq 0,\\
&N(m,(2m+n+1)p,\beta(m)+1)=\overline{T^p}^{p}(0),\ p\geq 1,\\
&N(m,(2m+n+1)p+r,\beta(m)+1)=\overline{Y_rT^p}^{p+1},\ r=1,\dots,2m+n,p\geq 0,\\
&L(i,(2m+n+1)p,2q)=\overline{T^pZ_i'Z_i''T^q}^{p+q+1},\ i=1,\dots,m,\ p,q\geq 0,\\
&L(i,(2m+n+1)p+r,2q)=\overline{Y_rT^pZ_i'Z_i''T^q}^{p+q+1},\\
&\quad i=1,\dots,m,\ r=1,\dots,2m+n,\ p,q\geq 0,\\
&L(i,(2m+n+1)p,2q+1)=\overline{T^pZ_i'Z_i''T^qV}^{p+q+1},\ i=1,\dots,m,\ p,q\geq 0,\\
&L(i,(2m+n+1)p+r,2q+1)=\overline{Y_rT^pZ_i'Z_i''T^qV}^{p+q+2},\\
&\quad i=1,\dots,m,\ r=1,\dots,2m+n,\ p,q\geq 0,\\
&P(i,2p,s)=\overline{T^pZ_{i,s}}^{p+1},\ i=0,\dots,m,\ s=\alpha(i),\dots,\beta(i),p\geq 0,\\
&P(i,2p+1,s)=\overline{T^pZ_{i,s}V}^{p+1},\ i=0,\dots,m,\ s=\alpha(i),\dots,\beta(i),\ p\geq 0,\\
&P(m,2p,\beta(m)+1)=\overline{T^p}^{p+1},\ p\geq 0,\\
&P(m,2p+1,\beta(m)+1)=\overline{T^pV}^{p+1},\ p\geq 0,\\
&Q(i,2p,2q)=\overline{T^qZ_i'Z_i''T^p}^{p+q+2},\ i=1,\dots,m,\ q>p\geq 0,\\
&Q(i,2p,2q+1)=\overline{VT^qZ_i'Z_i''T^p}^{p+q+2},\ i=1,\dots,m,\ q\geq p\geq 0,\\
&Q(i,2p+1,2q)=\overline{T^qZ_i'Z_i''T^pV}^{p+q+2},\ i=1,\dots,m,\ q>p\geq 0,\\
&Q(i,2p+1,2q+1)=\overline{VT^qZ_i'Z_i''T^pV}^{p+q+2},\ i=1,\dots,m,\ q>p\geq 0,\\
&Q'(i,2p)=\overline{T^pZ_i'}^{p+1},\ i=1,\dots,m,\ p\geq 0,\\
&Q'(i,2p+1)=\overline{T^pZ_i'V}^{p+1},\ i=1,\dots,m,\ p\geq 0,\\
&Q''(i,2p)=\overline{T^pZ_i''}^{p+1},\ i=1,\dots,m,\ p\geq 0,\\
&Q''(i,2p+1)=\overline{T^pZ_i''V}^{p+1},\ i=1,\dots,m,\ p\geq 0,
\end{aligned}
$$

and the Auslander–Reiten sequences in $\mathcal{U}(\mathrm{Hom}(M,\mathrm{mod}\,A))$ starting at these objects are of the form

$$
\begin{aligned}
&0\to M(i,p,q)\to M(i,p+1,q)\oplus M(i,p,q+1)\to M(i,p+1,q+1)\to 0,\\
&\quad i=1,\dots,m,\ p>q\geq 0,\\
&0\to M'(i,p)\to M(i,p+1,p)\to M''(i,p+1)\to 0,\ i=1,\dots,m,\ p\geq 0,\\
&0\to M''(i,p)\to M(i,p+1,p)\to M'(i,p+1)\to 0,\ i=1,\dots,m,\ p\geq 0,\\
&0\to N(i,p,s)\to N(i,p+1,s)\oplus N(i,p,s+1)\to N(i,p+1,s+1)\to 0,
\end{aligned}
$$

$$
\begin{aligned}
&\quad i=0,\dots,m,\ s=\alpha(i)-1,\dots,\beta(i),\ p\geq 0,\\
&0\to L(i,p,q)\to L(i,p+1,q)\oplus L(i,p,q-1)\to L(i,p+1,q-1)\to 0,\\
&\quad i=1,\dots,m,\ p\geq 0,\ q\geq 1,\\
&0\to P(i,p,s)\to P(i,p-1,s)\oplus P(i,p,s+1)\to P(i,p-1,s+1)\to 0,\\
&\quad i=0,\dots,m,\ s=\alpha(i)-1,\dots,\beta(i),\ p\geq 1,\\
&0\to Q(i,p,q)\to Q(i,p-1,q)\oplus Q(i,p,q-1)\to Q(i,p-1,q-1)\to 0,\\
&\quad i=1,\dots,m,\ q>p\geq 1,\\
&0\to Q'(i,p)\to Q(i,p,p-1)\to Q''(i,p-1)\to 0,\ i=1,\dots,m,\ p\geq 1,\\
&0\to Q''(i,p)\to Q(i,p,p-1)\to Q'(i,p-1)\to 0,\ i=1,\dots,m,\ p\geq 1,
\end{aligned}
$$

where we put

$$
\begin{aligned}
&M(i,p,p)=M'(i,p)\oplus M''(i,p),\ i=1,\dots,m,\ p\geq 0,\\
&N(i,p,\beta(i)+1)=M(i+1,p,0),\ i=0,\dots,m-1,\ p\geq 0,\\
&N(m,0,\beta(m)+1)=0,\\
&N(0,p,\alpha(0)-1)=N(m,p+2m+n+1,\beta(m)+1),\ p\geq 0,\\
&N(i,p,\alpha(i)-1)=L(i,p,0),\ i=1,\dots,m,\ p\geq 0,\\
&P(i,p,\beta(i)+1)=L(i+1,0,p),\ i=0,\dots,m-1,\ p\geq 0,\\
&P(0,p,\alpha(i)-1)=P(m,p+2,\beta(m)+1),\ p\geq 0,\\
&P(i,p,\alpha(i)-1)=Q(i,0,p),\ i=1,\dots,m,\ p\geq 0,\\
&Q(i,p,p)=Q'(i,p)\oplus Q''(i,p),\ i=1,\dots,m,\ p\geq 0.
\end{aligned}
$$

