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A necessary and sufficient condition for the existence of an exponential attractor

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Abstract: We give a necessary and sufficient condition for the existence of an exponential attractor. The condition is formulated in the context of metric spaces. It also captures the quantitative properties of the attractor, i.e., the dimension and the rate of attraction. As an application, we show that the evolution operator for the wave equation with nonlinear damping has an exponential attractor.

Keywords: global attractor, exponential attractor, fractal dimension, wave equation, nonlinear damping
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1 Introduction

Let W be a bounded, complete metric space and let $S : W \mapsto W$ be a Lipschitz continuous mapping. It is well known that the following characterization holds:

Theorem 1.1. The dynamical system (S^n, W) has a global attractor if and only if for an arbitrary sequence of points $x_n \in W$, the sequence $S^n x_n$ has a subsequence converging in W.

The usefulness of this theorem is two-fold. First, it gives us a deeper understanding of the concept of the global attractor. Second, since it contains the optimal existence condition, it is very useful in applications.

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Regarding the concept of an exponential attractor, there are several known criteria guaranteeing its existence. The first one is the 'squeezing property' used in [EFNT], where the notion of the exponential attractor was also introduced. Then one has the 'smoothing property' (see [EM]), which in the Hilbert space setting implies the 'squeezing property' (see [EM], [MP]), but works also in Banach spaces with a considerably simpler proof, see [EMZ].

Moreover, in applications one frequently finds that the studied operators satisfy various generalizations of the above mentioned 'smoothing' or 'squeezing' properties, cf. [P], [EMZ]. Thus the question arises whether, instead of repeating previously known proofs with some modificitions, one could find a common, general, and if possible, optimal condition to which all this would reduce.

We give such a condition in Theorem 2.1 below. One can note that the condition is formulated in a metric space setting. The construction in the 'if part' is not new, see e.g. [EMZ]. What we find interesting is that the condition is also necessary, and moreover, it optimally captures the quantitative properties of the exponential attractor, i.e., the dimension and the rate of attraction.

The paper is organized as follows: in section 2 we recall the definitions of basic concepts. Then we state and prove our main result, Theorem 2.1.

In section 3 we give an application: we observe that the 'generalized squeezing property' proved in [P] implies the existence of an exponential attractor for the evolution operator of the wave equation with nonlinear damping.

2 The condition

Let $(W, \rho(\cdot,\cdot))$ be a bounded, complete metric space. For A, $B \subset W$ we set $\operatorname{dist}(A,B) := \sup_{a\in A} \inf_{b\in B} \rho(a,b)$. The symbol $N(A,\varepsilon)$ stands for the smallest number of closed balls of radii ε with centers in A covering the set A. Finally, the fractal dimension of the set $A \subset W$ is defined as

$$d_f(A) := \limsup_{\varepsilon\to 0} \frac{\ln N(A,\varepsilon)}{-\ln \varepsilon}.$$

Let $S : W \mapsto W$ be a Lipschitz continuous mapping. A set $\mathcal{A} \subset W$ is called a global attractor for the dynamical system (S^n, W) if (i) $\mathcal{A}$ is compact, (ii) $S(\mathcal{A}) = \mathcal{A}$ and (iii) $\operatorname{dist}(S^n(W), \mathcal{A}) \to 0$ as $n \to \infty$.

A set $\mathcal{E} \subset W$ is called an exponential attractor if (i) $\mathcal{E}$ is compact, (ii) $S(\mathcal{E}) \subset \mathcal{E}$, (iii) $d_f(\mathcal{E}) < \infty$ and (iv) there exist c, $\gamma > 0$ such that

$$\operatorname{dist}(S^n(W), \mathcal{E}) \le c \exp(-\gamma n) \qquad \forall n \in \mathbb{N}. \tag{1}$$

Note that necesarily $\mathcal{A} \subset \mathcal{E}$. So the existence of $\mathcal{E}$ implies that $\mathcal{A}$ has finite fractal dimension.

Now we can state our main result.

Theorem 2.1. The following are equivalent:

(I) The dynamical system (S^n, W) has an exponential attractor.

