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A pathwise solution for nonlinear parabolic equations with stochastic perturbations

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Abstract: We analyse here a semilinear stochastic partial differential equation of parabolic type where the diffusion vector fields are depending on both the unknown function and its gradient $\partial_x u$ with respect to the state variable, $x \in \mathbb{R}^n$. A local solution is constructed by reducing the original equation to a nonlinear parabolic one without stochastic perturbations and it is based on a finite dimensional Lie algebra generated by the given diffusion vector fields.

Keywords: stochastic partial differential equations (SPDE), finite dimensional Lie algebras
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1 Introduction

In this paper we consider a nonlinear stochastic partial differential equation of parabolic type written in the Fisk-Stratonovich sense and described as follows

$$\begin{cases} d_t u = [\Delta u + f(t,x,u,\partial_x u)]dt + \sum_{i=1}^{m} g_i(x,u,\partial_x u) \bigcirc dw_i(t), \\ u(0,x) = 0, \ t \in (0,T], \ x \in \mathbb{R}^n, \ u \in \mathbb{R} \end{cases} \tag{1}$$

where $w = (w_1, \ldots, w_m)$ is an m-dimensional Wiener process on a filtered complete probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}\uparrow; P\}$. The initial condition in (1) can be taken as a bounded nonvanishing smooth function, provided the integral representation in (29) is written

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accordingly. The class of PDE as in (1) arises in a number of applications like filtering and pathwise stochastic control, mathematical finance, and one may find in [1] a complete list of the contributions to the subject.

The general remark in [1] remains true and the fully nonlinear drift $f(t, x, u, \partial_x u)$ is an obstruction for the use of the martingale theory even if the diffusion coefficients g_i are depending only on the unknown function u. The corresponding vectorial case, $u = (u_1, \ldots u_N)$, is analyzed in [2] and [3] using a finite dimensional Lie algebraic structure generated by the corresponding diffusion vector fields $g_i(x, u_1, \ldots, u_N)$. The method used in [2] and [3] allows one to define a smooth orbit in the space of the unknown functions $u \in \mathbb{R}^N$ and the influence of the stochastic integration is translated into the parameter-space of the fixed orbit provided the associated gradient system has a nonsingular algebraic representation as it is contained in [4]. The solution for (1) is obtained provided the method of gradient characteristics allows one to get a solution for the reduced stochastic differential equation

$$d_t u = \sum_{i=1}^{m} g_i(x, u, \partial_x u) \bigcirc dw_i(t), \quad t \in [0, T]. \tag{2}$$

The stochastic integration is shifted into the parameter space of a fixed orbit in the space of vector functions $(u(t,x), \partial_x u(t,x)) \in \mathbb{R}^{n+1}$ and the solution in (2) is obtained as a diffeomorphism application acting on the initial conditions $y_0(t,x) \overset{\text{def}}{=} (h(t,x), \partial_x h(t,x)) \in \mathbb{R}^{n+1}$ which are continuous and $\mathcal{F}_t$-adapted processes assuming the vector fields defining the characteristics are not commuting.

The characteristic system associated with the reduced stochastic differential equation (2) has to behave nicely and a local time (a stopping time which doesn't depend on the state $x \in \mathbb{R}^n$) is a good measure of this behaviour. In the paper [1] it is accomplished assuming that the given functions g_i are depending only on $p = \partial_x u$ which implies the corresponding vector fields are commuting and the solution can be represented explicitely using a standard orbit and a local time.

Here we point out that a local time and an extended orbit can be used provided the vector fields entering the characteristic system are generating a finite dimensional Lie algebra including the commuting vector fields. The same procedure was used in [4] to construct an orbit and a local time associated with evolution systems driven by diffusion vector fields which are not depending on the gradients $p_i = \partial_x u_i$, $i \in \{1, \ldots, N\}$. On the other hand, the meaning of a solution adopted here includes the verification of the SDE (1) along the continuous trajectories of the process $x = \hat{x}(t, \lambda) \in \mathbb{R}^n$, $t \in [0, T]$, obtained as a component of the solution $\hat{z}(t, \lambda) = (\hat{x}(t, x), \hat{u}(t, \lambda), \hat{p}(t, \lambda))$ fulfilling the characteristic system associated with the reduced SDE (2).

This indicates that the difference between the solution we consider in this paper and the weak solution appearing in [1] may come from the condition (iii) in the Definition 3.1 appearing in the main results. Here a solution $u(t,x)$ of the original parabolic equation (1) is obtained provided $y(t,x) = (u(t,x), \partial_x u(t,x))$ satisfies $y(t, \hat{x}(t, \lambda)) = \hat{y}(t, \lambda)$ and the extended system (46) has to be fulfilled along $x = \hat{x}(t, \lambda)$ not for any $x \in \mathbb{R}^n$. In particular, when $\partial_p g_i(x,u,p) = 0$, $i \in \{1,\dots,n\}$, this definition of a solution coincides with the usual one (see [4]) using any $x \in \mathbb{R}^n$.

2 Preliminaries

Consider the following Hamilton-Jacobi stochastic system

$$\begin{cases} d_t u = \sum_{i=1}^m g_i(x,u(t,x),p(t,x)) \bigcirc dw_i(t) \\ d_t p = \sum_{i=1}^m H_i(x,u(t,x),p(t,x),\partial_x p(t,x)) \bigcirc dw_i(t) \end{cases} \tag{3}$$

for $t \in [0,T]$, $u \in \mathbb{R}$, $p \in \mathbb{R}^n$, where

$$H_i(x,u,p,\partial_x p) \stackrel{\text{def}}{=} \left[\partial_x g_i + p\partial_u g_i + (\partial_x p)\partial_p g_i\right](x,u,p).$$

Let $k \in \mathbb{N}$ be a natural number, $B(0,\rho) \subset \mathbb{R}^{n+1}$ a fixed ball and denote $D = B(0,\rho)\times\mathbb{R}^n$, $z = (u,p,x)$. Define $C_b^k(D)$ the space consisting of all continuous and bounded functions $h(z): D \to \mathbb{R}$ admitting continuous and bounded partial derivatives up to order k with respect to $z \in D$. It is assumed

$$g_i \in C_b^4(D) \text{ and } f \in C([0,T]; C_b^2(D)),\ i \in \{1,\dots,m\} \tag{4}$$

Let the function $y_0(t,x) = (h(t,x), \partial_x h(t,x))$ be a continuous and $\mathcal{F}_t$-adapted process obeying $y_0(t,x), \partial_i y_0(t,x) \in B(0,\rho/2)$ for any $(t,x) \in [0,T]\times\mathbb{R}^n$, $i \in \{1,\dots,n\}$, where $\partial_i y_0 \stackrel{\text{def}}{=} \dfrac{\partial y_0}{\partial x_i}$. The method of characteristics applied to (3) involves an extended system of stochastic differential equations with an $\mathcal{F}_t$-adapted process $z_0^t(\lambda)$ as initial condition

$$\begin{cases} d_s z(s,\lambda) = \sum_{i=1}^m Z_i(z(s,\lambda)) \bigcirc dw_i(s), \quad s \in [0,t] \\ z(0,\lambda) = z_0^t(\lambda) \end{cases} \tag{5}$$

for each $t \in [0,T]$ fixed, where $Z_i(z) \in \mathbb{R}^{2n+1}$ fulfils

$$Z_i(z) \stackrel{\text{def}}{=} \begin{pmatrix} Y_i(z) \\ X_i(z) \end{pmatrix} \tag{6}$$

where $X_i(z) \stackrel{\text{def}}{=} -\partial_p g_i(x,u,p) \in \mathbb{R}^n$, $Y_i(z) \stackrel{\text{def}}{=} \begin{pmatrix} l_i(z) \\ L_i(z) \end{pmatrix}$, $l_i(z) \stackrel{\text{def}}{=} (g_i - \langle p, \partial_p g_i\rangle)(x,u,p)$ and $L_i(z) = (\partial_x g_i + p\partial_u g_i)(x,u,p) \in \mathbb{R}^n$, $i \in \{1,\dots,m\}$.