Proof. Since A' is a hereditary algebra of type $\tilde{\mathbb{A}}_{2m+n+2}$ and $M_1, \dots, M_m$ are pairwise orthogonal regular A'-modules of regular length 2, we may use the results of [2] in order to prove the lemma. In particular, [2, Theorem 2] immediately implies that the components containing modules from $\mathcal{U}(\operatorname{Hom}(M_i, \operatorname{mod} A'))$, $i = 1,\dots,m$, $\mathcal{T}_0$ and $\mathcal{I}$, are m components $\mathcal{C}_1, \dots, \mathcal{C}_m$ of 1st type. In order to describe the indecomposable objects of $\mathcal{U}(\operatorname{Hom}(M, \operatorname{mod} A))$ belonging to these components we need a more precise analysis. Denote by $R_{i,p}$, $i = 0,\dots,2m+n$, $p \geq 1$, the vertices of $\mathcal{T}_0$ in such a way that we have arrows $R_{i,p} \to R_{i,p+1}$ and $R_{i,p} \to R_{i+1,p-1}$ for any i and p (the latter arrow exists provided $p \geq 2$), where the addition on the first coordinate is always performed modulo $2m+n+1$. We may also assume that $M_i = R_{2i-1,2}$, $i = 1,\dots,m$. Similarly, we denote by $S_{i,p}$, $i = 0,\dots,2m+n$, $p \geq 1$, the vertices of $\mathcal{I}$ in such a way that we have arrows $S_{i,p} \to S_{i,p-1}$, $i = 0,\dots,2m+n$, $p \geq 2$, $S_{0,p} \to S_{1,p-2}$, $p \geq 3$, and $S_{i,p} \to S_{i+1,p}$, $i = 1,\dots,2m+n$, $p \geq 1$, and $S_{i,1}$ is the injective envelope of $R_{i,1}$, $i = 1,\dots,2m+n$.

Recall from [2, Section 4] that for each $i = 1,\dots,m$, the representatives of isomorphism classes of indecomposable objects of the vector space category $\operatorname{Hom}(M_i, \operatorname{mod} A')$ are $R_{2i-1,p}$, $p \geq 2$, $R_{2i,p}$, $p \geq 1$, $S_{j,p}$, $j = 2i-1, 2i$, $p \geq 1$. We also know that the vertices of $\mathcal{C}_1, \dots, \mathcal{C}_m$ are $R_{i,p}$, $i = 0,\dots,2m+n$, $p \geq 1$, $S_{i,p}$, $i = 0,\dots,2m+n$, $p \geq 1$, $\overline{R_{2i-1,p}}$, $i = 1,\dots,m$, $p \geq 2$, $\overline{R_{2i,p}}$, $i = 1,\dots,m$, $p \geq 1$, $\overline{S_{i,p}}$, $i = 1,\dots,2m$, $p \geq 1$, $\overline{R_{2i-1,p}R_{2i,q}}$, $i = 1,\dots,m$, $p-1 > q \geq 1$, $\overline{R_{2i,p}S_{2i-1,q}}$, $i = 1,\dots,m$, $p, q \geq 1$,

$\overline{S_{2i-1,p}S_{2i,q}}$, $i=1,\ldots,m$, $q>p\geq 1$, E_i, $i=1,\ldots,m$, where E_i is the unique simple injective object in $\mathcal{U}(\operatorname{Hom}(M_i,\operatorname{mod}A'))$. In the above description we express the indecomposable $A'[M_1]\cdots[M_k]$-modules in terms of the appropriate subspace category $\mathcal{U}(\operatorname{Hom}(M_i,\operatorname{mod}A'))$, $i=1,\ldots,m$. Finally, we have the following Auslander–Reiten sequences in $\mathcal{C}_1,\ldots,\mathcal{C}_m$:

$$0\to R_{i,1}\to R_{i,2}\to R_{i+1,1}\to 0,$$
$$i=0,1,3,\ldots,2m-1,2m+1,2m+2,\ldots,2m+n,$$
$$0\to R_{i,p}\to R_{i,p+1}\oplus R_{i+1,p-1}\to R_{i+1,p}\to 0,$$
$$i=0,2m+1,2m+2,\ldots,2m+n,\ p\geq 2,$$
$$0\to R_{2i-1,2}\to R_{2i-1,3}\oplus R_{2i,1}\oplus \overline{R_{2i-1,2}}\to \overline{R_{2i-1,3}R_{2i,1}}\to 0,\ i=1,\ldots,m,$$
$$0\to R_{2i-1,p}\to R_{2i-1,p+1}\oplus \overline{R_{2i-1,p}R_{2i,1}}\to \overline{R_{2i-1,p+1}R_{2i,1}}\to 0,$$
$$i=1,\ldots,m,\ p\geq 3,$$
$$0\to R_{2i,p}\to \overline{R_{2i-1,p+2}R_{2i,p}}\to \overline{R_{2i-1,p+2}}\to 0,\ i=1,\ldots,m,\ p\geq 1,$$
$$0\to S_{0,p}\to S_{0,p-1}\oplus S_{1,p-2}\to S_{1,p-3}\to 0,\ p\geq 4,$$
$$0\to S_{2i-1,1}\to \overline{R_{2i,1}S_{2i-1,1}}\to \overline{R_{2i,1}}\to 0,\ i=1,\ldots,m,$$
$$0\to S_{2i-1,p}\to S_{2i-1,p-1}\oplus \overline{R_{2i,1}S_{2i-1,p}}\to \overline{R_{2i,1}S_{2i-1,p-1}}\to 0,$$
$$i=1,\ldots,m,\ p\geq 2,$$
$$0\to S_{2i,1}\to \overline{S_{2i,1}}\to E_i\to 0,\ i=1,\ldots,m,$$
$$0\to S_{2i,p}\to \overline{S_{2i-1,p-1}S_{2i,p}}\to \overline{S_{2i-1,p-1}}\to 0,\ i=1,\ldots,m,\ p\geq 2,$$
$$0\to S_{i,p}\to S_{i,p-1}\oplus S_{i+1,p}\to S_{i+1,p-1}\to 0,\ i=2m+1,\ldots,2m+n,\ p\geq 2,$$
$$0\to \overline{R_{2i-1,p}}\to \overline{R_{2i-1,p+1}R_{2i,p-1}}\to R_{2i,p}\to 0,\ i=1,\ldots,m,\ p\geq 2,$$
$$0\to \overline{R_{2i,1}}\to \overline{R_{2i,2}}\to R_{2i+1,1}\to 0,\ i=1,\ldots,m,$$
$$0\to \overline{R_{2i,p}}\to \overline{R_{2i,p+1}}\oplus R_{2i+1,p-1}\to R_{2i+1,p}\to 0,\ i=1,\ldots,m,\ p\geq 2,$$
$$0\to \overline{S_{2i-1,p}}\to \overline{S_{2i-1,p-1}S_{2i,p}}\to S_{2i,p-1}\to 0,\ i=1,\ldots,m,\ p\geq 2,$$
$$0\to \overline{S_{2i,p}}\to \overline{S_{2i,p-1}}\oplus S_{2i+1,p}\to S_{2i+1,p-1}\to 0,\ i=1,\ldots,m,\ p\geq 2,$$
$$0\to \overline{R_{2i-1,p}R_{2i,q}}\to \overline{R_{2i-1,p+1}R_{2i,q}}\oplus \overline{R_{2i-1,p}R_{2i,q+1}}$$
$$\to \overline{R_{2i-1,p+1}R_{2i,q+1}}\to 0,\ i=1,\ldots,m,\ p-2>q\geq 1,$$
$$0\to \overline{R_{2i-1,p}R_{2i,p-2}}\to \overline{R_{2i-1,p+1}R_{2i,p-2}}\oplus \overline{R_{2i-1,p}}\oplus R_{2i,p-1}$$
$$\to \overline{R_{2i-1,p+1}R_{2i,p-1}}\to 0,\ i=1,\ldots,m,\ p\geq 3,$$
$$0\to \overline{R_{2i,p}S_{2i-1,1}}\to \overline{R_{2i,p+1}S_{2i-1,1}}\oplus \overline{R_{2i,p}}\to \overline{R_{2i,p+1}}\to 0,$$
$$i=1,\ldots,m,\ p\geq 1,$$
$$0\to \overline{R_{2i,p}S_{2i-1,q}}\to \overline{R_{2i,p+1}S_{2i-1,q}}\oplus \overline{R_{2i,p}S_{2i-1,q-1}}\to \overline{R_{2i,p+1}S_{2i-1,q-1}}\to 0,$$
$$i=1,\ldots,m,\ p\geq 1,\ q\geq 2,$$
$$0\to \overline{S_{2i-1,p}S_{2i,q}}\to \overline{S_{2i-1,p-1}S_{2i,q}}\oplus \overline{S_{2i-1,p}S_{2i,q-1}}\to \overline{S_{2i-1,p-1}S_{2i,q-1}}\to 0,$$
$$i=1,\ldots,m,\ q-1>p\geq 2,$$
$$0\to \overline{S_{2i-1,p}S_{2i,p+1}}\to \overline{S_{2i-1,p-1}S_{2i,p+1}}\oplus \overline{S_{2i-1,p}}\oplus S_{2i,p}\to \overline{S_{2i-1,p-1}S_{2i,p}}$$
$$\to 0,\ i=1,\ldots,m,\ p\geq 2,$$
$$0\to \overline{S_{2i-1,1}S_{2i,q}}\to \overline{S_{2i,q}}\oplus \overline{S_{2i-1,1}S_{2i,q-1}}\to \overline{S_{2i,q-1}}\to 0,$$

