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An essay on model theory

Ludomir Newelski*

Mathematical Institute,
University of Wroclaw,
pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland

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Abstract: Some basic ideas of model theory are presented and a personal outlook on its perspectives is given.

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1 Introduction

This essay is intended for non-model-theorists, or even non-logicians. Model theory has its own language, mostly unknown to common mathematicians, therefore a large part of this paper is devoted to explaining the basic notions. We will illustrate the definitions with examples. There are many excellent general textbooks on model theory, both introductory [CK, Sa] and advanced [Ho, Pi1, Ba, Ma]. The reader may consult them for details. The starting point in model theory is the notion of a model.

A **model**, or a structure, is a set M (called the universe) with some distinguished relations and functions on it, and also some constants (distinguished elements). So we write

$$M = (M; R_i, f_j, c_t)_{i \in I, j \in J, t \in T},$$

where I, J, T are some sets of indices, R_i is an n_i-ary relation on M, f_j is an k_j-argument function on M, with values in M, and $c_t \in M$. Examples are the field of complex numbers $\mathbb{C} = (\mathbb{C}; +, -, \cdot, 0, 1)$, the ordered field of reals $\mathbb{R} = (\mathbb{R}; +, -, \cdot, 0, 1, <)$ or a vector space $V = (V; +, -, 0, r)_{r \in F}$ over a field F (here r denotes a unary function, multiplication by scalar r).

* E-mail: newelski@math.uni.wroc.pl

The set of symbols $\{R_i, f_j, c_t\}_{i\in I, j\in J, t\in T}$ is called the language L or the signature of the model M. Besides M we consider also other models for L, that is models of the form

$$N = (N; R_i^N, f_j^N, c_t^N)_{i\in I, j\in J, t\in T},$$

where the symbols R_i, f_j, c_t of L are interpreted as relations R_i^N, functions f_j^N (of prescribed arity) and constants c_t^N. Using the symbols of L we can write expressions denoting elements of M (called terms) and formulas expressing properties of these elements.

Terms of the language L are formed from variables and constant symbols of L by applying to them function symbols of L. They denote elements of M or functions on M obtained by composing the functions interpreting the respective function symbols. For example in a field $F = (F, +, -, \cdot, 0, 1)$ terms are polynomial expressions like $x \cdot x - y + 1,\ 1 + 1 \cdot 0$.

Formulas of the language L are formed from the symbols of L, logical connectives, quantifiers, variables (ranging over M), parentheses and the equality symbol =. They express some properties of the elements of M. Sentences are formulas without free variables. They express properties of M. These formulas are called "first order", since quantifying is allowed only over the elements of the model (and not over subsets of the model).

For example, in a field $F = (F, +, -, \cdot, 0, 1)$ the formula $\varphi(x) = \exists y(y \cdot y = x)$ expresses the property of x being a square, while the sentence $(\forall x)\varphi(x)$ says, that every element is a square.

The rationale behind this approach is that a model M is regarded as an approximation to a part of mathematical world. This works well in algebra, which deals explicitly with algebraic structures of various kinds. They fit well into this scheme. More problematic is the case of analysis or geometry, where people consider some "higher order" properties of the mathematical world. For example, in analysis, the basic model is the ordered field of real numbers $\mathbb{R} = (\mathbb{R}; +, -, \cdot, 0, 1, <)$, and the sentence "every bounded set of reals has a supremum" can not be expressed by a sentence in the language $L = \{+, -, \cdot, 0, 1, <\}$ of the above kind, since it quantifies over subsets of the model (hence it is of second order).

On the other hand one could argue that mathematics may be formalized within set theory and in this very general set-up every mathematical sentence can be written in a first-order way. Model theory does not claim, however, to apply to all areas of mathematics. Still, despite its restricted outlook on mathematics, model theory has some general results on the logical foundations of mathematics and also some applications in classical mathematics.

The **theory** of M is the set $Th(M)$ of all sentences of L true in M. Two models M, N for L are called elementarily equivalent if $Th(M) = Th(N)$. We say that M is an elementary submodel of M (and N is an elementary extension of M, symbolically expressed as $M \prec N$) if M is a submodel of N (that is, M is a subset of N and the interpretations of the symbols of L in M are induced by the interpretations in N) and for every finite tuple $\overline{a}$ of elements of M and for every formula $\varphi(\overline{x})$ of L, $M \models \varphi(\overline{a})$ (that is, $\overline{a}$ satisfies $\varphi(\overline{x})$ in M) iff $N \models \varphi(\overline{a})$.