The ordinary stochastic differential system in (5) can be solved (see [5]) provided the following associated gradient system has a solution

$$\begin{cases} \partial_{t_i} z^t = Z_i(z^t),\ i \in \{1,\dots,m\},\ (t_1,\dots,t_m) \in D_m \\ z^t(0,\lambda) = z_0^t(\lambda) \stackrel{\text{def}}{=} (y_0(t,\lambda),\lambda) \in B(0,\rho_1)\times\mathbb{R}^n,\ t \in [0,T],\ \lambda \in \mathbb{R}^n\ \rho_1 = \rho/2, \end{cases} \tag{7}$$

where $D_m \stackrel{\text{def}}{=} \prod_1^m(-a_i, a_i)$, $y_0(t,\lambda) \stackrel{\text{def}}{=} (h(t,\lambda), \partial_\lambda h(t,\lambda))$.

The simpliest case is when the vector fields $\{Z_1, \ldots, Z_m\}$ commute using the standard Lie bracket, and if this is the case, we get the solution as a finite composition of local flows

$$z^t(\sigma,\lambda) = S_1(t_1) \circ \ldots \circ S_m(t_m)(z_0^t(\lambda)),\ \sigma \in D_m,\ \lambda \in \mathbb{R}^n,\ t \in [0,T] \tag{8}$$

where $S_j(\tau)(z_0)$, $\tau \in (-a_j, a_j)$, $z_0 \in B(0,\rho_1) \times \mathbb{R}^n$, is the local flow generated by the vector field Z_j, $j \in \{1,\ldots,m\}$, and $\sigma \stackrel{\text{def}}{=} (t_1, t_2, \ldots, t_m)$. Write the solution in (8) as $z^t(\sigma,\lambda) = (\hat{y}^t(\sigma,\lambda), \hat{x}^t(\sigma,\lambda))$, where $\hat{y}^t(\sigma,\lambda) = (\hat{u}^t(\sigma,\lambda), \hat{p}^t(\sigma,\lambda)) \in \mathbb{R}^{n+1}$ and $\hat{x}^t(\sigma,\lambda) \in \mathbb{R}^n$. Then find $\lambda = \psi^t(\sigma,x)$ as a unique solution of the algebraic equations

$$\hat{x}^t(\sigma,\lambda) = x,\ x \in \mathbb{R}^n,\ t \in [0,T],\ \sigma \in B(0,\tilde{\rho}) \subset D_m.$$

It holds

$$\hat{x}^t(\sigma, \psi^t(\sigma,x)) = x, \quad \psi^t(\sigma, \hat{x}^t(\sigma,\lambda)) = \lambda, \tag{9}$$

for any $x, \lambda \in \mathbb{R}^n$, $t \in [0,T]$ and $\sigma \in B(0,\tilde{\rho})$ provided $\tilde{\rho}$ is sufficiently small as will be explained later when constructing $y^t(\sigma,x)$ in (15).

Denote

$$y^t(\sigma,x) \stackrel{\text{def}}{=} \hat{y}^t(\sigma, \psi^t(\sigma,x)) \stackrel{\text{def}}{=} (u^t(\sigma,x), p^t(\sigma,x)),\ x \in \mathbb{R}^n,\ \sigma \in B(0,\tilde{\rho}), \tag{10}$$

for each $t \in [0,T]$ and it is easy to get the following equations

$$\partial_x u^t(\sigma,x) = p^t(\sigma,x), \quad y^t(0,x) = y_0(t,x) \stackrel{\text{def}}{=} (h(t,x), \partial_x h(t,x)). \tag{11}$$

$$\begin{cases} y^t(\sigma, \hat{x}^t(\sigma,\lambda)) = \hat{y}^t(\sigma,\lambda) \\ \partial_{t_i}\hat{y}^t(\sigma,\lambda) = \partial_{t_i} y^t(\sigma, \hat{x}(\sigma,\lambda)) + \partial_x y^t(\sigma, \hat{x}^t(\sigma,\lambda)) X_i(z^t(\sigma,\lambda)) \\ \qquad\qquad\quad = Y_i(z^t(\sigma,\lambda)),\ i \in \{1,\ldots,m\} \end{cases} \tag{12}$$

for each $t \in [0,T]$, $\lambda \in \mathbb{R}^n$, where the vector fields $X_i \in \mathbb{R}^n$, $Y_i \in \mathbb{R}^{n+1}$ are defined in (6) with $X_i, Y_i \in C_b^2(D)$. Using (11) we rewrite (12) as an equation expressing the partial derivative $\partial_{t_i} y^t(\sigma, \hat{x}^t(\sigma,\lambda))$ along $x = \hat{x}^t(\sigma,\lambda))$ and we get

$$\partial_{t_i} y^t(\sigma, \hat{x}^t(\sigma,\lambda)) = F_i(\hat{x}^t(\sigma,\lambda), \hat{y}^t(\sigma,\lambda), \partial_x p^t(\sigma, \hat{x}^t(\sigma,\lambda))),\ i \in \{1,\ldots,m\} \tag{13}$$

where the vector function $F_i \in \mathbb{R}^{n+1}$ is obtained from the original g_i as follows:

$$F_i(x,y,\partial_x p) = \begin{pmatrix} g_i(x,y) \\ [\partial_x g_i + p\partial_u g_i + (\partial_x p)\partial_p g_i](x,y) \stackrel{\text{def}}{=} H_i(x,y,\partial_x p) \end{pmatrix} \tag{14}$$

Using the diffeomorphism application $S(\sigma)(z_0)$ of $z_0 \in B(0,\rho_1) \times \mathbb{R}^n$ appearing in (8) we may and do write $y^t(\sigma,x)$ as a diffeomorphism mapping of $y_0^t(\sigma,x) \stackrel{\text{def}}{=} y_0(t, \psi^t(\sigma,x))$ as is shown in the following direct computations.

Write $z^t(\sigma,\lambda) \stackrel{\text{def}}{=} S(\sigma)(z_0^t(\lambda)) \stackrel{\text{def}}{=} (\hat{y}^t(\sigma,\lambda), \hat{x}^t(\sigma,\lambda)) \stackrel{\text{def}}{=} (\hat{G}(\sigma,\lambda; y_0(t,\lambda)), \hat{J}(\sigma, y_0(t,\lambda);\lambda))$ for any $z_0^t(\lambda) \stackrel{\text{def}}{=} (y_0(t,\lambda),\lambda) \in B(0,\rho_1) \times \mathbb{R}^n$ and find $\lambda = \psi^t(\sigma,x)$ as the unique solution of the equation $\hat{J}(\sigma, y_0(t,\lambda);\lambda) = x$, for $\sigma \in B(0,\tilde{\rho})$, $t \in [0,T]$, $x \in \mathbb{R}^n$. We may and do replace the equation $\hat{J}(\sigma, y_0(t,\lambda);\lambda) = x$ by an integral equation with respect to the unknown λ. The associated integral equation is obtained using $\hat{J}(\theta\sigma, y_0(t,\lambda);\lambda) = c(\theta,\sigma,t,\lambda)$ for $\theta \in [0,1]$ and

$$\frac{d}{d\theta}c = \sum_{i=1}^{m} \sigma_i X_i(z^t(\theta\sigma,\lambda)),\ c(0) = \lambda.$$

The equation $\hat{J}(\sigma, y_0(t,\lambda);\lambda) = x$ is converted into $c(1,\sigma,t,\lambda) = x$ and we get the following integral equation

$$\lambda + \sum_{i=1}^{m} \sigma_i \int_0^1 X_i(z^t(s\sigma,\lambda))ds = x$$

for the unknown λ. Here the standard procedure of taking iterations will lead us to the unique solution $\lambda = \psi^t(\sigma,x)$ for each $x \in \mathbb{R}^n$ provided $\sigma \in B(0,\tilde{\rho}) \subset D_m$ and $\tilde{\rho} > 0$ is sufficiently small. Denote $y_0^t(\sigma,x) \overset{\text{def}}{=} y_0(t,\psi^t(\sigma,x))$ and we get

$$y^t(\sigma,x) = \hat{G}(\sigma,\psi^t(\sigma,x);\ y_0^t(\sigma,x)) \tag{15}$$

where $\frac{\partial \hat{G}}{\partial y_0}(\sigma,\lambda;y_0)$ is a $(n+1)\times(n+1)$ nonsingular matrix.

In addition, $y = y_0^t(\sigma,x)$ and $\lambda = \psi^t(\sigma,x)$ are continuously differentiable with respect to $t \in (0,T]$ provided $y_0(t,\lambda)$ exhibits the same smoothness. The mapping $y^t(\sigma,x)$ is first order continuously differentiable with respect to $t \in [t',t''] \subset (0,T]$ and second order continuously differentiable with respect to (σ,x).