$$i = 1, \ldots, m,\ q \geq 3,$$
$$0 \to \overline{S_{2i-1,1}S_{2i,2}} \to \overline{S_{2i,2}} \oplus \overline{S_{2i-1,1}} \oplus S_{2i,1} \to \overline{S_{2i,1}} \to 0,\ i = 1, \ldots, m.$$

In order to prove the required result, we need to present the vertices of $\mathcal{C}_1, \ldots, \mathcal{C}_m$ in terms of the subspace category $\mathcal{U}(\mathrm{Hom}(M, \mathrm{mod}\, A))$. Recall that in the proof of Lemma 3.1 we calculated the vector space category $\mathrm{Hom}(M, \mathrm{mod}\, A_0)$ and identified it with a full subcategory of $\mathrm{Hom}(M, \mathrm{mod}\, A)$. Since $A' = A_0[M]$ we can use those calculations and the algorithms of constructing $\mathcal{T}_0$ and $\mathcal{I}$ to express the modules $R_{i,p}$, $i = 0, \ldots, 2m+n$, $p \geq 1$, $S_{i,p}$, $i = 0, \ldots, 2m+n$, $p \geq 1$, in terms of $\mathcal{U}(\mathrm{Hom}(M, \mathrm{mod}\, A))$. Namely, we have

$$\begin{aligned}
&R_{0,(2m+n+1)p+r} = \overline{Y_r T^p}^{p+1},\ p \geq 0,\ r = 1, \ldots, 2m+n,\\
&R_{0,(2m+n+1)p} = \overline{T^p}^{p}(0),\ p \geq 1,\\
&R_{2i-1,(2m+n+1)p+2m+n+2-2i} = \overline{T^p Z_{i-1,1}}^{p},\ i = 1, \ldots, m,\ p \geq 0,\\
&R_{2i-1,(2m+n+1)p+r+2m+n+2-2i} = \overline{Y_r T^p Z_{i-1,1}}^{p+1},\\
&\qquad i = 1, \ldots, m,\ r = 1, \ldots, 2m+n,\ p \geq 0,\\
&R_{2i,(2m+n+1)p+2m+n+1-2i} = \overline{T^p Z_i'}^{p},\ i = 1, \ldots, m,\ p \geq 0,\\
&R_{2i,(2m+n+1)p+r+2m+n+1-2i} = \overline{Y_r T^p Z_i'}^{p+1},\\
&\qquad i = 1, \ldots, m,\ r = 1, \ldots, 2m+n,\ r \geq 0,\\
&R_{2m+i,(2m+n+1)p+n+1-i} = \overline{T^p Z_{m,i}}^{p},\ i = 1, \ldots, n,\ p \geq 0,\\
&R_{2m+i,(2m+n+1)p+r+n+1-i} = \overline{Y_r T^p Z_{m,i}}^{p+1},\\
&\qquad i = 1, \ldots, n,\ r = 1, \ldots, 2m+n,\ p \geq 0,\\
&S_{0,2p-1} = \overline{T^{p-1}}^{p},\ p \geq 1,\\
&S_{0,2p} = \overline{T^{p-1}V}^{p},\ p \geq 1,\\
&S_{2i-1,2p-1} = \overline{T^{p-1}Z_{i-1,1}}^{p},\ i = 1, \ldots, m,\ p \geq 1,\\
&S_{2i-1,2p} = \overline{T^{p-1}Z_{i-1,1}V}^{p},\ i = 1, \ldots, m,\ p \geq 1,\\
&S_{2i,2p-1} = \overline{T^{p-1}Z_i'}^{p},\ i = 1, \ldots, m,\ p \geq 1,\\
&S_{2i,2p} = \overline{T^{p-1}Z_i'V}^{p},\ i = 1, \ldots, m,\ p \geq 1,\\
&S_{2m+i,2p-1} = \overline{T^{p-1}Z_{m,i}}^{p},\ i = 1, \ldots, m,\ p \geq 1,\\
&S_{2m+i,2p} = \overline{T^{p-1}Z_{m,i}V}^{p},\ i = 1, \ldots, m,\ p \geq 1.
\end{aligned}$$