(II) There exist constants a, $b > 0$, $\eta \in (0,1)$ and $K > 1$ such that

$$N(S^n(W), a\eta^n) \le bK^n \qquad \forall\, n \in \mathbb{N}. \tag{2}$$

Furthermore, the existence of an exponential attractor implies (2) with $\eta = \exp(-\gamma)$ and $K = \exp(\gamma d)$ where γ is the constant in (1) and d is arbitrary such that $d_f(\mathcal{E}) < d$.

Conversely, if (2) holds then one can construct an exponential attractor such that

$$d_f(\mathcal{E}) \le \frac{\ln K}{-\ln \eta} \tag{3}$$

and (1) holds with $\gamma = -\ln(\eta)$.

Proof ((I) $\Longrightarrow$ (II)). If $d_f(\mathcal{E}) < d$, then $N(\mathcal{E}, c\exp(-\gamma n)) \le c_1(c^{-1}\exp(\gamma n))^d$ holds with a suitable c_1 for all $n \in \mathbb{N}$. This together with (1) implies

$$N(S^n(W), 2c[\exp(-\gamma)]^n) \le \frac{c_1}{c^d}[\exp(\gamma d)]^n,$$

i.e., (2) holds with $\eta = \exp(-\gamma)$ and $K = \exp(\gamma d)$ as required.

The proof of the converse implication will be preceded by two simple lemmas.

Lemma 2.2. Let $A \subset W$ and let there exist a, $b > 0$, $\theta \in (0,1)$ and $N > 1$ such that

$$N(A, a\theta^n) \le bN^n \qquad \forall n \in \mathbb{N}.$$

Then $d_f(A) \le \ln N/(-\ln\theta)$.

Proof. A straightforward estimate; see for example the last part of the proof of [P, Lemma 4.1].

Lemma 2.3. Let (2) hold. Then (S^n, W) has a global attractor $\mathcal{A}$ with $d_f(\mathcal{A}) \le \ln K/(-\ln\eta)$.

Proof. Let $x_n \in W$ be an arbitrary sequence. It is easy to see that (2) enables one to extract a Cauchy subsequence from $S^n x_n$. Hence $\mathcal{A}$ exists by Theorem 1.1.

Since $S^n(\mathcal{A}) = \mathcal{A}$, (2) further implies that $N(\mathcal{A}, a\eta^n) \le bK^n$. The dimension estimate of $\mathcal{A}$ follows by Lemma 2.2.

Proof (Theorem 2.1, (II) $\Longrightarrow$ (I)). Let E_n be the set of the centers of the balls from the coverings which verify (2). Setting

$$\mathcal{E}_0 = \bigcup_{m,k\ge 1} S^m(E_k)$$

$$\mathcal{E} = \mathcal{E}_0 \cup \mathcal{A}$$

we claim that $\mathcal{E}$ is the exponential attractor we are looking for.

Indeed, $S(\mathcal{E}) \subset \mathcal{E}$, since both $\mathcal{A}$ and $\mathcal{E}_0$ have this property. To verify the compactness, it is enough to note that any sequence intersecting infinitely many $S^m(E_k)$ belongs to infinitely many $S^n(X)$ and hence has an accumulation point in $\mathcal{A}$ by Theorem 1.1.

Further, since $\mathrm{dist}(S^n(X), E_n) \le a\eta^n$, the attracting property (1) holds with $c = a$ and $\gamma = -\ln(\eta)$ as desired.

Now it remains to estimate the dimension. Since $S^m(E_k) \subset S^{m+k}(X)$, we have

$$\mathcal{E}_0 \subset \bigcup_{m+k<n} S^m(E_k) \cup S^n(X)$$

and hence using (2)

$$N(\mathcal{E}_0, a\eta^n) \le \sum_{k+m<n} \#[S^m(E_k)] + N(S^n(X), a\eta^n) \le b(n^2+1)K^n.$$

as $\#[S^m(E_k)] = \#E_k \le bK^k$. Consequently $N(\mathcal{E}_0, a\eta^n) \le \tilde{b}\tilde{K}^n$ holds with some suitable $\tilde{b}$ for arbitrary $\tilde{K} > K$. Lemma 2.2 then implies that $d_f(\mathcal{E}_0) \le \ln K/(-\ln \eta)$. Since by Lemma 2.3 the same estimate holds for $\mathcal{A}$, we are done.