Define the exit time of the ball $\tau(\omega) : \Omega \to [0,T]$ such that

$$\sigma(t) = w(t\wedge\tau) \in B(0,\tilde{\rho}) \subset D_m,\ \text{for any } t \in [0,T] \tag{16}$$

and write

$$\tilde{y}(t,x) = y^t(\sigma(t),x), \hat{y}(t,\lambda) = \hat{y}^t(\sigma(t),\lambda), \hat{x}(t,\lambda) = \hat{x}^t(\sigma(t),\lambda). \tag{17}$$

Let $\chi_\tau(t)$ be the characteristic function of τ, i.e.

$$\chi_\tau(t) = \begin{cases} 1 & \text{for } \tau > t,\ t \in [0,T] \\ 0 & \text{for } \tau \leqslant t \end{cases} \tag{18}$$

Using a smooth approximation of the Wiener process $w(t)$, $t \in [0,T]$ we get a stochastic differential of $\tilde{y}(t,x)$ along $x = \hat{x}(t,\lambda)$, say $D_t\tilde{y}(t,\hat{x}(t,\lambda))$ as follows

$$D_t\tilde{y}(t,\hat{x}(t,\lambda)) = d_t\hat{y}(t,\lambda) - \frac{\partial \tilde{y}}{\partial x}(t,\hat{x}(t,\lambda))d_t\hat{x}(t,\lambda) \tag{19}$$

and on the other hand we have

$$\begin{aligned} D_t\tilde{y}(t,\hat{x}(t,\lambda)) =& \frac{\partial y^t}{\partial t}(\sigma(t),\hat{x}(t,\lambda))dt \\ &+ \sum_{i=1}^{m} \chi_\tau(t)F_i(\hat{x}(t,\lambda),\hat{y}(t,\lambda),\partial_x\tilde{p}(t,\hat{x}(t,\lambda))\bigcirc dw_i(t), \end{aligned} \tag{20}$$

for any $t \in [t', t''] \subset (0, T]$, where $\hat{y}(t, \lambda) = \tilde{y}(t, \hat{x}(t, \lambda))$ is used.

The stochastic differential equation (1) is fulfilled for $u = \tilde{u}(t, x)$ along $x = \hat{x}(t, \lambda)$ adding $\chi_\tau(t)$ in the diffusion part of (1) and provided $y = \tilde{y}(t, x) \overset{\text{def}}{=} (\tilde{u}(t, x), \tilde{p}(t, x)) = (\tilde{u}(t, x), \partial_x \tilde{u}(t, x))$ satisfies $\tilde{y}(0, x) = 0$ and

$$\begin{aligned} D_t \tilde{y}(t, \hat{x}(t, \lambda)) =& [\Delta_x \tilde{y}(t, \hat{x}(t, \lambda)) + F(t, \hat{x}((t, \lambda), \hat{y}(t, \lambda), \partial_x \tilde{p}(t, \hat{x}(t, \lambda))] dt \\ &+ \sum_{i=1}^{m} \chi_\tau(t) F_i(\hat{x}(t, \lambda), \hat{y}(t, \lambda), \partial_x \tilde{p}(t, \hat{x}(t, \lambda)) \bigcirc dw_i(t) \end{aligned} \tag{21}$$

where $D_t \tilde{y}(t, \hat{x}(t, \lambda))$ is computed in (20). By abuse of notation we shall denote $D_t \tilde{y}(t, \hat{x}(t, \lambda)) = d_t \tilde{y}(t, \hat{x}(t, \lambda))$ for $t \in [t', t''] \subset (0, T]$.

Using (20) in (21) we get a parabolic equation

$$\frac{\partial y^t}{\partial t}(\sigma(t), \hat{x}(t, \lambda)) = \Delta_x \tilde{y}(t, \hat{x}(t, \lambda)) + F(t, \hat{x}(t, \lambda), \hat{y}(t, \lambda), \partial_x \tilde{p}(t, \hat{x}(t, \lambda))) \tag{22}$$

for $t \in [t', t''] \subset (0, T]$, $\lambda \in \mathbb{R}^n$. Here $F \in \mathbb{R}^{n+1}$ is obtained from the original $f \in \mathbb{R}$ as

$$F(t, x, y, \partial_x p) \overset{\text{def}}{=} \begin{pmatrix} f(t, x, y) \\ [\partial_x f + p \partial_u f + (\partial_x p) \partial_p f](t, x, y) \end{pmatrix}. \tag{23}$$

Using $\hat{y}^t(\sigma, \lambda) \overset{\text{def}}{=} \hat{G}(\sigma, \lambda; y_0(t, \lambda))$ and $y^t(\sigma, x) \overset{\text{def}}{=} \hat{y}^t(\sigma, \psi^t(\sigma, x))$ we compute explicitely the derivatives appearing in (22) and we get

$$\begin{aligned} \frac{\partial y^t}{\partial t}(\sigma(t), \hat{x}(t, \lambda)) =& \frac{\partial \hat{y}^t}{\partial t}(\sigma(t), \lambda) + \frac{\partial \hat{y}^t}{\partial \lambda}(\sigma(t), \lambda) \frac{\partial \psi^t}{\partial t}(\sigma(t), \hat{x}(t, \lambda)) \\ =& \frac{\partial \hat{G}}{\partial y_0}(\sigma(t), \lambda; y_0(t, \lambda)) \frac{\partial y_0}{\partial t}(t, \lambda) + \frac{\partial \hat{y}^t}{\partial \lambda}(\sigma(t), \lambda) \frac{\partial \psi^t}{\partial t}(\sigma(t), \hat{x}(t, \lambda)) \end{aligned} \tag{24}$$

$$\partial_i \tilde{y}(t, \hat{x}(t, \lambda)) = \frac{\partial \hat{y}^t}{\partial \lambda}(\sigma(t), \lambda) \partial_i \psi^t(\sigma(t), \hat{x}(t, \lambda)),\ i \in \{1, \ldots, n\}, \tag{25}$$

$$\begin{aligned} \partial_i^2 \tilde{y}(t, \hat{x}(t, \lambda)) &= \partial_i \left[\frac{\partial \hat{y}^t}{\partial \lambda}(\sigma(t), \tilde{\psi}(t, x)) \partial_i \tilde{\psi}(t, x) \right] \bigg|_{x = \hat{x}(t, \lambda)} \\ &= \left[\left(\frac{\partial^2 \hat{y}^t}{\partial \lambda^2}(\sigma(t), \lambda \right) \partial_i \tilde{\psi}(t, \hat{x}(t, \lambda)), \partial_i \tilde{\psi}(t, \hat{x}(t, \lambda)) \right] \\ &\quad + \frac{\partial \hat{y}^t}{\partial \lambda}(\sigma(t), \lambda) \cdot \partial_i^2 \tilde{\psi}(t, \hat{x}(t, \lambda)). \end{aligned} \tag{26}$$

where $\tilde{\psi}(t, x) \overset{\text{def}}{=} \psi^t(\sigma(t), x)$ and $\dfrac{\partial \tilde{\psi}}{\partial x}(t, \hat{x}(t, \lambda)) \cdot \dfrac{\partial \hat{x}(t, \lambda)}{\partial \lambda} = I_n$.