Given $T = Th(M)$ we consider the class of all models of T. Describing the structure of models of T is one of the main tasks of model theory. Usually, we consider only infinite models M. More specifically, people consider here the questions on the number of models of T in a given cardinality, and on the fine structure of the models (classification problems). Also, various ways of constructing models are investigated here. We say that T is categorical in cardinality κ ($\kappa \geq \aleph_0$) if T has a single model of power κ, up to isomorphism. The following theorem of Morley [Mo], answering a conjecture of Łoś, was a breakthrough in model theory.

Theorem 1.1. If a countable theory T is categorical in some uncountable power, then it is categorical in every uncountable power.

The non-trivial proof of this theorem uses, among others, the topological and combinatorial properties of the space of types of T, which we will define below. When investigating a model M, we are interested mainly in those subsets of M, that are definable.

A **definable** set in M is any subset of M^n of the form

$$\varphi(M,\overline{a}) = \{\overline{b} \in M^n : M \models \varphi(\overline{b},\overline{a})\},$$

where $\varphi(\overline{x},\overline{y})$ is a formula of L and $\overline{a}$ is a tuple of elements of M (called parameters in this context). If $\overline{a} \subseteq A \subseteq M$, we say that $\varphi(M,\overline{a})$ is A-definable. A-definable subsets of M^n form a Boolean algebra of sets $Def^n_A(M)$. One of the basic goals of model theory is to understand which sets are definable in M. The nicest case is where all definable sets in M are definable by formulas without quantifiers. In this case we say that M (and its theory $Th(M)$) has elimination of quantifiers. Many important structures eliminate quantifiers. For example, there are algebraically closed fields, the ordered field of reals, atomless Boolean algebras and infinite dense linear orders. In several cases, there is a "partial" quantifier elimination, where the definable sets in M are defined by formulas of a specified simple form. For example, in any module definable sets are Boolean combinations of pp-definable sets (which are projections of solutions sets of systems of linear equations). In algebraically closed fields, definable sets are Boolean combinations of affine algebraic varieties (and are called constructible sets in algebraic geometry). Understanding the definable sets enables the further model-theoretic analysis of M.

Types of elements are the central notion of model theory. A **type** $p(x)$ in variable x, over $A \subseteq M$, is any set of formulas $\varphi(x,\overline{a})$, where $\overline{a} \subseteq A$ and $\varphi(x,\overline{y})$ is a formula of L. The type $p(x)$ determines the set

$$p(M) = \{b \in M : M \models \varphi(b,\overline{a}) \text{ for every } \varphi(x,\overline{a}) \in p\} = \bigcap_{\varphi \in p} \varphi(M).$$

We call the set $p(M)$ type-definable (over A). We say that any element of $p(M)$ realizes the type $p(x)$.

We say that the type $p(x)$ is consistent, if the set $\{\varphi(M,\overline{a}) : \varphi(x,\overline{a}) \in p(x)\}$ generates a proper filter in $Def^1_A(M)$. A maximal consistent type over A is called complete. $S_x(A)$ denotes the set of all complete types over A, in variable x. $S_x(A)$ may be identified with the Stone space of ultrafilters of $Def_A^n(M)$, and under this identification it becomes a compact topological space (with the Stone topology). Similarly we define types in any tuple of variables, e.g. $\langle x_\alpha \rangle_{\alpha \in \mu}$, where μ is a cardinal.

Assume $N \succ M$ and $a \in N$. The type of a over A is the set

$$tp^N(a/A) = \{\varphi(x, \overline{b}) : \varphi(x, \overline{y}) \text{ is an } L\text{-formula}, \overline{b} \subseteq A \text{ and } N \models \varphi(a, \overline{b})\}.$$

Therefore $tp^N(a/A) \in S_x(A)$. Moreover, any type $p(x) \in S_x(A)$ is of this form (that is, it is realized in some $N \succ M$). Therefore the type $p(x)$ may be regarded as a list of properties of a "variable" element x of a model.