3 The wave equation with nonlinear damping

In [P] it was shown that the evolution of the wave equation with nonlinear damping satisfies a certain type of 'generalized squeezing property' (GSP), see [P, Lemma 3.1]. This further implies a finite fractal dimension of the global attractor, see [P, Lemma 4.1].

The essence of the proof of Lemma 4.1 of [P] was to observe that (GSP) guarantees the condition of the type (2). Thus, in view of Theorem 2.1 one can expect that the exponential attractor exists as well. The aim of this section is to explain how suitable modifications of the proofs given in [P] lead to this result.

The equation under the study reads

$$\begin{aligned} u_{tt} + g(u_t) - \triangle u + f(u) &= 0 && \text{in } \Omega \times (0,\infty), \\ u &= u_0 && \text{on } \Omega \times \{0\}, \\ u_t &= u_1 && \text{on } \Omega \times \{0\}, \\ u &= 0 && \text{on } \partial\Omega \times (0,\infty). \end{aligned} \tag{4}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^n$, $n = 2$ or 3. The nonlinearities f, g are required to be C^1 functions obeying the growth conditions

$$\begin{aligned} g(0) &= 0, \\ |f'(u)| &\le c_1(1+|u|)^{p-2}, \\ c_2(1+|u|)^{q-2} \le g'(u) &\le c_3(1+|u|)^{q-2}, \end{aligned} \tag{5}$$

where p and q satisfy

$$2 \le p, \qquad 2 \le q < 4 \qquad \text{if } n = 2, \tag{6}$$

$$2 \le p < 4, \quad 2 \le q < 3 + 1/3 \qquad \text{if } n = 3. \tag{7}$$

Moreover, we require that

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1, \tag{8}$$

where λ_1 is the first eigenvalue of the Laplace operator on Ω with zero Dirichlet boundary condition.

By (6), (7), the equation is subcritical. Thus for any $(u_0, u_1) \in E := W_0^{1,2}(\Omega) \times L^2(\Omega)$, there exist unique weak solution in the class

$$u \in C([0,+\infty); W_0^{1,2}(\Omega)), \quad u_t \in C([0,+\infty); L^2(\Omega)).$$

See for example [F] for the existence and [P] for the uniqueness. Condition (8) ensures asymptotic boundedness of solutions and moreover, that the global attractor $\mathcal{A}$ exists, see [F].

Let $S(t) : E \to E$ denote the solution operators to (4). The operators $S(t)$ are locally Lipschitz continuous, cf. [P, (1.5)].

Let B be an arbitrary closed and bounded subset of E, which is positively invariant and uniformly absorbing with respect to $S(t)$. For example, we can take $B = cl_E(\cup_{t\geq t_0} S(t)B_0)$ where B_0 is any open ball containing the global attractor $\mathcal{A}$ and t_0 is sufficiently large number.

The essential feature of the proof is that we establish the existence of the exponential attractor first in the space of trajectories. We define

$$\begin{aligned} B_\ell &:= \{\chi : [0,\ell] \to E : \ \chi(0) \in B, \ \chi(t) = S(t)\chi(0) \ \forall t \in (0,\ell]\}, \\ W_\ell &:= cl_{L^2(0,\ell;E)}(B_\ell). \end{aligned} \tag{9}$$

The set B_ℓ is simply a set of all solutions to (4) on the time interval $[0,\ell]$ with the initial condition in B.

The set W_ℓ is also a set of solutions to (4). So let $\chi_0 \in W_\ell$. Then there exist $\chi_n \in B_\ell$ so that $\chi_n \to \chi_0$ in $L^2(0,\ell;E)$. Without loss of generality $\chi_n(t) \to \chi_0(t)$ for almost every $t \in (0,\ell)$. Using now the continuity of the solution operator $S(t)$, we see that χ_0 is solution to (4) though in general only on $(0,\ell]$. The set W_ℓ is considered with the topology $L^2(0,\ell;E)$. Note that W_ℓ, as a closed subset of a Hilbert space, is a complete metric space.