It is easily seen that

$$\frac{\partial \hat{y}^t}{\partial \lambda}(\sigma(t), \lambda) \overset{\text{def}}{=} \frac{\partial \hat{G}}{\partial y_0}(\sigma(t), \lambda; y_0(t, \lambda)) \cdot \partial_\lambda y_0(t, \lambda) + \frac{\partial \hat{G}}{\partial \lambda}(\sigma(t), \lambda; y_0(t, \lambda))$$

allows one to write

$$\Delta_x\tilde{y}(t,\hat{x}(t,\lambda)) = \frac{\partial\hat{G}}{\partial y_0}(\sigma(t),\lambda;y_0(t,\lambda))\sum_{i=1}^{n}\left(\frac{\partial^2 y_0(t,\lambda)}{\partial\lambda^2}\partial_i\tilde{\psi}(t,\hat{x}(t,\lambda)),\partial_i\tilde{\psi}(t,\hat{x}(t,\lambda))\right) + \frac{\partial\hat{y}^t}{\partial\lambda}(\sigma(t),\lambda)\Delta_x\tilde{\psi}(t,\hat{x}(t,\lambda)) + R(\sigma(t),\lambda,y_0(t,\lambda),\partial_\lambda y_0(t,\lambda)).$$

Using (22) we get the following PDE for $y_0(t,\lambda)$

$$\begin{aligned}\partial_t y_0(t,\lambda) = &\sum_{i=1}^{n}\left(\frac{\partial^2 y_0}{\partial\lambda^2}(t,\lambda)\partial_i\tilde{\psi}(t,\hat{x}(t,\lambda)),\partial_i\tilde{\psi}(t,\hat{x}(t,\lambda))\right)\\ &+ R_0(\sigma(t),\lambda,y_0(t,\lambda),\partial_\lambda y_0(t,\lambda))\left[\Delta_x\tilde{\psi}(t,\hat{x}(t,\lambda) - \frac{\partial\psi^t(\sigma(t),\hat{x}(t,\lambda))}{\partial t}\right]\\ &+ F_0(t,\sigma(t),\hat{x}(t,\lambda),y_0(t,\lambda),\partial_\lambda y_0(t,\lambda))\end{aligned} \tag{27}$$

for any $t\in[t',t'']\subset(0,T]$, $\lambda\in\mathbb{R}^n$, with $y_0(0,\lambda)=0$. Here F_0 is a Lipschitz continuous function of $y_0,\partial_i y_0\in B(0,\rho_1)$ uniformly with respect to $(t,\sigma,x)\in[0,T]\times B(0,\tilde{\rho})\times\mathbb{R}^n$.

Assuming that the vector fields $-X_j(z)\overset{\text{def}}{=}\frac{\partial g_j}{\partial p}(x,u,p)$ are constant, i.e. $X_j(z)=b_j\in\mathbb{R}^n$, $j\in\{1,\ldots,n\}$, then a direct computation leads us to the following equations

$$\begin{aligned}&\hat{x}(t,\lambda)=\lambda+\sum_{j=1}^{m}b_j w_j(t\wedge\tau),\ \tilde{\psi}(t,x)=x-\sum_{j=1}^{m}b_j w_j(t\wedge\tau)=\psi(\sigma(t),x)\\ &\partial_i\tilde{\psi}(t,x)=e_i,\ i\in\{1,\ldots,n\},\ \text{and } \Delta_x\tilde{\psi}(t,\hat{x}(t,\lambda))=0.\end{aligned} \tag{28}$$

The solution of the parabolic system in (27) under the additional hypothesis $X_j(z)=b_j\in\mathbb{R}^n$, $j\in\{1,\ldots,m\}$, is found solving the following integral equations

$$\begin{cases} y_0(t,\lambda)=\int_0^t\left[\int_{\mathbb{R}^n}F_0(s,\sigma(s),\hat{x}(s,\mu),y_0(s,\mu),\partial_\mu y_0(s,\mu))P(t-s,\lambda,\mu)d\mu\right]ds\\ \partial_\lambda y_0(t,\lambda)=\int_0^t\left[\int_{\mathbb{R}^n}F_0(s,\sigma(s),\hat{x}(s,\mu),y_0(s,\mu),\partial_\mu y_0(s,\mu))\partial_\lambda P(t-s,\lambda,\mu)d\mu\right]ds\end{cases} \tag{29}$$

for $t\in[0,a]$, $\lambda\in\mathbb{R}^n$, $0<a\leqslant T$, where $P(\tau,x,y)$, $\tau>0$, is the fundamental solution of the parabolic equation

$$\partial_\tau P(\tau,x,y)=\Delta_x P(\tau,x,y),\ \text{for any } \tau>0, x,y\in\mathbb{R}^n$$

The analysis given above is summarized in the following

Lemma 2.1. Let f, g_i be given such that the hypothesis (4) is fulfilled. Assume the vector fields $\{Z_1(z),\ldots,Z_m(z)\}$ defined in (6) are commuting and $X_j(z)\overset{\text{def}}{=}-\frac{\partial g_j}{\partial p}(z)=b_j\in\mathbb{R}^n$, $j\in\{1,\ldots,m\}$. Then there exists a local pathwise solution fulfilling stochastic PDE (1) along $x=\hat{x}(t,\lambda)$ given in (28) i.e. there exists an $\mathcal{F}_t$-adapted and continuous process

$\tilde{y}(t,x):[0,a]\times\mathbb{R}^n\to\mathbb{R}^{n+1}$ which is second order continuously differentiable with respect to $x\in\mathbb{R}^n$ for each $t\in(0,a]$ such that $\tilde{y}(0,x)=0$, $\tilde{y}(t,\hat{x}(t,\lambda))=\hat{y}(t,\lambda)$ and

$$d_t\tilde{y}(t,\hat{x}(t,\lambda)) = [\Delta_x\tilde{y}(t,\hat{x}(t,\lambda)) + F(t,\hat{x}(t,\lambda),\hat{y}(t,\lambda),\partial_x\tilde{p}(t,\hat{x}(t,\lambda)))]\,dt + \sum_{i=1}^{m}\chi_\tau(t)F_i(\hat{x}(t,\lambda),\hat{y}(t,\lambda),\partial_x\tilde{p}(t,\hat{x}(t,\lambda)))\bigcirc dw_i(t)$$

for $t\in(0,\alpha]$, where $\tilde{y}(t,x)=(\tilde{u}(t,x),\partial_x\tilde{u}(t,x))$

Remark 2.2. If the original functions $g_i(x,y)=g_i(p)$, $p\in\mathbb{R}^n$, $i\in\{1,\dots,m\}$, then $\{Z_1(z),\dots Z_m(z)\}$ commute. □

Remark 2.3. Generally, the vector fields $\{Z_1(z),\dots,Z_m(z),\ z\in D\}$ are not commuting and the basic gradient system in (7) is not solvable by the explicit solution given in (8). Assuming that the real Lie algebra $L(Z_1,\dots,Z_m)\subset C^\infty(D;R^{2n+1})$ determined by $\{Z_1,\dots Z_m\}$ is finite dimensional, there is an extended gradient system associated with (7) for which the explicit solution is written as an orbit of $L(Z_1,\dots Z_m)$ and it allows one to extend the computation contained in Lemma 2.1 to include the noncommuting case. □

Using the same notations as above we replace the hypothesis (4) with the following one

$$\begin{cases} f\in C([0,T];C_b^2(D)),\ g_i\in C_b^\infty(D),\ i\in\{1,\dots,m\},\\ \text{and } L(Z_1,\dots,Z_m)\subset C^\infty(D;\mathbb{R}^{2n+1}) \text{ is finite dimensional},\\ \text{i.e. there exists a system of generators } \{Z_1,\dots,Z_m,Z_{m+1},\dots,Z_M\}\subset L\\ \text{with } M\geqslant m, \text{ such that any } Z\in L \text{ can be written } Z(z)=\sum_{j=1}^{M}\alpha_jZ_j(z), z\in D,\\ \text{where the real constants } \alpha_j \text{ are depending on } Z.\end{cases} \tag{30}$$

Remark 2.4. We notice that any linear functions of $z\stackrel{\text{def}}{=}(u,p,x)$, $g_i(z)=a_i+\langle b_i,z\rangle$, $i\in\{1,\dots,m\}$, generate linear vector fields $Z_i(z)$, $i\in\{1,\dots,m\}$ of $z\in\mathbb{R}^{2n+1}$ and the corresponding Lie algebra $L(Z_1,\dots,Z_m)$ is finite dimensional. □

Assuming the hypothesis (30) fulfilled, we choose an extended system of generators $\{Z_1,\dots Z_m,Z_{m+1},\dots Z_M\}\subset L$ and define the following orbit in L.

$$z^t(\sigma,\lambda)=S_1(t_1)\circ\dots\circ S_M(t_M)(z_0^t(\lambda)), \tag{31}$$

where $\sigma\stackrel{\text{def}}{=}(t_1,\dots,t_M)\in D_M$, $D_M\stackrel{\text{def}}{=}\prod_{j=1}^M(-a_j,a_j)$, $z_0^t(\lambda)\stackrel{\text{def}}{=}(y_0(t,\lambda),\lambda)\in B(0,\rho_1)\times\mathbb{R}^n$, $y_0(t,\lambda)\stackrel{\text{def}}{=}(h(t,\lambda),\ \partial_\lambda h(t,\lambda))$. Here $S_j(\tau)(z_0)$ with $\tau\in(-a_j,a_j)$, $z_0\in B(0,\rho_1)\times\mathbb{R}^n$, is the local flow generated by the smooth vector field Z_j, $j\in\{1,\dots,M\}$. It is a matter of a finite dimensional Lie algebra to get some smooth vector fields $\{q_1,\dots,q_M\}\subset$

$C^\infty(D_M;\mathbb{R}^m)$ (they are analytic functions) such that (see [4], Th.2, p.31)

$$\frac{\partial z^t}{\partial \sigma}(\sigma,\lambda)\cdot q_j(\sigma) = Z_j(z^t(\sigma,\lambda)),\ j\in\{1,\dots,M\},\ z^t(0,\lambda) \overset{\text{def}}{=} z_0^t(\lambda) \tag{32}$$

for each $t\in[0,T]$, $\lambda\in\mathbb{R}^n$.