For $A \subseteq M$, $Aut(M/A)$ denotes the group of automorphisms of M fixing A pointwise. $Aut(M)$ is $Aut(M/\emptyset)$. Clearly, if $a, b \in M$ are in the same orbit of $Aut(M/A)$, then they have the same type over A. In some models the converse is true, at least when $|A| < \kappa$ for some cardinal κ. Such models are called strongly κ-homogeneous. A model M is called κ-saturated if for every $A \subseteq M$ of power $< \kappa$, every type in $S_x(A)$ is realized in M.

Let $\overline{\kappa}$ be a "large" cardinal (that is, a cardinal larger than any cardinal we really would want to deal with). We say that a model $\mathfrak{C}$ is a **monster model**, if $\mathfrak{C}$ is $\overline{\kappa}$-saturated and strongly $\overline{\kappa}$-homogeneous. Any theory T has a monster model, usually denoted by $\mathfrak{C}$. Any subset of $\mathfrak{C}$ of power $< \overline{\kappa}$ is called small, and we tacitly assume that all sets of parameters we consider and all models (except $\mathfrak{C}$) are small. $\mathfrak{C}$ is useful because of the following properties.

- Every (small) model of T is isomorphic to an elementary submodel of $\mathfrak{C}$.
- For every (small) $A \subseteq \mathfrak{C}$, the types in $S_x(A)$ correspond precisely to the orbits of $Aut(\mathfrak{C}/A)$.

So if we want to study small submodels of T, we can restrict ourselves to elementary submodels of $\mathfrak{C}$. This concept will be used throughout the rest of the paper.

If we think of a type $p(x)$ as an "idea" of an element x of a model of T, then $\mathfrak{C}$ may be regarded as an "ideal" model, as all these "ideal" elements exist there. If we agree, that what really matters is not an element a of $\mathfrak{C}$, but its orbit under $Aut(\mathfrak{C}/A)$, then we see that the type $tp(a/A)$ fully describes a in $\mathfrak{C}$ with respect to A. Model theorists believe that the properties of M and $T = Th(M)$ show up most clearly in the monster model $\mathfrak{C}$ (or, more modestly, in a sufficiently saturated model, as opposed to a somewhat accidental model M we started with). For example, in the case of an algebraically closed field K, the monster model of $Th(K)$ may be any algebraically closed field of the same characteristic, of power $\geq \overline{\kappa}$. In the case of a vector space V over a field F, the monster model of $Th(V)$ may be any vector space over F of power $\geq \overline{\kappa}$. It would be harder to point a monster model of the theory of ordered field of reals, since it is unstable. We explain this property below.

If $M = \{a_\alpha : \alpha < \mu\}$, then the isomorphism type of M is determined by the sequence of types $tp(a_\alpha/a_{<\alpha}), \alpha < \mu$, where $a_{<\alpha} = \{a_\beta : \beta < \alpha\}$. So complete types are the real building bricks of any model. The more types there are in a theory, the more freedom there is to construct models of T, and the more complicated these models can be.

Definition 1.2. Let $\kappa \geq \aleph_0$.

(1) We say that T is κ-stable if for every $A \subseteq \mathfrak{C}$ of power $\leq \kappa$, $|S_x(A)| \leq \kappa$.

(2) We say that T is superstable, if for some κ, T is stable in each power $\geq \kappa$.

(3) We say that T is stable if T is stable in some power κ. Otherwise we say T is unstable.

For a countable L, if T is $\aleph_0$-stable, then T is stable in each infinite power, hence T is superstable. In turn, superstability implies stability. So if T is unstable, then it has many types in each power and consequently many complicated models. Shelah proved that in this case (and more generally, for unsuperstable T), T has 2^κ (that is, the maximal possible number of) models in each power $\kappa > |T|$ and that we can not classify them in any reasonable way, uniformly in κ. Moreover, he described exactly within the hierarchy of stability those theories whose models in uncountable powers are classifiable. This is the content of his monumental work [Sh]. By classification of (uncountable) models of T we mean assigning uniformly to each uncountable model of T, a system of cardinal invariants which describe this model up to isomorphism (like for example, the transcendental dimension characterizes an algebraically closed field).

We say that a model M is stable (κ-stable, κ-categorical, etc.) if its theory $Th(M)$ is also stable. The distinction between models given by the stability hierarchy is meaningful in algebra. Below, we give some examples of stable and unstable structures.