Moreover, we have

$$R_{i,p} = \tau_{A_0}^{-i} Y_j,\ i, p \geq 1,\ i + p \leq 2m+n.$$

As a consequence we have the following formulas

$$\begin{aligned}
&\overline{R_{2i-1,(2m+n+1)p+2m+n+2-2i}} = \overline{T^p Z_i''}^{p},\ i = 1, \ldots, m,\ p \geq 0,\\
&\overline{R_{2i-1,(2m+n+1)p+r+2m+n+2-2i}} = \overline{Y_r T^p Z_i''}^{p+1},\\
&\qquad i = 1, \ldots, m,\ r = 1, \ldots, 2m+n,\ p \geq 0,\\
&\overline{R_{2i,(2m+n+1)p+2m+n+1-2i}} = \overline{T^p Z_{i,0}}^{p},\ i = 1, \ldots, m,\ p \geq 0,\\
&\overline{R_{2i,(2m+n+1)p+r+2m+n+1-2i}} = \overline{Y_r T^p Z_{i,0}}^{p+1},
\end{aligned}$$

$$i = 1, \dots, m,\ r = 1, \dots, 2m+n,\ p \geq 0,$$
$$\overline{S_{2i-1,2p-1}} = \overline{T^{p-1}Z_i''}^p,\ i = 1, \dots, m,\ p \geq 1,$$
$$\overline{S_{2i-1,2p}} = \overline{T^{p-1}Z_i''V}^p,\ i = 1, \dots, m,\ p \geq 1,$$
$$\overline{S_{2i,2p-1}} = \overline{T^{p-1}Z_{i,0}}^p,\ i = 1, \dots, m,\ p \geq 1,$$
$$\overline{S_{2i,2p}} = \overline{T^{p-1}Z_{i,0}V}^p,\ i = 1, \dots, m,\ p \geq 1,$$
$$\overline{R_{2i-1,(2m+n+1)p+2m+n+2-2i}R_{2i,s}} = \overline{T^pZ_{i-1,s+1}}^p,$$
$$i = 1, \dots, m,\ s = 1, \dots, 2m+n-2i,\ p \geq 0,$$
$$\overline{R_{2i-1,(2m+n+1)p+2m+n+2-2i}R_{2i,(2m+n+1)q+2m+n+1-2i}} = \overline{T^qZ_i'Z_i''T^p}^{p+q},$$
$$i = 1, \dots, m,\ p > q \geq 0,$$
$$\overline{R_{2i-1,(2m+n+1)p+2m+n+2-2i}R_{2i,(2m+n+1)q+s+2m+n+1-2i}} = \overline{Y_sT^qZ_i'Z_i''T^p}^{p+q+1},$$
$$i = 1, \dots, m,\ s = 1, \dots, 2m+n,\ p > q \geq 0,$$
$$\overline{R_{2i-1,(2m+n+1)p+r+2m+n+2-2i}R_{2i,s}} = \overline{Y_rT^pZ_{i-1,s+1}}^{p+1},$$
$$i = 1, \dots, m,\ r = 1, \dots, 2m+n,\ s = 1, \dots, 2m+n-2i,\ p \geq 0,$$
$$\overline{R_{2i-1,(2m+n+1)p+r+2m+n+2-2i}R_{2i,(2m+n+1)q+2m+n+1-2i}} = \overline{T^qZ_i'Z_i''T^pY_r}^{p+q+1},$$
$$i = 1, \dots, m,\ r = 1, \dots, 2m+n,\ p \geq q \geq 0,$$
$$\overline{R_{2i-1,(2m+n+1)p+r+2m+n+2-2i}R_{2i,(2m+n+1)q+s+2m+n+1-2i}}$$
$$= \overline{Y_sT^qZ_i'Z_i''Y_rT^p}^{p+q+2},\ i = 1, \dots, m,\ r, s = 1, \dots, 2m+n-2i,$$
$$p, q \geq 0, (2m+n+1)p + r > (2m+n+1)q + s,$$
$$\overline{R_{2i,r}S_{2i-1,2q-1}} = \overline{T^{q-1}Z_{i-1,r+1}}^q,$$
$$i = 1, \dots, m,\ r = 1, \dots, 2m+n-2i,\ q \geq 1,$$
$$\overline{R_{2i,r}S_{2i-1,2q}} = \overline{T^{q-1}Z_{i-1,r+1}V}^q,$$
$$i = 1, \dots, m,\ r = 1, \dots, 2m+n-2i,\ q \geq 1,$$
$$\overline{R_{2i,(2m+n+1)p+2m+n+1-2i}S_{2i-1,2q-1}} = \overline{T^pZ_i'Z_i''T^{q-1}}^{p+q},$$
$$i = 1, \dots, m,\ p \geq 0,\ q \geq 1,$$
$$\overline{R_{2i,(2m+n+1)p+2m+n+1-2i}S_{2i-1,2q}} = \overline{T^pZ_i'Z_i''T^{q-1}V}^{p+q},$$
$$i = 1, \dots, m,\ p \geq 0,\ q \geq 1,$$
$$\overline{R_{2i,(2m+n+1)p+r+2m+n+1-2i}S_{2i-1,2q-1}} = \overline{Y_rT^pZ_i'Z_i''T^{q-1}}^{p+q+1},$$
$$i = 1, \dots, m,\ r = 1, \dots, 2m+n,\ p \geq 0,\ q \geq 1,$$
$$\overline{R_{2i,(2m+n+1)p+r+2m+n+1-2i}S_{2i-1,2q}} = \overline{Y_rT^pZ_i'Z_i''T^{q-1}V}^{p+q+1},$$
$$i = 1, \dots, m,\ r = 1, \dots, 2m+n,\ p \geq 0,\ q \geq 1,$$
$$\overline{S_{2i-1,2p-1}S_{2i,2q-1}} = \overline{T^{q-1}Z_i'Z_i''T^{p-1}}^{p+q},\ i = 1, \dots, m,\ q > p \geq 1,$$
$$\overline{S_{2i-1,2p-1}S_{2i,2q}} = \overline{VT^{q-1}Z_i'Z_i''T^{p-1}}^{p+q},\ i = 1, \dots, m,\ q \geq p \geq 1,$$
$$\overline{S_{2i-1,2p}S_{2i,2q-1}} = \overline{T^{q-1}Z_i'Z_i''T^{p-1}V}^{p+q},\ i = 1, \dots, m,\ q > p \geq 1,$$
$$\overline{S_{2i-1,2p}S_{2i,2q}} = \overline{VT^{q-1}Z_i'Z_i''T^{p-1}V}^{p+q},\ i = 1, \dots, m,\ q > p \geq 1,$$

where on the left hand side are objects of $\mathcal{U}(\mathrm{Hom}(M_i, \mathrm{mod}\, A'))$ and on the right hand side we have objects of $\mathcal{U}(\mathrm{Hom}(M, \mathrm{mod}\, A))$. Above, we used that $Z_{i-1,1}$ and Z_i' are (up to isomorphism) the only indecomposable A_0-modules X with $\mathrm{Hom}_A(M, X) \neq 0 \neq \mathrm{Hom}_A(M_i, X)$ and the following equalities in $\mathcal{U}(\mathrm{Hom}(M_i, \mathrm{mod}\, A_0))$

$$\overline{Z_{i-1,1}} = Z_i''$$
$$\overline{Z_i'} = Z_{i,0}$$
$$\overline{Z_{i-1,1}\tau_{A_0}^{-2i}Y_r} = Z_{i-1,r+1},\ r = 1, \dots, 2m + n - 2i,$$

obtained in the proof of Lemma 3.1. We also have $\overline{Z_{i-1,1}Z_i'} = \overline{Z_{i-1,1}} \oplus Z_i' = Z_i'' \oplus Z_i'$. Since for $i = 1, \dots, m$, $\overline{R_{2i-1,p}}$, $p = 2, \dots, 2m + n + 1 - 2i$, and $\overline{R_{2i,p}}$, $j = 1, \dots, 2m + n - 2i$, and $\overline{R_{2i-1,p}R_{2i,q}}$, $1 \leq q < p - 1 \leq 2m + n - 2i$, are indecomposable objects of $\mathcal{U}(\mathrm{Hom}(M_i, \mathrm{mod}\, A_0))$, which are zero objects in $\mathcal{U}(\mathrm{Hom}(M, \mathrm{mod}\, A))$, the claim follows.

