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Conformal Geometry and the Composite Membrane Problem

Abstract

We show that a certain eigenvalue minimization problem in two dimensions for the Laplace operator in conformal classes is equivalent to the composite membrane problem. We again establish such a link in higher dimensions for eigenvalue problems stemming from the critical GJMS operators. New free boundary problems of unstable type arise in higher dimensions linked to the critical GJMS operator. In dimension four, the critical GJMS operator is exactly the Paneitz operator.

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We wish to study here a minimization problem for eigenvalues of the Laplace operator on Riemann surfaces and its higher dimensional generalizations. We begin with the case of Riemann surfaces.
Let (Ω^2, g_0) be a Riemann surface which is bounded and has a smooth boundary. The surface is endowed with a metric g_0. We consider all metrics g that are in the same conformal class as g_0 and write $g \in [g_0]$. That is $g \in [g_0]$ if and only if

$$g = e^{2u} g_0. \tag{1}$$

We denote the Laplace-Beltrami operator for g by Δ_g. We will now fix two constraints and consider only those metrics $g \in [g_0]$ that satisfy the conditions:

1. There is a constant A, such that for all metrics g conformal to g_0 through (1) we have, $||u||_{L^\infty(\Omega)} \leq A$.
2. We will prescribe the volume of the conformal class, that is we prescribe $M > 0$ so that for all metrics

$$\int_\Omega dV_g = \int_\Omega e^{2u} dV_{g_0} = M.$$

All metrics conformal to g_0, that is $g \in [g_0]$, and satisfying the two constraints above will be said to lie in class C. Now under the two constraints above we seek to minimize the first eigenvalue of $-\Delta_g$ with zero boundary conditions on $\partial\Omega$. That is we seek to minimize:

$$\inf_{g \in C} \inf_{\phi \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla_g \phi|_g^2 dV_g}{\int_\Omega |\phi|^2 dV_g} \tag{2}$$

We next discuss a Physical problem, the Composite membrane problem. The article [4] treats the Composite Membrane problem in the flat case. The Physical problem is as follows:

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Build a body in $\mathbb{R}^n$ with prescribed shape and prescribed mass, out of materials with prescribed and varying density, so that the body has maximum rigidity. This last requirement is the demand that the first eigenvalue for the Laplace operator with Dirichlet boundary conditions is as small as possible.
We now re-formulate the Physical problem described above in Mathematical terms. We consider a domain $\Omega \subseteq \mathbb{R}^n$, with $n \geq 2$. Let $\rho(z)$ be a bounded function defined on Ω such that it satisfies two requirements for given $\lambda > 0$, $\Lambda < \infty$ and $M > 0$. First we demand

$$0 < \lambda \leq \rho(z) \leq \Lambda < \infty, \tag{3}$$

and secondly,

$$\int_\Omega \rho(z)\ dz = M. \tag{4}$$

We now seek to minimize the integral below, subject to the constraints on ρ, (3) and (4). We minimize,

$$\inf_{\int_\Omega \rho(z)dz=M}\ \inf_{\phi\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla\phi|^2\ dz}{\int_\Omega |\phi|^2\rho(z)\ dz}. \tag{5}$$

The key result proved in Theorem 13 of [4], is that a minimizer to the minimization problem (5) exists. A sharp difference between the minimization problem stated above and a traditional eigenvalue problem for the Laplace operator is that, in our problem we are searching for a minimizing pair (ϕ, ρ) as opposed to the eigenfunction by itself. The second difference between a traditional eigenvalue problem and (5) is that the minimization process requires the infimum over two constraints, a double infimum, that complicates matters and produces a non-linearity in the problem. Re-stating Theorem 13 of [4] we have

Theorem 1.

There is a minimizer for our variational integral (5) given by a minimizing pair (ϕ, ρ) such that,

1. *The minimizing function $\rho(z)$ is given by,*

$$\rho(z) = \lambda\chi_D + \Lambda\chi_{D^c},$$

where $D \subseteq \Omega$ is a sub-level set of the minimizing eigenfunction ϕ, that is there exists $c > 0$ with

$$D = \{z \in \Omega |\ \phi(z) \leq c\}.$$

D^c denotes the complement of D in Ω.

This says that the body can be built out of just two types of materials, the lowest density and the highest density material.

2. *The eigenfunction ϕ and consequently the accompanying set D and thus the function $\rho(z)$ need not be unique. In the case of the ball, by re-arrangement methods one does have uniqueness, see [4].*

One of the ingredients that is used to establish the existence of a minimizer in the theorem stated above, see [4], is the Bathtub principle [12].
Knowing that $\rho(z)$ has a special form given by part (1) in the Theorem stated above, one can re-formulate the composite membrane problem (5) by studying now the variational problem with $\alpha > 0$ given by:

$$\inf_{D\subseteq\Omega,\ |D|=B}\ \inf_{\phi\in H_0^1(\Omega)} \frac{\int_\Omega (|\nabla h|^2 + \alpha\chi_D h^2)\ dz}{\int_\Omega |h|^2\ dz}.$$

The precise relation between λ, Λ, M and α, B is given in Theorem 13 of [4]. We end by remarking that the composite membrane problem leads to a free boundary problem, i.e the consideration of the regularity of ∂D.