(1) $\aleph_0$-stable structures are, for example, algebraically closed fields, vector spaces, an infinite set with no structure, differentially closed fields in characteristic zero, and all structures interpretable within them. For example, all algebraic groups and the additive group of rationals.
(2) The additive group of integers is superstable and not $\aleph_0$-stable. It is harder for more natural examples here, since for example, by a result of Cherlin, Shelah [CS] and Macintyre [Mc], every superstable field is algebraically closed, hence $\aleph_0$-stable.
(3) Separably closed (non-perfect) fields and differentially closed fields of positive characteristic are stable and not superstable. There is also a conjecture that every stable field is separably closed.
(4) All modules and all abelian groups are stable, some of them are superstable and $\aleph_0$-stable.
(5) The ordered field of reals, all infinite Boolean algebras and all infinite linear orderings are unstable. However, the ordered field of reals is "o-minimal", that is, every definable subset of $\mathbb{R}$ is a finite union of points and intervals. There is a large theory of o-minimal linearly ordered structures [D]. It is related to real algebraic geometry.
(6) Pseudofinite fields and fields with "generic" automorphism are unstable, however their instability is, in a way, not so complicated (that is, they still have some nice properties of stable structures). They and their theories are called simple. Another example of a simple structure is the random graph [Wa].

There are some attempts to treat some geometric and analytic structures as model-theoretic objects. This is so with compact complex spaces [Pi2] or Lie groups [NPi]. This approach consists in discerning certain subsets of a given structure as their basic relations (in the case of compact complex spaces, these are the analytic sets).

2 Geometric model theory

Examples of $\aleph_1$-categorical theories are algebraically closed fields K and vector spaces V. In both cases we have a dependence relation induced by a closure operator and the models are determined up to isomorphism by their dimension, that is, the size of a maximal independent subset. In case of algebraically closed fields this is the algebraic closure. In the case of vector spaces, this is the linear span operator.

In his proof of Theorem 1.1 Morley noticed, that these ways of characterizing models may be generalized to a much broader context. Thereafter it became frequent in model theory to find a broader context for some specific algebraic phenomena. I like to think of this as finding deeper roots for these phenomena. We will survey some of these developments in this section. First, we approach the algebraic and definable closure. As usual we will work within a monster model $\mathfrak{C}$.

We call a formula $\varphi(x,\bar{b})$ (or a type $p(x)$) algebraic if the set $\varphi(\mathfrak{C},\bar{b})$ ($p(\mathfrak{C})$ respectively) is finite. We say that a is algebraic over $A \subseteq \mathfrak{C}$ if it satisfies an algebraic formula $\varphi(x)$ with parameters from A. We say that a is definable over A if, moreover, $\varphi(\mathfrak{C})$ is a singleton (hence φ defines a in $\mathfrak{C}$). We define the **algebraic** and **definable** closures of A as the sets $acl(A)$ and $dcl(A)$ of the elements of $\mathfrak{C}$ algebraic or definable over A, respectively. In the case of algebraically closed fields acl agrees with the usual algebraic closure, while in vector spaces acl is the linear span. $acl : \mathcal{P}(\mathfrak{C}) \to \mathcal{P}(\mathfrak{C})$ has the following general properties.

(1) $A \subseteq acl(A)$ and $acl(acl(A)) = acl(A)$.

(2) $A \subseteq B$ implies $acl(A) \subseteq acl(B)$.

(3) (finite character) $acl(A) = \bigcup\{acl(A') : A' \subseteq A \text{ is finite}\}$.

Moreover, in the case of algebraically closed fields and vector spaces (and also the ordered field of reals) we have the exchange property.

(4) If $a \in acl(A \cup \{b\}) \setminus acl(A)$, then $b \in acl(A \cup \{a\})$.

Any operator $cl : \mathcal{P}(\mathfrak{C}) \to \mathcal{P}(\mathfrak{C})$ satisfying (1)–(4) is called a (combinatorial) pregeometry (or a matroid) on $\mathfrak{C}$ (see [CR]). In a pregeometry we can define well-behaved notions of cl-independence and cl-dimension. Morley proved that within any uncountably categorical structure M we can find a definable set X, where acl is a pregeometry. Then M is characterized by the acl-dimension of X.

Shelah generalized this approach to arbitrary stable theories. He found a well-behaved notion of independence (called forking independence), that allowed him to obtain his general classification results and became the core of model theory. There are many ways of defining forking independence, for the purposes of this paper we choose the shortest one (which unfortunately looks a bit artificial).