4 Missing lemmas

In this section we present some of the missing calculations from the previous sections.

Lemma 4.1. Let $(\mathbb{K}, |-|)$ be a vector space category and let T be an indecomposable object of $\mathbb{K}$ such that $\dim_K |T| = 2$. Then the isomorphism class of $\overline{T^p}^{p-1}$, $p \geq 2$, does not depend on the choice of a basis in $|T|$.

Proof. Let $\mathfrak{t}_1$, $\mathfrak{t}_2$ and $\mathfrak{s}_1$, $\mathfrak{s}_2$, be two bases of $|T|$. We want to show that the triples (T^p, K^{p-1}, φ) and (T^p, K^{p-1}, ψ) are isomorphic, where

$$\varphi(\mathfrak{e}_i) = \mathfrak{t}_2^{(i)} + \mathfrak{t}_1^{(i+1)},\ i = 1, \dots, p - 1,$$

and

$$\psi(\mathfrak{e}_i) = \mathfrak{s}_2^{(i)} + \mathfrak{s}_1^{(i+1)},\ i = 1, \dots, p - 1.$$

Thus we need to construct an invertible linear map $\Phi : K^{p-1} \to K^{p-1}$ and an automorphism Ψ of T^p such that $\psi\Phi = |\Psi|\varphi$. Let $\mathfrak{s}_1 = a_{1,1}\mathfrak{t}_1 + a_{1,2}\mathfrak{t}_2$ and $\mathfrak{s}_2 = a_{2,1}\mathfrak{t}_1 + a_{2,2}\mathfrak{t}_2$. For each $k \geq 1$, we define a $k \times k$ matrix $B^{(k)} = (b_{i,j}^{(k)})$ in the following way. We set $b_{1,1}^{(1)} = 1$. If $k > 1$ and $B^{(k-1)} = (b_{i,j}^{(k-1)})$ is already defined, then we put

$$b_{1,j}^{(k)} = a_{2,2}b_{1,j}^{(k-1)},\ j = 1, \dots, k - 1,$$
$$b_{1,k}^{(k)} = a_{2,1}b_{1,k-1}^{(k-1)},$$
$$b_{i,j}^{(k)} = a_{1,2}b_{i-1,j}^{(k-1)} + a_{2,2}b_{i,j}^{(k-1)},\ j = 1, \dots, k - 1,\ i = 2, \dots, k - 1,$$
$$b_{i,k}^{(k)} = a_{1,1}b_{i-1,k-1}^{(k-1)} + a_{2,1}b_{i,k-1}^{(k-1)},\ i = 2, \dots, k - 1,$$
$$b_{k,j}^{(k)} = a_{1,2}b_{k-1,j}^{(k-1)},\ j = 1, \dots, k - 1,$$
$$b_{k,k}^{(k)} = a_{1,1}b_{k-1,k-1}^{(k-1)}.$$

By easy induction on k we get

$$b_{1,j}^{(k)} = a_{2,1}b_{1,j-1}^{(k-1)},\ j = 2, \dots, k - 1,$$

$$b_{i,j}^{(k)} = a_{1,1}b_{i-1,j-1}^{(k-1)} + a_{2,1}b_{i,j-1}^{(k-1)},\ j = 2,\dots,k-1,\ i = 2,\dots,k-1,$$
$$b_{k,j}^{(k)} = a_{1,1}b_{k-1,j-1}^{(k-1)},\ j = 2,\dots,k-1.$$

We show that $\det B^{(k)} = (a_{1,1}a_{2,2} - a_{1,2}a_{2,1})^{\binom{k}{2}}$, $k \geq 2$. If $k = 2$ then it is obvious. Assume $k > 2$. Since $a_{1,2} \neq 0$ or $a_{2,2} \neq 0$, we may assume without loss of generality that $a_{2,2} \neq 0$. Subtracting in turn the ith row multiplied by $\frac{a_{1,2}}{a_{2,2}}$ from the $(i+1)$th row, $i = 1,\dots,k-2$, we get, using the definition of $B^{(k)}$, the following matrix

$$\begin{bmatrix} a_{2,2}b_{1,1}^{(k-1)} & \cdots & a_{2,2}b_{1,k-1}^{(k-1)} & b_{1,k}^{(k)} \\ \vdots & & \vdots & \vdots \\ a_{2,2}b_{i,1}^{(k-1)} & \cdots & a_{2,2}b_{i,k-1}^{(k-1)} & \sum_{j=1}^{i}(-\frac{a_{1,2}}{a_{2,2}})^{i-j}b_{j,k}^{(k)} \\ \vdots & & \vdots & \vdots \\ a_{2,2}b_{k-1,1}^{(k-1)} & \cdots & a_{2,2}b_{k-1,k-1}^{(k-1)} & \sum_{j=1}^{k-1}(-\frac{a_{1,2}}{a_{2,2}})^{k-1-j}b_{j,k}^{(k)} \\ a_{1,2}b_{k-1,1}^{(k-1)} & \cdots & a_{1,2}b_{k-1,k-1}^{(k-1)} & b_{k,k}^{(k)} \end{bmatrix}.$$

Note that if we remove the last column, then the $(k-1)$th and kth rows of the obtained matrix are linearly dependent. Thus expanding with respect to the last column we get

$$\det B^{(k)} = b_{k,k}^{(k)}a_{2,2}^{k-1}\det B^{(k-1)} - \Big(\sum_{j=1}^{k-1}\big(-\frac{a_{1,2}}{a_{2,2}}\big)^{k-1-j}b_{j,k}^{(k)}\Big)a_{1,2}a_{2,2}^{k-2}\det B^{(k-1)}.$$

Using the inductive hypothesis, the easily verified formula

$$b_{i,k}^{(k)} = \binom{k-1}{i-1}a_{1,1}^{i-1}a_{2,1}^{k-i},\ i = 1,\dots,k,$$

and the equality $\binom{k-1}{2} + (k-1) = \binom{k}{2}$, and performing the standard calculations we get the desired result. In particular we have $\det B^{(k)} \neq 0$.