In addition, the real Lie algebra $L(q_1,\dots q_m)$ determined by the first m vector fields $\{q_1,\dots q_m\}$ is finite dimensional with $\{q_1,\dots,q_M\}$ as a fixed system of generators.

It shows that the fixed orbit of L defined in (31) is the solution for the extended gradient system in (32). Write the vector fields Z_j, $j\in\{1,\dots,M\}$ as

$$Z_j(y,x) \overset{\text{def}}{=} \begin{pmatrix} Y_j(x,y) \\ X_j(x,y)\end{pmatrix} \text{ with } Y_j\in\mathbb{R}^{n+1},\ X_j\in\mathbb{R}^n, \tag{33}$$

for $y\in B(0,\rho)\subset\mathbb{R}^{n+1}$, $x\in\mathbb{R}^n$ and similarly

$$z^t(\sigma,\lambda) = (\hat{y}^t(\sigma,\lambda),\hat{x}^t(\sigma,\lambda)),\ \sigma\in B(0,\tilde{\rho})\subset D_M,\ \lambda\in\mathbb{R}^n,\ t\in[0,T]. \tag{34}$$

Define a smooth mapping $\psi^t(\sigma,x): B(0,\tilde{\rho})\times\mathbb{R}^n\to\mathbb{R}^n$ such that

$$\hat{x}^t(\sigma,\psi^t(\sigma,x)) = x,\quad \psi^t(\sigma,\hat{x}^t(\sigma,\lambda)) = \lambda \tag{35}$$

for each $t\in[0,T]$, and the smooth mapping

$$y^t(\sigma,x) \overset{\text{def}}{=} \hat{y}^t(\sigma,\psi^t(\sigma,x)),\quad \sigma\in B(0,\tilde{\rho})\subset D_M,\ x\in\mathbb{R}^n, \tag{36}$$

fulfills $y^t(\sigma,x) = (u^t(\sigma,x),p^t(\sigma,x))$ with $p^t(\sigma,x) = \partial_x u^t(\sigma,x)$ and it will be the solution for the following Hamilton - Jacobi gradient system

$$\begin{aligned} &\frac{\partial y^t}{\partial \sigma}(\sigma,x)q_j(\sigma) + \frac{\partial y^t}{\partial x}(\sigma,x)\cdot X_j(x,y^t(\sigma,x)) = Y_j(x,y^t(\sigma,x)) \\ &y^t(0,x) = y_0(t,x) \overset{\text{def}}{=} (h(t,x),\partial_x h(t,x)),\ j\in\{1,\dots,M\}\end{aligned} \tag{37}$$

for $(t,x)\in[0,T]\times\mathbb{R}^n$ and $\sigma\in B(0,\tilde{\rho})\subset\mathbb{R}^M$.

Lemma 2.5. Let the smooth functions $g_i(x,u,p)$, $i\in\{1,\dots,m\}$ be given such that the hypothesis (30) is fulfilled. Consider the solution $z^t(\sigma,\lambda)\overset{\text{def}}{=}(\hat{y}^t(\sigma,\lambda),\hat{x}^t(\sigma,\lambda))$ in (31) associated with the gradient system in (32) and define $y^t(\sigma,x)$ as in (36). Then $y^t(\sigma,x) = (u^t(\sigma,x),p^t(\sigma,x))$ with $p^t(\sigma,x) = \partial_x u^t(\sigma,x)$ and

$$\begin{cases} \left\langle \dfrac{\partial u^t}{\partial \sigma}(\sigma,x), q_i(\sigma)\right\rangle = g_i(x,y^t(\sigma,x)) \\ \dfrac{\partial p^t}{\partial \sigma}(\sigma,x)q_i(\sigma) = \partial_x[g_i(x,y^t(\sigma,x))] \overset{\text{def}}{=} H_i(x,y^t(\sigma,x),\partial_x p^t(\sigma,x)) \\ y^t(0,x) = y_0(t,x) = (h(t,x),\partial_x h(t,x)),\ i\in\{1,\dots,m\}\end{cases} \tag{38}$$

for $\sigma \in B(0,\tilde{\rho}) \subset \mathbb{R}^M$, $x \in \mathbb{R}^n$, $t \in [0,T]$, where $y_0(t,x)$, $\partial_i y_0(t,x) \in B(0,\rho_1)$. In addition, there exists a smooth diffeomorphism $G(\sigma,\lambda;y_0)$, $y_0 \in B(0,\rho_1) \subset \mathbb{R}^{n+1}$, $\rho_1 = \rho/2$, such that

$$y^t(\sigma,x) = G(\sigma,\psi^t(\sigma,x);y_0^t(\sigma,x)), \quad G \in C_b^2(\hat{D}) \tag{39}$$

where $\hat{D} = B(0,\tilde{\rho}) \times \mathbb{R}^n \times B(0,\rho_1)$, $y_0^t(\sigma,x) \stackrel{\text{def}}{=} y_0(t,\psi^t(\sigma,x))$.

Proof. By hypothesis, the computations performed in (32)–(37) are valid and using (37) for $j \in \{1,\dots,m\}$ we get $p^t(\sigma,x) = \partial_x u^t(\sigma,x)$ and the system (38) are fulfilled, recalling that the vector fields, X_j, Y_j are defined in (6) for $j \in \{1,\dots,m\}$.

Using (35) we get $y^t(\sigma;\hat{x}^t(\sigma,\lambda)) = \hat{y}^t(\sigma,\lambda)$ and relying on (31) one obtains

$$y^t(\sigma,x) = G(\sigma,\psi^t(\sigma,x);y_0^t(\sigma,x)) \tag{40}$$

where $G(\sigma,\lambda;y_0(t,\lambda)) \stackrel{\text{def}}{=} \hat{y}^t(\sigma,\lambda)$. The proof is complete. □

Remark 2.6. The smooth mapping $\psi^t(\sigma,x)$, $t \in [t',t''] \subset (0,T]$, fulfilling (35) is continuously differentiable in the variable t provided $y_0(t,\lambda) \stackrel{\text{def}}{=} (h(t,\lambda),\partial_\lambda h(t,\lambda))$ is smooth. It can be easily seen writting the application $\hat{x}^t(\sigma,\lambda)$ as $\hat{x}^t(\sigma,\lambda) = J(\sigma,y_0(t,\lambda);\lambda)$ where the smooth mapping $\lambda = \psi^t(\sigma,x)$ in Lemma 2.5 is obtained as the unique solution of the following algebraic equations

$$x = J(\sigma,y_0(t,\psi^t(\sigma,x));\psi^t(\sigma,x)).$$

According to the last equations we get a smooth mapping $\psi^t(\sigma,x)$ with respect to $t \in [t',t'']$ provided $y_0(t,\lambda)$ is continuously differentiable of $(t,\lambda) \in [t',t''] \times \mathbb{R}^n$ and as a consequence $y_0^t(\sigma,x) \stackrel{\text{def}}{=} y_0(t,\psi^t(\sigma,x))$ is continuously differentiable with respect to $t \in [t',t'']$. Here we use the same standard arguments as we did before for commuting vector fields. □

Using the same elementary computation we get $y^t(\sigma,x)$ defined in (40) as a continuously differentiable mapping of $t \in [t',t''] \subset (0,T]$, $\sigma \in B(0,\tilde{\rho}) \subset \mathbb{R}^M$ and $x \in \mathbb{R}^n$.