Assume that we have defined the meaning of "a being independent of B over A", where $A, B \subseteq \mathfrak{C}$ and $a \in \mathfrak{C}$. This notion should be $Aut(\mathfrak{C})$-invariant, therefore it should depend only on $tp(a/A \cup B)$, or even on the relationship between $tp(a/A)$ and $tp(a/A \cup B)$. It is always true, that $tp(a/A) \subseteq tp(a/A \cup B)$, so what we are looking for is a distinguished family of types in $S_x(A \cup B)$ extending $tp(a/A)$ in a "free, unrestricted way".

For example, in algebraically closed fields, if $a \notin acl(A)$, then "a being independent of B over A" should mean that a is not a root of a non-zero polynomial with parameters from $A \cup B$. This condition determines a unique type in $S_x(A \cup B)$ extending $tp(a/A)$, called the transcendental type.

In the general stable case we proceed as follows. Assume $A, B, C \subseteq \mathfrak{C}$. The group $Aut(\mathfrak{C}/A)$ acts on $\mathfrak{C}$, hence also on $Def_{\mathfrak{C}}(\mathfrak{C})$ and on $S_x(\mathfrak{C})$ (identified with the Stone space of $Def_{\mathfrak{C}}(\mathfrak{C})$). Assume $p \in S_x(A)$ and $q \in S_x(\mathfrak{C})$ extends p. We say that q is a free (or: non-forking) extension of p if the orbit of q under the action of $Aut(\mathfrak{C}/A)$ has size $< \overline{\kappa}$ (in fact, its size is $\leq 2^{|T|}$ then). If $r \in S_x(A \cup B)$ extends p, then we say that r is a free extension of p, if some $q \in S_x(\mathfrak{C})$ extending r is a free extension of p. Otherwise we say that r forks over A. Finally we say that a is **(forking) independent** of B over A (symbolically: $a \perp_A B$) if $tp(a/A \cup B)$ is a free extension of $tp(a/A)$. $A \perp_C B$ means that for every finite tuple $\overline{a} \subseteq A$, $\overline{a} \perp_C B$.

As an example consider here the theory of differentially closed fields. So let $\mathfrak{C}$ be a monster model of this theory (that is, a "big" differentially closed field), $K \subseteq L \subseteq \mathfrak{C}$ are its differential subfields and $a \in \mathfrak{C}$. Let $f_{a/K}(X)$ be the differential non-zero polynomial of minimal order and then of minimal degree, vanishing on a (if no such polynomial exists we stupilate $f_K(X) = 0$). Define $f_{a/L}(X)$ likewise. In this case the type $tp(a/L)$ is a free extension of $tp(a/K)$ iff the polynomials $f_{a/K}(X)$ and $f_{a/L}(X)$ have the same order (we stipulate the order of the zero polynomial as ∞). This describes forking in differentially closed fields.

In the stable case the relation of forking independence enjoys many nice properties, like symmetry, transitivity, monotonicity, etc. Moreover, existence of an abstract independence relation in $\mathfrak{C}$ with these properties implies that $\mathfrak{C}$ is stable and this abstract independence agrees with forking independence. So forking of types is intimately related to stability.

We say that a type $p \in S_x(A)$ is **regular**, if the forking closure defined by:

$$a \in cl(B) \iff a \not\perp_A B$$

is a pregeometry on $p(\mathfrak{C})$. In this case we have a good notion of dimension inside $p(\mathfrak{C})$. Regular types are one of the main ingredients of Shelah's classification theory. In the important cases each type may be "decomposed" into finitely many regular types, and then a model is characterized by the tree of dimensions of some regular types.

Considering the "geometric" properties of forking on definable or type-definable sets in $\mathfrak{C}$ led to development of geometric model theory. The most prominent figures here are B. Zilber, E. Hrushovski and A. Pillay. The turning point was the discovery that often the geometry on a regular type is [locally] modular and finding in non-trivial cases definable groups in $\mathfrak{C}$ ("group configuration theorems").