We define Φ and Ψ by the following formulas

$$\Phi(\mathfrak{e}_i) = \sum_{j=1}^{p-1} b_{j,i}^{(p-1)}\mathfrak{e}_j,\ i = 1,\dots,p-1,$$
$$\Psi(t_1,\dots,t_p) = \Big(\sum_{i=1}^{p} b_{1,i}^{(p)}t_i,\dots,\sum_{i=1}^{p} b_{p,i}^{(p)}t_i\Big).$$

Then

$$|\Psi|(\mathfrak{t}_1^{(i)}) = \sum_{j=1}^{p} b_{j,i}^{(p)}\mathfrak{t}_1^{(j)},\ j = 1,\dots,p,$$
$$|\Psi|(\mathfrak{t}_2^{(i)}) = \sum_{j=1}^{p} b_{j,i}^{(p)}\mathfrak{t}_2^{(j)},\ i = 1,\dots,p.$$

Using the equalities $\mathfrak{s}_1^{(i)} = a_{1,1}\mathfrak{t}_1^{(i)} + a_{1,2}\mathfrak{t}_2^{(i)}$ and $\mathfrak{s}_2^{(i)} = a_{2,1}\mathfrak{t}_1^{(i)} + a_{2,2}\mathfrak{t}_2^{(i)}$, $i = 1, \dots, p$, we obtain

$$\begin{aligned}(\psi\Phi)(\mathfrak{e}_i) = {} & a_{2,1}b_{1,i}^{(p-1)}\mathfrak{t}_1^{(1)} + \sum_{j=2}^{p-1}(a_{1,1}b_{j-1,i}^{(p-1)} + a_{2,1}b_{j,i}^{(p-1)})\mathfrak{t}_1^{(j)} + a_{1,1}b_{p-1,i}\mathfrak{t}_1^{(p)} \\ & + a_{2,2}b_{1,i}^{(p-1)}\mathfrak{t}_2^{(1)} + \sum_{j=2}^{p-1}(a_{1,2}b_{j-1,i}^{(p-1)} + a_{2,2}b_{j,i}^{(p-1)})\mathfrak{t}_2^{(j)} + a_{1,2}b_{p-1,i}\mathfrak{t}_2^{(p)}, \\ & i = 1, \dots, p-1, \\ (|\Psi|\varphi)(\mathfrak{e}_i) = {} & \sum_{j=1}^{p} b_{j,i+1}^{(p)}\mathfrak{t}_1^{(j)} + \sum_{j=1}^{p} b_{j,i}^{(p)}\mathfrak{t}_2^{(j)},\ i = 1, \dots, p-1.\end{aligned}$$

The equality $\psi\Phi = |\Psi|\varphi$ follows from the above formulas and the definition and properties of $b_{i,j}^{(k)}$. Since $\det B^{(k)} \neq 0$, the maps Φ and Ψ are invertible. This finishes the proof.

Lemma 4.2. Let $p \geq 1$. The objects $\overline{Y_kY_lU'U''T^{2p}}^{2p+2}$, $k < l$, in $\mathcal{U}(\mathbb{K}_{m,n})$ have trivial endomorphism rings.

Proof. Fix basis elements $\mathfrak{y}'$ in $|Y_k|$, $\mathfrak{y}''$ in $|Y_l|$, $\mathfrak{u}'$ in $|U'|$, $\mathfrak{u}''$ in $|U''|$ and $\mathfrak{t}_0$, $\mathfrak{t}_1$, $\mathfrak{t}_2$ in $|T|$ in an appropriate way (see Section 2 for details). We may assume without loss of generality that $\mathfrak{t}_0 = \mathfrak{t}_1 + \mathfrak{t}_2$. Recall that $\overline{Y_kY_lU'U''T^{2p}}^{2p+2}$ is the triple $(Y_k \oplus Y_l \oplus U' \oplus U'' \oplus T^{2p}, K^{2p+2}, \varphi)$, $p \geq 1$, where φ is given by

$$\begin{aligned}\varphi(\mathfrak{e}_1) &= \mathfrak{y}' + \mathfrak{t}_1^{(1)}, \\ \varphi(\mathfrak{e}_i) &= \mathfrak{t}_1^{(i-1)} + \mathfrak{t}_2^{(i-1)} + \mathfrak{t}_1^{(i)},\ i = 2, \dots, p, \\ \varphi(\mathfrak{e}_{p+1}) &= \mathfrak{t}_1^{(p)} + \mathfrak{t}_2^{(p)} + \mathfrak{u}', \\ \varphi(\mathfrak{e}_{p+2}) &= \mathfrak{u}' + \mathfrak{u}'' + \mathfrak{t}_1^{(p+1)} + \mathfrak{t}_2^{(p+1)}, \\ \varphi(\mathfrak{e}_i) &= \mathfrak{t}_1^{(i-2)} + \mathfrak{t}_1^{(i-1)} + \mathfrak{t}_2^{(i-1)},\ i = p+3, \dots, 2p+1, \\ \varphi(\mathfrak{e}_{2p+2}) &= \mathfrak{t}_1^{(2p)} + \mathfrak{y}''.\end{aligned}$$

An endomorphism of $\overline{Y_kY_lU'U''T^{2p}}^{2p+2}$ is a pair (Ψ, Φ) consisting of a linear map $\Phi : K^{2p+2} \to K^{2p+2}$ and an endomorphism Ψ of $Y_k \oplus Y_l \oplus U' \oplus U'' \oplus T^{2p}$ such that $\varphi\Phi = |\Psi|\varphi$. We obviously have

$$\Phi(\mathfrak{e}_i) = \sum_{j=1}^{2p+2} a_{i,j}\mathfrak{e}_j,\ i = 1, \dots, 2p+2,$$

thus

$$\begin{aligned}(\varphi\Phi)(\mathfrak{e}_i) = {} & a_{i,1}\mathfrak{y}' + a_{i,2p+2}\mathfrak{y}'' + (a_{i,p+1} + a_{i,p+2})\mathfrak{u}' + a_{i,p+2}\mathfrak{u}'' \\ & + \sum_{j=1}^{p}(a_{i,j} + a_{i,j+1})\mathfrak{t}_1^{(j)} + \sum_{j=p+1}^{2p}(a_{i,j+1} + a_{i,j+2})\mathfrak{t}_1^{(j)} + \sum_{j=1}^{2p} a_{i,j+1}\mathfrak{t}_2^{(j)}.\end{aligned}$$