Remark 2.7. The next step is to find $\sigma = \sigma(t)$, $t \in [0,T]$ as a continuous and $\mathcal{F}_t$-adapted process such that $\sigma(t) \in B(0,\tilde{\rho}) \subset D_M \subset \mathbb{R}^M$ for any $t \in [0,T]$ and $z(t,\lambda) = (\hat{y}(t,\lambda),\hat{x}(t,\lambda)) \stackrel{\text{def}}{=} S(\sigma(t);z_0(t,\lambda)) = (\hat{y}^t(\sigma(t),\lambda),\hat{x}^t(\sigma(t),\lambda))$ satisfies the following stochastic differential equations

$$\begin{cases} d_t\hat{y}(t,\lambda) = \sum_{i=1}^m \chi_\tau(t) Y_i(\hat{x}(t,\lambda),\hat{y}(t,\lambda)) \bigcirc dw_i(t) + \dfrac{\partial \hat{y}^t}{\partial t}(\sigma(t),\lambda)dt \\ d_t\hat{x}(t,\lambda) = \sum_{i=1}^m \chi_\tau(t) X_i(\hat{x}(t,\lambda),\hat{y}(t,\lambda)) \bigcirc dw_i(t) + \dfrac{\partial \hat{x}^t}{\partial t}(\sigma(t),\lambda)dt, \end{cases} \tag{41}$$

for $t \in [t',t''] \subset (0,T]$, where the diffeomorphism application $S(\sigma;z_0)$ is defined in (31) and the corresponding vector fields X_j, Y_j, $j \in \{1,\dots,M\}$, are given in (33). Here we use the characteristic function $\chi_\tau(t)$, $t \in [0,T]$, of a stopping time $\tau(\omega) : \Omega \to [0,T]$ taken with respect to the given filtration $\{\mathcal{F}_t\}\uparrow$ such that the solution associated with

$$d_t\sigma = \sum_{i=1}^{m} \chi_\tau(t) q_i(\sigma) \bigcirc dw_i(t), \quad t \in [0,T],\ \sigma(0) = 0, \tag{42}$$

for q_j, $j \in \{1,\dots,M\}$, defined in (32), verifies $\sigma(t) \in B(0,\tilde{\rho})$. Using (42) and $\hat{y}^t(\sigma,\lambda) = G(\sigma,\lambda; y_0(t,\lambda))$, $\hat{x}^t(\sigma,\lambda) = J(\sigma, y_0(t,\lambda);\lambda)$ for some smooth mappings G and J we get the equations (41) fulfilled, applying the stochastic rule of differentiation for the process $\sigma = \sigma(t)$, $y = y_0(t,\lambda)$ and using G and J. $\square$

In addition, using the mapping $y^t(\sigma,x)$ defined in Lemma 2.5 we get $y^t(\sigma(t),\hat{x}(t,\lambda)) = \hat{y}(t,\lambda)$ and it allows us to express the differential $d_t\tilde{y}(t,x)$ along $x = \hat{x}(t,\lambda)$ of $\tilde{y}(t,x) \overset{\text{def}}{=} y^t(\sigma(t),x)$ as $d_t\tilde{y}(t,\hat{x}(t,\lambda)) = [d_t y^t(\sigma(t),x)]_{x=\hat{x}(t,\lambda)}$ and we get

Lemma 2.8. Let the hypothesis (30) be fulfilled and define $\tilde{y}(t,x) \overset{\text{def}}{=} y^t(\sigma(t),x)$, $\hat{x}(t,\lambda) = \hat{x}^t(\sigma(t),\lambda)$, where $\hat{y}^t(\sigma)$, $\hat{x}^t(\sigma,\lambda)$ and $y^t(\sigma,x)$ are defined in Lemma 2.5. Then the continuous and $\mathcal{F}_t$-adapted process $\tilde{y}(t,x)$, $t \in [t',t''] \subset (0,T]$, satisfies the following SDE $d_t\tilde{y}(t,\hat{x}(t,\lambda)) = d_t\hat{y}(t,\lambda) - \dfrac{\partial\tilde{y}}{\partial x}(t,\hat{x}(t,\lambda)) d_t\hat{x}(t,\lambda)$ where $\hat{y}(t,\lambda) = \tilde{y}(t,\hat{x}(t,\lambda))$ and

$$\begin{aligned} d_t\tilde{y}(t,\hat{x}(t,\lambda)) =& \sum_{j=1}^{m} \chi_\tau(t) \frac{\partial y^t}{\partial\sigma}(\sigma(t),\hat{x}(t,\lambda)) q_j(\sigma(t)) \bigcirc dw_j(t) + \frac{\partial y^t}{\partial t}(\sigma(t),\hat{x}(t,\lambda))dt \\ =& \sum_{j=1}^{m} \chi_\tau(t) F_j(\hat{x}(t,\lambda),\hat{y}(t,\lambda),\partial_x\tilde{p}(t,\hat{x}(t,\lambda))) \bigcirc dw_j(t) \\ &+ \frac{\partial y^t}{\partial t}(\sigma(t),\hat{x}(t,\lambda))dt \end{aligned} \tag{43}$$

where the continuous process $\sigma = \sigma(t)$ satisfies the equation (42) in Remark 2.7 and

$$F_i(x,y,\partial_x p) = \begin{pmatrix} g_i(x,y) \\ H_i(x,y,\partial_x p) \end{pmatrix}, \quad i \in \{1,\dots,m\}.$$

Proof. By hypothesis, the properties given in Lemma 2.5 hold true and we get a continuously differentiable mapping $y^t(\sigma,x) \overset{\text{def}}{=} \hat{y}^t(\sigma,\psi^t(\sigma,x))$, for $t \in [t',t''] \subset (0,T]$, $\sigma \in B(0,\tilde{\rho}) \subset D_M \subset \mathbb{R}^M$, $x \in \mathbb{R}^n$ such that $\tilde{y}(t,\hat{x}(t,\lambda)) = \hat{y}(t,\lambda)$. Using a smooth approximation of the Wiener process $w(t)$, $t \in [0,T]$, and the Hamilton-Jacobi gradient system in (37) we get the conclusion. The proof is complete. $\square$

Remark 2.9. Using $\hat{y}^t(\sigma,\lambda) = G(\sigma,\lambda; y_0(t,\lambda))$ and $\hat{x}^t(\sigma,\lambda) = J(\sigma, y_0(t,\lambda);\lambda)$ as in Remark 2.7 we are looking for $y_0(t,\lambda)$ such that

$$\begin{aligned} d_t\tilde{y}(t,\hat{x}(t,\lambda)) =& [\Delta_x\tilde{y}(t,\hat{x}(t,\lambda)) + F(t,\hat{x}(t,\lambda),\hat{y}(t,\lambda),\partial_x\tilde{p}(t,\hat{x}(t,\lambda)))]dt \\ &+ \sum_{j=1}^{m} \chi_\tau(t) F_i(\hat{x}(t,\lambda),\hat{y}(t,\lambda),\partial_x\tilde{p}(t,\hat{x}(t,\lambda))) \bigcirc dw_j(t) \end{aligned} \tag{44}$$

where

$$F(t,x,y,\partial_x p) \stackrel{\text{def}}{=} \begin{pmatrix} f(t,x,y) \\ [\partial_x f + p\partial_u f + (\partial_x p)\partial_p f](t,x,y) \end{pmatrix} \quad \square \tag{45}$$

3 Main results

Let $f(t,x,u,p)$, $g_i(x,u,p)$, $i \in \{1,\dots,m\}$ be given such that the hypothesis (30) is fulfilled for $t \in [0,T]$, $x \in \mathbb{R}^n$, $y \stackrel{\text{def}}{=} (u,p) \in B(0,\rho) \subset \mathbb{R}^{n+1}$ with $\rho > 0$ fixed. Consider the original SPDE (1) and associate the following extended SPDE for the unknown $y(t,x) \stackrel{\text{def}}{=} (u(t,x), \partial_x u(t,x))$

$$\begin{cases} d_t y(t,x) = [\Delta_x y(t,x) + F(t,x,y(t,x),\partial_x y(t,x))]dt \\ \qquad\qquad + \sum_{i=1}^{m} \chi_\tau(t) F_i(x,y(t,x),\partial_x y(t,x)) \bigcirc dw_i(t),\ t \in (0,T] \\ y(0,x) = 0,\ x \in \mathbb{R}^n \end{cases} \tag{46}$$

where $F, F_i \in \mathbb{R}^{n+1}$ are defined by f and g_i correspondingly

$$\begin{aligned} F(t,x,y,\partial_x y) &\stackrel{\text{def}}{=} \begin{pmatrix} f(t,x,y) \\ [\partial_x f + p\partial_u f + (\partial_x p)\partial_p f](t,x,y) \end{pmatrix} \\ F_i(x,y,\partial_x y) &\stackrel{\text{def}}{=} \begin{pmatrix} g_i(x,y) \\ [\partial_x g_i + p\partial_u g_i + (\partial_x p)\partial_p g_i](x,y) \end{pmatrix},\ i \in \{1,\dots,m\} \end{aligned} \tag{47}$$

and $\tau : \Omega \to [0,T]$ is a stopping time with respect to the given filtration $\{\mathcal{F}_t\}\uparrow$, i.e. $\{\tau > t\} \in \mathcal{F}_t$, $t \in [0,T]$.