We say that a pregeometry cl on a set X is [**locally**] **modular**, if for all cl-closed sets $A, B \subseteq X$ [with $dim(A \cap B) > 0$] we have

$$dim(A) + dim(B) = dim(A \cup B) + dim(A \cap B).$$

A non-trivial modular pregeometry of total dimension ≥ 4 is essentially isomorphic to a projective geometry over a division ring F. A regular type $p(x)$ is called [locally] modular if the pregeometry on $p(\mathfrak{C})$ is such. It turns out that in the $\aleph_0$-categorical case every regular type $p(x)$ is either trivial or locally modular. In the latter case, if $p(x)$ is modular, then the pregeometry on $p(\mathfrak{C})$ is projective over a finite field, while in the locally modular, non-modular case it is affine over a finite field. This enabled us to understand the abstract $\aleph_0$-categorical $\aleph_0$-stable structures in terms of some classical mathematical objects (like vector spaces over finite fields).

The transcendental type in an algebraically closed field is an example of a regular type, which is not locally modular. This led Ziber to conjecture that if in an $\aleph_1$-categorical theory we have a non-trivial regular type, then either it is locally modular, or an algebraically closed field is interpretable in $\mathfrak{C}$. This conjecture was refuted by Hrushovski [H2]. However it is an example of an approach in model theory consisting in re-discovering in some very general circumstances some classical mathematical objects.

Many results of geometric model theory generalize some basic results of algebraic geometry. No wonder, in a way algebraic geometry studies definable sets in an $\aleph_0$-stable structure: an algebraically closed field. The prominent example are the group existence theorems of Hrushovski, generalizing the group chunk theorem of Andre Weil. Hrushovski proved that very often, if forking is sufficiently complicated, then a definable group exists in $\mathfrak{C}$, and then the complication of forking is explained within this group. This attracted the attention of model-theorists to the theory of stable groups, which again generalizes some basic properties of algebraic groups.

In the 90's model theory has found some applications in differential algebra and diophantine geometry. For example, Hrushovski gave his celebrated model-theoretic proof of the Mordell-Lang conjecture for function fields. Some further results in this spirit were obtained by Pillay [Pi3] and Scanlon [Sc]. Modular types play the main role in these applications. The proofs rely on model-theoretic analysis of fields with some additional operators (like derivation or automorphism). The main point is to understand the geometry of the regular types there. If this geometry is locally modular, it is considered "simple". Moreover, a small family of regular types is pointed so that each non-locally modular type is strongly related ("non-orthogonal") to a member of this family. Also, some advanced model-theory of stable groups plays a role there.

Geometric model theory enabled a progress on Vaught conjecture, one of the main old open problems of model theory. **Vaught conjecture** says, that if a countable theory T has $< 2^{\aleph_0}$ countable models, then T has countably many of them. For an exposition see [L]. In model theory Vaught conjecture has been proved by Shelah for $\aleph_0$-stable theories [SHM] and by Buechler for superstable theories of finite rank [Bu]. An intermediate step here was so-called Saffe's conjecture, proved in [Ne1]. People believe that proving Vaught conjecture for a given theory yields a classification of countable models of it when such a reasonable classification exists. In this way the program of proving Vaught conjecture becomes a countable counterpart of the Shelah classification theory (which deals with uncountable models).

3 Perspectives

In this section I point some possible developments in model theory, related to my research. Describing models we need some auxiliary objects. Above we mentioned complete types. They form a compact topological space. Another kind of such auxiliary objects are so-called imaginary elements of a model (shortly, imaginaries).

Assume M is a model and $E(\overline{x}, \overline{y})$ is a formula defining an equivalence relation on M^n (for some $n = |\overline{x}| = |\overline{y}|$). Then we adjoin to M the quotient set M^n/E as an additional, imaginary sort, and call its elements (that is, the classes of E) imaginaries or imaginary elements. We do it for all possible E. In this way M is extended to a larger, many-sorted structure denoted by M^{eq}. M is a distinguished "real" sort of M^{eq}. Moreover, this extension does not affect M, since any subset of M definable in M^{eq} is definable in M itself. However, M^{eq} has many advantages. The main one is the presence of names of definable subsets.

Namely, assume $\varphi(x, y)$ is a formula. φ determines a family $\varphi(M, b), b \in M$, of uniformly definable subsets of M. If we fix φ in mind, then each $b \in M$ determines $\varphi(M, b)$, so b could be regarded a "name" of $\varphi(M, b)$. However, this b may be not uniquely determined by $\varphi(M, b)$, since there may be some other $b' \in M$ with $\varphi(M, b) = \varphi(M, b')$. To get a unique name we proceed as follows.