On the other hand,

$$|\Psi|(\mathfrak{y}') = b\mathfrak{y}' + b'\mathfrak{y}'' + \sum_{j=1}^{2p} b_j(\mathfrak{t}_1^{(j)} + \mathfrak{t}_2^{(j)}),$$

$$|\Psi|(\mathfrak{y}'') = c\mathfrak{y}'' + \sum_{j=1}^{2p} c_j(\mathfrak{t}_1^{(j)} + \mathfrak{t}_2^{(j)}),$$

$$|\Psi|(\mathfrak{u}') = d\mathfrak{u}' + \sum_{j=1}^{2p} d_j\mathfrak{t}_1^{(j)},$$

$$|\Psi|(\mathfrak{u}'') = e\mathfrak{u}'' + \sum_{j=1}^{2p} e_j\mathfrak{t}_2^{(j)},$$

$$|\Psi|(\mathfrak{t}_1^{(i)}) = \sum_{j=1}^{2p} f_{i,j}\mathfrak{t}_1^{(j)},$$

$$|\Psi|(\mathfrak{t}_2^{(i)}) = \sum_{j=1}^{2p} f_{i,j}\mathfrak{t}_2^{(j)},$$

and hence

$$(|\Psi|\varphi)(\mathfrak{e}_1) = b\mathfrak{y}' + b'\mathfrak{y}'' + \sum_{j=1}^{2p}(b_j + f_{1,j})\mathfrak{t}_1^{(j)} + \sum_{j=1}^{2p} b_j\mathfrak{t}_2^{(j)},$$

$$(|\Psi|\varphi)(\mathfrak{e}_i) = \sum_{j=1}^{2p}(f_{i-1,j} + f_{i,j})\mathfrak{t}_1^{(j)} + \sum_{j=1}^{2p} f_{i-1,j}\mathfrak{t}_2^{(j)},\ i = 2, \dots, p,$$

$$(|\Psi|\varphi)(\mathfrak{e}_{p+1}) = d\mathfrak{u}' + \sum_{j=1}^{2p}(d_j + f_{p,j})\mathfrak{t}_1^{(j)} + \sum_{j=1}^{2p} f_{p,j}\mathfrak{t}_2^{(j)},$$

$$(|\Psi|\varphi)(\mathfrak{e}_{p+2}) = d\mathfrak{u}' + e\mathfrak{u}'' + \sum_{j=1}^{2p}(d_j + f_{p+1,j})\mathfrak{t}_1^{(j)} + \sum_{j=1}^{2p}(e_j + f_{p+1,j})\mathfrak{t}_2^{(j)},$$

$$(|\Psi|\varphi)(\mathfrak{e}_i) = \sum_{j=1}^{2p}(f_{i-2,j} + f_{i-1,j})\mathfrak{t}_1^{(j)} + \sum_{j=1}^{2p} f_{i-1,j}\mathfrak{t}_2^{(j)},$$
$$i = p + 3, \dots, 2p + 1,$$

$$(|\Psi|\varphi)(\mathfrak{e}_{2p+2}) = c\mathfrak{y}'' + \sum_{j=1}^{2p}(c_j + f_{2p,j})\mathfrak{t}_1^{(j)} + \sum_{j=1}^{2p} c_j\mathfrak{t}_2^{(j)}.$$

Comparing the coefficients of $\mathfrak{y}'$ and $\mathfrak{y}''$ in the equations $(\varphi\Phi)(\mathfrak{e}_i) = (|\Psi|\varphi)(\mathfrak{e}_i)$, $i = 1, \dots, 2p + 2$, we get

$$a_{1,1} = b,\ a_{i,1} = 0,\ i = 2, \dots, 2p + 2,$$
$$a_{1,2p+2} = b',\ a_{i,2p+2} = 0, i = 2, \dots, 2p + 1,\ a_{2p+2,2p+2} = c.$$

Similarly, by comparing the coefficients of $\mathfrak{u}'$ and $\mathfrak{u}''$, we obtain

$$a_{p+1,p+1} = d,\ a_{p+2,p+1} = d - e,\ a_{i,p+1} = 0,\ i \neq p + 1, p + 2,$$

$$a_{p+2,p+2} = e,\ a_{i,p+2} = 0,\ i \neq p+2.$$

Now we look at the coefficients of $\mathfrak{t}_1^{(j)}$, $\mathfrak{t}_2^{(j)}$, $j = 1, \ldots, p$, in the equations $(\varphi\Phi)(\mathfrak{e}_i) = (|\Psi|\varphi)(\mathfrak{e}_i)$, $i = 1, \ldots, p+1$. In this way we get

$$\begin{aligned}
a_{1,j} + a_{1,j+1} &= b_j + f_{1,j},\ j = 1, \ldots, p, \\
a_{i,j} + a_{i,j+1} &= f_{i-1,j} + f_{i,j},\ j = 1, \ldots, p,\ i = 2, \ldots, p, \\
a_{p+1,j} + a_{p+1,j+1} &= f_{p,j} + d_j,\ j = 1, \ldots, p, \\
a_{1,j+1} &= b_j,\ j = 1, \ldots, p, \\
a_{i,j+1} &= f_{i-1,j},\ j = 1, \ldots, p,\ i = 2, \ldots, p+1.
\end{aligned}$$

Subtracting the appropriate equations we obtain

$$\begin{aligned}
a_{i,j} &= f_{i,j},\ j = 1, \ldots, p,\ i = 1, \ldots, p, \\
a_{p+1,j} &= d_j,\ j = 1, \ldots, p.
\end{aligned}$$

Analyzing the above equations we get

$$\begin{aligned}
a_{i,j} &= f_{i,j} = a_{p+1+i-j,p+1} = 0,\ 1 \leq i < j \leq p, \\
a_{i,i} &= f_{i,i} = b = d,\ i = 1, \ldots, p+1, \\
a_{i,j} &= f_{i,j} = a_{i+1-j,1} = 0,\ 1 \leq j < i \leq p, \\
b_j &= a_{p+1-j,p+1} = 0,\ j = 1, \ldots, p, \\
d_j &= a_{p+1,j} = a_{p+2-j,1} = 0,\ j = 1, \ldots, p.
\end{aligned}$$

Looking at the coefficients of $\mathfrak{t}_1^{(j)}$, $\mathfrak{t}_2^{(j)}$, $j = 1, \ldots, p$, in the equations $(\varphi\Phi)(\mathfrak{e}_i) = (|\Psi|\varphi)(\mathfrak{e}_i)$, $i = p+2, \ldots, 2p+2$, and taking into account that $d_j = 0$, $j = 1, \ldots, p$, we get

$$\begin{aligned}
a_{p+2,j} + a_{p+2,j+1} &= f_{p+1,j},\ j = 1, \ldots, p, \\
a_{i,j} + a_{i,j+1} &= f_{i-2,j} + f_{i-1,j},\ j = 1, \ldots, p,\ i = p+3, \ldots, 2p+1, \\
a_{2p+2,j} + a_{2p+2,j+1} &= f_{2p,j} + c_j,\ j = 1, \ldots, p, \\
a_{p+2,j+1} &= e_j + f_{p+1,j},\ j = 1, \ldots, p, \\
a_{i,j+1} &= f_{i-1,j},\ j = 1, \ldots, p,\ i = p+3, \ldots, 2p+1, \\
a_{2p+2,j+1} &= c_j,\ j = 1, \ldots, p.
\end{aligned}$$