We finally introduce operators $L : W_\ell \to W_\ell$ and $e : W_\ell \to E$ as

$$\begin{aligned} L\chi(s) &:= S(\ell)\chi(s) \qquad \forall s \in (0,\ell], \\ e(\chi) &:= \chi(\ell). \end{aligned}$$

Clearly L assigns to any trajectory the trajectory starting from its endpoint, while e assigns to a given trajectory the endpoint itself. Note that L and e are well defined since the elements of W_ℓ have continuous representatives. Both L and e are Lipschitz continuous as follows from the Lipschitz continuity of $S(t)$ by an easy computation. See [P, Lemma 2.1] or [MP, Lemma 2.1] for the proof.

While operators $S(t)$ capture the evolution of (4) in the space of the initial conditions E, the operator L describes the evolution of the equation in the auxiliary space of trajectories W_ℓ.

Lemma 3.1. Let ℓ be sufficiently large. Then the operator L satisfies on W_ℓ the 'generalized squeezing property' (GSP). Namely, there exist $C > 0$, $\theta \in (0, 1/4)$ and a finite-dimensional orthonormal projector P in E_ℓ such that for any χ, $\psi \in W_\ell$,

either

$$\|L\chi - L\psi\|_{E_\ell} \leq C \left(\|P(\chi - \psi)\|^2_{E_\ell} + \|P(L\chi - L\psi)\|^2_{E_\ell}\right)^{1/2} \tag{GSP1}$$

or

$$\|L\chi - L\psi\|_{E_\ell} < \theta\|\chi - \psi\|_{E_\ell}. \tag{GSP2}$$

Proof. In Lemma 3.1 of [P] it was shown that the GSP holds for the operator L_ℓ ($= L$ in the present notation) on the set $\mathcal{A}_\ell$ - the set of all the trajectories starting from the global attractor $\mathcal{A}$. But the proof only uses the fact the trajectories are (pointwise) bounded in E, which certainly holds for W_ℓ.

Lemma 3.2. Let W be a closed, bounded subset of a Hilbert space and let the mapping $S : W \to W$ satisfy on W the GSP. Then the dynamical system (S^n, W) has an exponential attractor.

Proof. In Lemma 4.1 of [P] it was shown that if a Lipschitz continuous mapping L satisfies the GSP on a set A such that $L(A) = A$, then A has finite fractal dimension.

The key step here was that if $B = B(x, R) \cap W$ is a ball, then $N(L(B), 3R/4)$ can be bounded independently of x, R. This follows by a slight generalization of [EFNT, Lemma 2.1]. Inductively we thus obtain that $N(L^n(W), (3/4)^n R) \leq K^n$, i.e. condition (2), which by Theorem 2.1 implies the existence of an exponential attractor.

Combining the last two lemmas we conclude that the dynamical system (L^n, W) has an exponential attractor provided that ℓ is sufficiently large. Using the Lipschitz continuity of e it is easy to verify that $\mathcal{E} := e(\mathcal{E}_\ell)$ is an exponential attractor for the dynamical system $(S^n(\ell), B)$.

We have thus proved

Theorem 3.3. Let (5), (6), (7) and (8) hold and let $B \subset E := W^{1,2}_0(\Omega) \times L^2(\Omega)$ be a closed, bounded, and positively invariant set. Then for $\ell > 0$ sufficiently large, the dynamical system $(S^n(\ell), B)$ has an exponential attractor.

Remark. One can also obtain an exponential attractor for the continuous dynamical system $(S(t), B)$. However, some information on the continuity of $S(t)$ with respect to t is needed. Thus we either have to restrict ourselves to smoother solutions (e.g. bounded in $W^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$), or to obtain the attractor only in some weaker space as for example $L^2(\Omega)\times W^{-1,2}(\Omega)$.

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