We define an equivalence relation $E_\varphi(y, y')$ on M by

$$E_\varphi(b, b') \Leftrightarrow \varphi(M, b) = \varphi(M, b').$$

E_φ is 0-definable and the imaginary element b/E_φ serves as a canonical name for the set $\varphi(M, b)$. In M^{eq} the set $\varphi(M, b)$ is definable over the imaginary element b/E_φ by an explicit formula.

In this way, if we have definable groups $H \triangleleft G$ in M, then the quotient group G/H lives in M^{eq}.

Many objects in model theory are constructed not in M, but rather in M^{eq}. For example, this is so with the groups from the Hrushovski's group existence theorems. Likewise, several objects derived from forking live on the imaginary level. For exampe this is so with canonical bases of types (these are "canonical names" of type-definable sets). However, for a model-theorist M^{eq} is an equally friendly world to work in as M, and in many ways more convenient. So this is what we (tacitly) do.

Sometimes every element $a \in M^{eq}$ is interdefinable with some $a' \in M$ (that is, $a \in dcl(a')$ and $a' \in dcl(a)$). Then we say that M eliminates imaginaries. In this case there is no need to work with M^{eq}, since M is ample enough. Again, some important structures eliminate imaginaries. This is so with algebraically closed fields [Po], the ordered field of reals (and other o-minimal structures), differentially closed fields.

Algebraic geometry deals with algebraic varieties. Model theoreticaly, these objects live on the imaginary level. By elimination of imaginaries in algebraically closed fields, each algebraic variety V is isomorphic (via a definable bijection) with a definable subset X of some cartesian power of the original field F. However in the transition from V to X we lose the geometric structure of V (in the sense of algebraic geometry), so sometimes it is better to work on the imaginary level even if the structure eliminates imaginaries.

In general we are far from a complete description of models of a given theory T, particularly if T is unstable. But also for stable T we do not have a satisfactory description of countable models (Vaught conjecture is open). We may try to describe at least some parts of M or M^{eq}. One such attempt of mine are profinite strucures. Again, we work in $\mathfrak{C}^{eq}$ for a monster model $\mathfrak{C}$ of T (we may omit eq in $\mathfrak{C}^{eq}$). We assume T and the language L are countable.

Assume X is a 0-definable set in $\mathfrak{C}$ and $f_n, n < \omega$, are 0-definable functions with $f_n : X \to acl(\emptyset)$. Let

$$U = \{\langle f_n(a)\rangle_{n<\omega} : a \in X\}.$$

So $U \subseteq acl(\emptyset)^\omega$ is a type-definable set in $\mathfrak{C}$ (consisting of infinite tuples), with finite projections on each coordinate (since each $f_n[X]$ is finite, being a definable subset of $acl(\emptyset)$).

We equip $acl(\emptyset)$ with the discrete topology and $acl(\emptyset)^\omega$ with the product topology. Then U as a subspace of $acl(\emptyset)^\omega$ becomes a compact topological space, acted upon by $Aut(\mathfrak{C})$. Let $Aut^*(U)$ be the group of permutations of U induced by $Aut(\mathfrak{C})$. $Aut^*(U)$ is a profinite group acting continuously on U. U itself is a profinite topological space (that is, an inverse limit of finite discrete spaces). We call the pair $(U, Aut^*(U))$ a profinite structure interpretable in $\mathfrak{C}$.

For example assume G is a 0-definable group in $\mathfrak{C}$, that is a definable set on which there is a 0-definable group law. The **connected component** of G, denoted by G^0, is the intersection of all 0-definable normal subgroups of finite index in G. We can find a decreasing sequence of such groups $H_n, n < \omega$ with

$$G^0 = \bigcap_n H_n.$$

Then the quotient group G/G^0 is the projective limit of the finite groups $G/H_n, n < \omega$, so we can think of G/G^0 as the set $\{\langle f_n(a)\rangle_{n<\omega} : a \in G\}$, where $f_n(a) = a/H_n \in \mathfrak{C}^{eq}$. So $U := G/G^0$ is a profinite structure and a profinite group interpretable in $\mathfrak{C}$.

More generally, a profinite group interpretable in $\mathfrak{C}$ is a profinite structure $(U, Aut^*(U))$ interpretable in $\mathfrak{C}$, with a group operation on U, whose graph is closed and $Aut^*(U)$-invariant. Then U is a profinite group in the usual sense (that is, a projective limit of finite groups). For more details see [Ne4, Ne5].