Subtracting the appropriate equations we obtain

$$\begin{aligned}
a_{p+2,j} &= -e_j,\ j = 1, \ldots, p, \\
a_{i,j} &= f_{i-2,j},\ j = 1, \ldots, p,\ i = p+3, \ldots, 2p+2.
\end{aligned}$$

Analyzing the above equations we easily get

$$\begin{aligned}
a_{i,j} &= f_{i-1,j-1} = a_{i+j-1,1} = 0,\ j = 2, \ldots, 2p+3-i,\ i = p+3, \ldots, 2p+1, \\
a_{i,j} &= f_{i-2,j} = a_{i+j-p-1,p+1} = 0, \\
&\qquad j = 2p+4-i, \ldots, p,\ i = p+4, \ldots, 2p+2,
\end{aligned}$$

$$f_{p+1,j} = a_{p+j+2,1} = 0,\ j = 1, \ldots, p,$$
$$c_j = a_{p+j+2,p+1} = 0,\ j = 1, \ldots, p.$$

Moreover, since $a_{p+2,j+1} = f_{p+1,j} + a_{p+2,j}$, $j = 1, \ldots, p$, and $a_{p+2,1} = 0$, we get

$$e_j = a_{p+2,j+1} = 0,\ j = 1, \ldots, p.$$

In particular, we have

$$d - e = a_{p+2,p+1} = 0.$$

Now we compare the coefficients of $\mathfrak{t}_1^{(j)}$, $\mathfrak{t}_2^{(j)}$, $j = p+1, \ldots, 2p$, in the equations $(\varphi\Phi)(\mathfrak{e}_i) = (|\Psi|\varphi)(\mathfrak{e}_i)$, $i = 1, \ldots, p+1$. We obtain

$$a_{1,j+1} + a_{1,j+2} = b_j + f_{1,j},\ j = p+1, \ldots, 2p,$$
$$a_{i,j+1} + a_{i,j+2} = f_{i-1,j} + f_{i,j},\ j = p+1, \ldots, 2p,\ i = 2, \ldots, p,$$
$$a_{p+1,j+1} + a_{p+1,j+2} = f_{p,j} + d_j,\ j = p+1, \ldots, 2p,$$
$$a_{1,j+1} = b_j,\ j = p+1, \ldots, 2p,$$
$$a_{i,j+1} = f_{i-1,j},\ j = p+1, \ldots, 2p,\ i = 2, \ldots, p+1.$$

Subtracting the appropriate equations we get

$$a_{i,j+2} = f_{i,j},\ j = p+1, \ldots, 2p,\ i = 1, \ldots, p,$$
$$a_{p+1,j+2} = d_j,\ j = p+1, \ldots, 2p.$$

Analyzing the above equations we get

$$a_{i,j} = f_{i,j-2} = a_{i+j-p-2,p+2} = 0,\ j = p+3, \ldots, 2p+3-i,\ i = 1, \ldots, p,$$
$$a_{i,j} = f_{i,j-2} = a_{i+j-2p-2,2p+2} = 0,\ j = 2p+4-i, \ldots, 2p+1\ i = 1, \ldots, p,$$
$$b_j = a_{j-p,p+2} = 0,\ j = p+1, \ldots, 2p,$$
$$d_j = a_{p+1,j+2} = a_{j+1-p,2p+2} = 0,\ j = p+1, \ldots, 2p.$$

Note that we have

$$b' = a_{1,2p+2} = 0.$$

Finally, if we compare the coefficients of $\mathfrak{t}_1^{(j)}$, $\mathfrak{t}_2^{(j)}$, $j = p+1, \ldots, 2p$, in the equations $(\varphi\Phi)(\mathfrak{e}_i) = (|\Psi|\varphi)(\mathfrak{e}_i)$, $i = p+2, \ldots, 2p+2$, and use the fact that $d_j = 0$, $j = p+1, \ldots, 2p$, we obtain

$$a_{p+2,j+1} + a_{p+2,j+2} = f_{p+1,j},\ j = p+1, \ldots, 2p,$$
$$a_{i,j+1} + a_{i,j+2} = f_{i-2,j} + f_{i-1,j},$$
$$j = p+1, \ldots, 2p,\ i = p+3, \ldots, 2p+1,$$
$$a_{2p+2,j+1} + a_{2p+2,j+2} = f_{2p,j} + c_j,\ j = p+1, \ldots, 2p,$$
$$a_{p+2,j+1} = e_j + f_{p+1,j},\ j = p+1, \ldots, 2p,$$
$$a_{i,j+1} = f_{i-1,j},\ j = p+1, \ldots, 2p,\ i = p+3, \ldots, 2p+1,$$
$$a_{2p+2,j+1} = c_j,\ j = p+1, \ldots 2p.$$

Subtracting the appropriate equations we get

$$a_{p+2,j+2} = -e_j,\ j = p+1, \ldots, 2p,$$
$$a_{i,j+2} = f_{i-2,j},\ j = p+1, \ldots, 2p,\ i = p+3, \ldots, 2p+2.$$

Analyzing the above equations we obtain

$$a_{i,j} = f_{i-1,j-1} = a_{i+2p+2-j,2p+2},\ j = i+1, \ldots, 2p+1,\ i = p+3, \ldots, 2p,$$
$$a_{i,i} = f_{i-1,i-1} = a_{2p+2,2p+2} = c,\ i = p+3, \ldots, 2p+1,$$
$$a_{i,j} = f_{i-2,j-2} = a_{i+p+2-j,p+2},\ j = p+3, \ldots, i-1,\ i = p+4, \ldots, 2p+2,$$
$$f_{p+1,j} = a_{3p+3-j,2p+2} = 0,\ j = p+2, \ldots, 2p,$$
$$f_{p+1,p+1} = a_{2p+2,2p+2} = c,$$
$$c_j = a_{3p+3-j,p+2} = 0,\ j = p+1, \ldots, 2p.$$

Moreover, since $a_{p+2,j+1} = f_{p+1,j} - a_{p+2,j+2}$, $j = p+1, \ldots, 2p$, and $a_{p+2,2p+2} = 0$, we get

$$a_{p+2,p+2} = c,$$
$$a_{p+2,j} = 0,\ j = p+3, \ldots, 2p+1,$$
$$e_j = 0, j = p+1, \ldots, 2p.$$

In particular, we have

$$e = a_{p+2,p+2} = c.$$

It follows from the above calculations that

$$a_{i,j} = 0,\ i \neq j,$$
$$b_j = c_j = d_j = e_j = 0,\ i = 1, \ldots, 2p,$$
$$f_{i,j} = 0,\ i \neq j,$$
$$a_{i,i} = b = c = d = e = f_{j,j},\ i = 1, \ldots, 2p+2,\ j = 1, \ldots, 2p,$$

and this finishes the proof.

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