Definition 3.1. We say that $y(t,x) = (u(t,x), p(t,x)) \in \mathbb{R}^{n+1}$, $t \in [0,a]$, $a \leqslant T$, $x \in \mathbb{R}^n$ is a solution for (46) along $x = \hat{x}(t,\lambda)$ if $p(t,x) = \partial_x u(t,x)$, $y(t,x) \in B(0,\rho) \subset \mathbb{R}^{n+1}$ and $y(t,\hat{x}(t,\lambda)) = \hat{y}(t,\lambda)$ such that

(i) $y(t,x)$, $t \in [0,a]$, is a continuous and $\mathcal{F}_t$-adapted process for each $x \in \mathbb{R}^n$ and is continuously differentiable with respect to x such that $\partial_i y(t,x) \stackrel{\text{def}}{=} \frac{\partial y}{\partial x_i}(t,x) \in B(0,\rho)$ is a continuous process for $t \in [0,a]$, $i \in \{1,\dots,n\}$;

(ii) $y(t,x)$ is second order continuously differentiable with respect to $x \in \mathbb{R}^n$ and $\Delta_x y(t,x)$ is a continuous and $\mathcal{F}_t$-adapted process for any $t \in (0,a]$;

(iii) there exists a stopping time $\tau : \Omega \to [0,T]$ such that SPDE (46) is fulfilled along $x = \hat{x}(t,\lambda)$ i.e. it holds $d_t y(t,\hat{x}(t,\lambda)) = d_t \hat{y}(t,\lambda) - \frac{\partial y}{\partial x}(t,\hat{x}(t,\lambda)) \cdot d_t \hat{x}(t,\lambda)$ and

$$\begin{aligned} d_t y(t,\hat{x}(t,\lambda)) = {} & [\Delta_x y(t,\hat{x}(t,\lambda)) + F(t,\hat{x}(t,\lambda),\hat{y}(t,\lambda),\partial_x p(t,\hat{x}(t,\lambda))]\, dt \\ & + \sum_{i=1}^{m} \chi_\tau(t) F_i(\hat{x}(t,\lambda),\hat{y}(t,\lambda),\partial_x p(t,\hat{x}(t,\lambda)) \bigcirc dw_i(t) \end{aligned} \tag{48}$$

for any $t \in [t', t''] \subset (0, a]$, $\lambda \in \mathbb{R}^n$, where the continuous processes $\hat{x}(t, \lambda)$ and $\hat{y}(t, \lambda)$ are defined by (41) in Remark 2.7. □

A solution for SPDE (46) along $x = \hat{x}(t, \lambda)$ is constructed using the continuous processes $y = \hat{y}(t, \lambda)$, $x = \hat{x}(t, \lambda)$ fulfilling the stochastic differential equations given in Remark 2.7, and by definition $\hat{y}(t, \lambda) \stackrel{\text{def}}{=} G(\sigma(t), \lambda); y_0(t, \lambda))$, $\hat{x}(t, \lambda) \stackrel{\text{def}}{=} J(\sigma(t), y_0(t, \lambda); \lambda)$, where the continuous process $\sigma = \sigma(t)$, $t \in [0, T]$, satisfies the following SDE

$$d_t\sigma = \sum_{j=1}^{m} \chi_\tau(t) q_j(\sigma) \bigcirc dw_j(t),\ t \in [0, T],\ \sigma(0) = 0. \tag{49}$$

Here the smooth vector fields $q_j \in \mathbb{R}^M$, $j \in \{1, \ldots, M\}$, are given in (32) and the diffeomorphism mappings $y = G(\sigma, \lambda; y_0) \in B(0, \rho)$, $x = J(\sigma, y_0; \lambda)$ for $(\sigma, \lambda, y_0) \in B(0, \tilde{\rho}) \times \mathbb{R}^n \times B(0, \rho_1)$, $\rho_1 = \frac{\rho}{2}$, are defined in Lemma 2.5.

We choose $y_0(t, \lambda) \stackrel{\text{def}}{=} (h(t, \lambda), \partial_\lambda h(t, \lambda))$ as a smooth function fulfilling $y_0(t, \lambda)$, $\partial_i y_0(t, \lambda) \in B(0, \rho_1)$, for $t \in [0, T]$, $\lambda \in \mathbb{R}^n$, and define $\lambda = \psi^t(\sigma, x)$, $(\sigma, x) \in B(0, \tilde{\rho}) \times \mathbb{R}^n$, $t \in [0, T]$, as the unique solution fulfilling $J(\sigma, y_0(t, \lambda); \lambda) = x$.

Define the smooth function $y^t(\sigma, x) = G(\sigma, \psi^t(\sigma, x); y_0^t(\sigma, x))$, $y_0^t(\sigma, x) \stackrel{\text{def}}{=} y_0(t, \psi^t(\sigma, x))$, for $(\sigma, x, t) \in B(0, \tilde{\rho}) \times \mathbb{R}^n \times [0, T]$, as in Lemma 2.8 and write

$$\tilde{y}(t, x) = y^t(\sigma(t), x),\ \hat{y}^t(\sigma, \lambda) = G(\sigma, \lambda; y_0(t, \lambda)) \tag{50}$$

for $t \in [0, T]$, $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\sigma \in B(0, \tilde{\rho}) \subset \mathbb{R}^M$, where $\sigma = \sigma(t)$ satisfies SDE (49).

Theorem 3.2. Let f, g_i, $i \in \{1, \ldots, m\}$, be given such that the hypothesis (30) is fulfilled and $\partial_p g_i(x, u, p) = b_i \in \mathbb{R}^n$, $i \in \{1, \ldots, m\}$, are constant vectors. Then the mapping $y = \tilde{y}(t, x)$, $t \in [0, a]$, $x \in \mathbb{R}^n$, defined in (50) is a solution of s.p.d.e. (46) along $x = \hat{x}(t, \lambda)$ provided $y = y_0(t, \lambda)$ is a solution of the following parabolic equation

$$\begin{cases} \partial_t y_0(t, \lambda) = \Delta_\lambda y_0(t, \lambda) + F_0(t, \lambda, y_0(t, \lambda), \partial_\lambda y_0(t, \lambda)),\ \forall t \in (0, a], \\ y_0(0, \lambda) = 0,\ \lambda \in \mathbb{R}^n, \end{cases} \tag{51}$$

where $F_0(t, \lambda, y_0, \partial_\lambda y_0)$ is obtained as a continuous function performing the computations (53)–(58) given below.