Describing profinite structures interpretable in $\mathfrak{C}$ may be regarded as describing a “piece” of $\mathfrak{C}$. Notice that any finite structure may be interpreted in $\mathfrak{C}$.

I was considering profinite structures interpretable in $\mathfrak{C}$ for a superstable theory T with $< 2^{\aleph_0}$ countable models. I proved that such structures are very well behaved. For example, I proved that under these assumptions any profinite group interpretable in $\mathfrak{C}$ is abelian-by-finite. For details see [Ne3, Ne2]. Profinite structures may be a tool to treat some phenomena in simple theories. On the other hand, several model-theoretic arguments have their counterparts for small profinite structures, where sometimes they are demonstrated more directly than in the original set-up, without using all advanced machinery of geometric model theory.

Another direction in model theory may be connected with more advanced topological properties of the space of types. Morley and subsequently Shelah were considering just various variants of the Cantor-Bendixson rank on a space of types. Here recall, that for a compact topological space X, the Cantor-Bendixson rank of a point $a \in X$ (denoted by $CB(a)$) is the minimal α such that $a \notin X^{(\alpha+1)}$, where $X^{(\alpha+1)}$ is the $\alpha+1$-Cantor-Bendixson derivative of X. In fact, formally the approach of Shelah here was more combinatorial than topological (that is, he defined his plethora of ranks by combinatorial means). An essential ingredient in my research on Vaught conjecture and profinite structures was systematic consideration of the Baire category properties of certain closed sets in the space of types. Maybe considering the topology of the space of types in greater detail may help to understand models ?

I will describe a recent result of mine in this spirit. It is related to a series of my results on Lascar strong types [Ne6]. Assume G is a definable group in $\mathfrak{C}$, covered by countably many sets $X_n, n < \omega$. If the sets X_n are definable, then on the level of the space of types they are clopen, and cover a compact set corresponding to G. So by compactness, finitely many of the sets X_n suffice to cover G.

Now assume only that the sets X_n are type-definable. Then the compact space corresponding to G is covered by the corresponding closed sets, and we can not hope to get a finite number of X_n covering G. However, I proved that in this case we have the following.

Theorem 3.1. For some finite number k, finitely many of the sets X_n generate G (as a group) in at most k steps.

In the proof of this theorem (and in [Ne6]) I use an open analysis of a compact space with respect to its covering by a family of subsets. This new tool may find applications elsewhere.

Then quite surprisingly I proved that there is a finite upper bound on k in this theorem. When G is abelian, k is bounded by 2, while for arbitrary G, k is bounded by 3 (see [NPe] for more details). So for example we have the following theorem.

Theorem 3.2. Assume G is an $\aleph_0$-saturated abelian group covered by countably many 0-type-definable sets $X_n, n < \omega$. Then for some n we have

$$G = X_{\leq n} - X_{\leq n},$$

where $X_{\leq n} = \bigcup_{m \leq n} X_m$.

This is a model-theoretic theorem, however we can rephrase it in a model-theory-free way. We consider a countable compactification X of the discrete space ω of natural numbers. So X is a metric space, and for $Y \subseteq X$ and $\epsilon > 0$, $B(Y, \epsilon)$ denotes the open ball in X around Y, of diameter ϵ.

Theorem 3.3. Assume G is an abelian group, X is a countable compactification of ω and $f : G \to \omega$. Then there is a finite set $Y \subseteq X$ such that for every $\epsilon > 0$,

$$G = f^{-1}[B(Y, \epsilon)] - f^{-1}[B(Y, \epsilon)].$$

We can think of f in this theorem as a colouring of G into countably many colours. Then the theorem says that if we arrange these colours as isolated points of a countable compact space X, then for some finite set $Y \subseteq X$, for each $\epsilon > 0$, we can throw away all the elements of G with colours outside $B(Y, \epsilon)$, and what remains still generates G in 2 steps. Apparently, the higher the CB-rank of X is, the "smaller" the sets $f^{-1}[B(Y, \epsilon)]$ ("generating G") are. I want just to point that Theorems 3.2 and 3.3 are really restatements of each other and that so sometimes model theory enables us to express in an elegant form some properties of mathematical structures.

Putting it in another way, it may appear to some that considering abstract saturated models is remote from the "real mathematical world". I think however that the results of model theory tell us something also about this "real world". And sometimes they do it in a more elegant and transparent manner.

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