Proof. By hypotheses, the properties stipulated in Lemma 2.8 hold true for $y = \tilde{y}(t, x)$ and $(\hat{y}^t(\sigma, \lambda), \hat{x}^t(\sigma, \lambda)) \stackrel{\text{def}}{=} z^t(\sigma, \lambda)$ defined in Lemma 2.5 as smooth mappings $\hat{y}^t(\sigma, \lambda) = G(\sigma, \lambda; y_0(t, \lambda))$, $\hat{x}^t(\sigma, \lambda) = J(\sigma, y_0(t, \lambda); \lambda)$. By definition $\tilde{y}(t, x) = \hat{y}^t(\sigma(t), \psi^t(\sigma(t), x))$, $\psi^t(\sigma(t), \hat{x}(t, \lambda)) \equiv \lambda$, and $\tilde{y}(t, x)$ is a solution for SPDE (46) along $x = \hat{x}(t, \lambda)$ provided $\tilde{y}(0, x) \stackrel{\text{def}}{=} y_0(0, x) = 0$, $x \in \mathbb{R}^n$, and

$$\begin{aligned} \frac{\partial \hat{y}^t}{\partial t}(\sigma(t), \lambda) + \frac{\partial \hat{y}^t}{\partial \lambda}(\sigma(t), \lambda) \frac{\partial \psi^t}{\partial t}(\sigma(t), \hat{x}(t, \lambda)) =& \Delta_x \tilde{y}(t, \hat{x}(t, \lambda)) \\ &+ F(t, \hat{x}(t, \lambda), \hat{y}(t, \lambda), \partial_x \tilde{p}(t, \hat{x}(t, \lambda))) \end{aligned} \tag{52}$$

for any $t \in (0, a]$, $\lambda \in \mathbb{R}^n$.

By a direct computation we get

$$\frac{\partial \hat{y}^t}{\partial t}(\sigma(t), \lambda) = \frac{\partial G}{\partial y_0}(\sigma(t), \lambda; y_0(t, \lambda))\partial_t y_0(t, \lambda) \tag{53}$$

$$\frac{\partial \tilde{y}}{\partial x}(t, x) = \frac{\partial \hat{y}^t}{\partial \lambda}(\sigma(t), \psi^t(\sigma(t), x))\frac{\partial \psi^t}{\partial x}(\sigma(t), x) \tag{54}$$

and $\tilde{\psi}(t, x) \overset{\text{def}}{=} \psi^t(\sigma(t), x) = x - \sum_{j=1}^m b_j w_j(t \wedge \tau)$.

As a consequence, it follows

$$\frac{\partial \psi^t}{\partial t}(\sigma(t), \hat{x}(t, \lambda)) = 0, \quad \frac{\partial \psi^t}{\partial x}(\sigma(t), x) = E_n, \quad \frac{\partial^2 \psi^t}{\partial x_i^2}(\sigma(t), x) = 0 \tag{55}$$

for any $i \in \{1, \ldots, n\}$, and

$$\frac{\partial^2 \tilde{y}}{\partial x_i^2}(t, \hat{x}(t, \lambda)) = \frac{\partial^2 \hat{y}^t}{\partial \lambda_i^2}(\sigma(t), \lambda), \quad \frac{\partial \tilde{y}}{\partial x}(t, \hat{x}(t, \lambda)) = \frac{\partial \hat{y}^t}{\partial \lambda}(\sigma(t), \lambda)). \tag{56}$$

Using

$$\frac{\partial \hat{y}^t}{\partial \lambda}(\sigma(t, \lambda)) = \frac{\partial G}{\partial y_0}(\sigma(t), \lambda; y_0(t, \lambda))\partial_\lambda y_0(t, \lambda) + \frac{\partial G}{\partial \lambda}(\sigma(t), \lambda; y_0(t, \lambda)) \tag{57}$$

and

$$\begin{aligned}\frac{\partial^2 \hat{y}^t}{\partial \lambda_i^2}(\sigma(t), \lambda) =& \frac{\partial G}{\partial y_0}(\sigma(t), \lambda; y_0(t, \lambda))\partial_i^2 y_0(t, \lambda) \\ &+ 2\frac{\partial^2 G_0}{\partial \lambda_i \partial y_0}(\sigma(t), \lambda; y_0(t, \lambda))\partial_i y_0(t, \lambda) + \frac{\partial^2 G}{\partial \lambda_i^2}(\sigma(t), \lambda; y_0(t, \lambda)) \\ &+ \left[\frac{\partial^2 G}{\partial y_0^2}(\sigma(t), \lambda; y_0(t, \lambda))\partial_i y_0(t, \lambda), \partial_i y_0(t, \lambda))\right]\end{aligned} \tag{58}$$

we rewrite PDE (52) as in (51) and the proof is complete. □

Definition 3.3. We say that $u = u(t, x) \in \mathbb{R}$, $t \in [0, a]$, $x \in \mathbb{R}^n$, is a solution of SPDE (1) along $x = \hat{x}(t, \lambda)$ provided $y(t, x) = (u(t, x), \partial_x u(t, x))$, $t \in [0, a]$, $x \in \mathbb{R}^n$, is a solution of the extended SPDE (46) along $x = \hat{x}(t, \lambda)$. □

Theorem 3.4. Let f, g_i be given fulfilling the hypothesis (30) and $\partial_p g_i(x, u, p) = b_i \in \mathbb{R}^n$, $i \in \{1, \ldots, m\}$ are constant vectors. Let $y_0(t, \lambda) \overset{\text{def}}{=} (h(t, \lambda), \partial_\lambda h(t, \lambda))$ be the unique solution fulfilling the associated PDE (51) given in Theorem 3.2. Define $\tilde{y}(t, x) \overset{\text{def}}{=} \hat{y}^t(\sigma(t), \psi(\sigma(t), x))$, $t \in [0, a]$, $x \in \mathbb{R}^n$, where $\psi(\sigma(t), x) \overset{\text{def}}{=} x - \sum_{j=1}^m b_j w_j(t \wedge \tau)$. Then $\tilde{y}(t, x) = (\tilde{u}(t, x), \partial_x \tilde{u}(t, x))$, $t \in [0, a]$, $x \in \mathbb{R}^n$, is a solution of SPDE (46) along $x = \hat{x}(t, \lambda) \overset{\text{def}}{=} \hat{x}^t(\sigma(t), \lambda)$ and in particular, $u = \tilde{u}(t, x)$, $t \in [0, a]$, $x \in \mathbb{R}^n$, is a solution of SPDE (1) along $x = \hat{x}(t, \lambda)$.

Proof. By hypotheses, the conclusion of Theorem 3.2 holds true and associate PDE (51) for the unknown $y_0(t, \lambda) = (h(t, \lambda), \partial_\lambda h(t, \lambda)) \in \mathbb{R}^{n+1}$, $t \in [0, a]$.

The parabolic system in (51) is solved using the associated integral equations for the unknowns $y_0(t,\lambda)$, $\partial_\lambda y_0(t,\lambda)$ via the fundamental solution of a linear parabolic equation

$$\partial_\tau P(\tau,\lambda,\mu) = \Delta_\lambda P(\tau,\lambda,\mu),\ \tau > 0,\ \lambda,\mu \in \mathbb{R}^n. \tag{59}$$

Let $(\hat{y}^t(\sigma,\lambda), \hat{x}^t(\sigma,\lambda)) = z^t(\sigma,\lambda)$ be the solution given in Lemma 2.5 and define $\tilde{y}(t,x) = \hat{y}^t(\sigma(t), \psi(\sigma(t),x))$, $t \in [0,a]$, $x \in \mathbb{R}^n$ where $\sigma = \sigma(t)$ satisfies the SDE $d_t\sigma(t) = \sum_{j=1}^m \chi_\tau(t) q_j(\sigma(t)) \bigcirc dw_j(t)$, $t \in [0,T]$, $\sigma(0) = 0$ given in the Remark 2.7. Applying Theorem 3.2 we get the conclusion. The proof is complete. □

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References

[1] Pierre Lions and Panagiotis E. Souganidis: *Uniqueness of weak solutions of fully nonlinear stochastic partial differential equations*, C.R. Acad. Sci, Paris, t.331, Série I, 2000, pp. 783–790.

[2] Bogdan Iftimie: *Qualitative Theory for Diffusion Equations with Applications in Physics, Economy and Techniques*, Doctoral Thesis, Institute of Mathematics, Romanian Academy of Sciences, 2001.

[3] R. Racke: *Lectures on Nonlinear Evolution Equations, Aspects of Mathematics*, Vieweg, Berlin, 1992.

[4] Constantin Varsan: *On Evolution Systems of Differential Equations with Stochastic Perturbations*, Preprint No. 4/2001, IMAR, ISSN-02503638.

[5] Constantin Varsan: *Applications of Lie Algebras to Hyperbolic and Stochastic Differential Equations*, Kluwer Academic Publishers, Holland, 1999.

[6] Constantin Varsan and Cristina Sburlan: *Basics of Equations of Mathematical Physics and Differential Equations*, Ex Ponto, Constantza, 2000.