For the sequel we record a metric analog of (5). For functions $\rho(z)$ satisfying $0 < \lambda \le \rho(z) \le \Lambda < \infty$, we will consider the variational integral:

$$\inf_{\int_\Omega \rho \ dV_{g_0} = M} \inf_{\phi \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla_{g_0}\phi|^2 \ dV_{g_0}}{\int_\Omega |\phi|^2 \rho \ dV_{g_0}}. \tag{6}$$

We now wish to tie together the eigenvalue minimization problem in Conformal classes and the Composite Membrane problem. Though the Composite Membrane problem has been posed and understood in $\mathbb{R}^n$, with $n \ge 2$, we can only establish a link with eigenvalue minimization in Conformal classes when $n = 2$.
We have the proposition:

Proposition 2.

The eigenvalue minimization problem (2) is equivalent to the Composite Membrane Problem (6).

Proof. The proof follows by simply noting that the numerator in the Rayleigh quotient in (2) is a conformal invariant as we are in two dimensions. That is

$$\int_\Omega |\nabla_g \phi|_g^2 dV_g = \int_\Omega |\nabla_{g_0}\phi|_{g_0}^2 dV_{g_0}.$$

Set $\rho = e^{2u}$, and so the problem (2) by virtue of the constraints can be re-written as: For given $\lambda = e^{-2A} > 0$, $\Lambda = e^{2A} < \infty$, $M > 0$ such that $0 < \lambda \le \rho \le \Lambda < \infty$ solve the minimization problem,

$$\inf_{\int_\Omega \rho dV_{g_0} = M} \inf_{\phi \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla_{g_0}\phi|^2 \ dV_{g_0}}{\int_\Omega |\phi|^2 \rho \ dV_{g_0}}.$$

This last variational integral is exactly the one associated with the metric analog of the Composite Membrane problem (6). In the case our background metric $g_0 = dx^2 + dy^2$ is the flat metric, then this last minimization problem (6) is exactly the one treated in Theorem 13, [4], and discussed above (5), i.e. the Composite Membrane Problem. □

Existence and regularity questions associated to the composite membrane problem is treated in many articles, [4], [5], [14], [2], [13], [6] and [7] among others. Thus these articles now give complete information about the minimization of eigenvalues in conformal classes, the existence and regularity of the limit metric and the associated eigenfunction and more importantly the optimal $C^{1,1}$ regularity of the minimizing eigenfunction. The limit metric and thus the associated eigenfunction need not be unique due to a symmetry breaking phenomena [4]. We remind the reader again, that unlike a traditional minimization problem for eigenfunctions, we have to find a minimizing pair (u_∞, ϕ_∞). We remark that our regularity results rely on blow-up analysis and so curvature has no role to play in regularity issues. We may summarize the results in [4], [5], [6] and [7] as applied to eigenvalue minimization (2) in the form of a theorem. The proofs essentially follow by using $\rho = e^{2u}$. We also restrict to the case g_0 is flat.

Theorem 3.

There exists a limit metric $\rho_\infty g_0 = e^{2u_\infty} g_0$ and an associated limit eigenfunction ϕ_∞, such that,

1. $$e^{2u_\infty} = \lambda \chi_D + \Lambda \chi_{D^c}$$
   where $D \subseteq \Omega$ and D^c denotes the complement of D in Ω.
2. *D is a sub-level set of the eigenfunction ϕ_∞, that is there exists $c > 0$ such that,*
   $$D = \{z \in \Omega | \ \phi_\infty(z) \le c\}.$$
3. *The limiting eigenfunction ϕ_∞ belongs to $C^{1,1}(\overline{\Omega})$. In particular $\phi_\infty \in W^{2,2}(\overline{\Omega})$. The $C^{1,1}$ regularity of ϕ_∞ is optimal.*

4. *D^c has finitely many components, and the free boundary ∂D^c consists of finitely many, simple, closed real-analytic curves.*

5. *Due to symmetry breaking, the function u_∞ associated to the limiting metric and the eigenfunction ϕ_∞ are not necessarily unique. If Ω is a disk, then u_∞ and the associated eigenfunction ϕ_∞ are unique. Additional hypotheses on convex Ω guarantees uniqueness of (u_∞, ϕ_∞), see [6].*

6. *If Ω is simply-connected, then D is connected.*

We now pass to a higher dimensional analog of the problem stated above. This concerns the critical GJMS operator and its conformal covariance properties. The GJMS hierarchy of conformally covariant operators was constructed in [10] and include the Paneitz operator and the Yamabe operator. The book [1] explains the construction and contains many references.
Specifically we consider (Ω^n, g_0) and the associated critical GJMS operator $P^{g_0}_{n/2}$. For us the covariance properties established in [10], [11] prove important. The operator $P^{g_0}_{n/2}$ has the property that if one considers the metric $g = e^{2u} g_0$, then the critical GJMS operator in the new metric $P^g_{n/2}$ satisfies the relation, (see [10], [11])

$$P^g_{n/2}(\phi) = e^{-nu} P^{g_0}_{n/2}(\phi). \tag{7}$$

The operator $P^g_{n/2}$ is an elliptic, self-adjoint operator with leading term $(-\Delta_g)^{n/2}$. So in particular it is fourth order in dimension 4. This fourth order operator in dimension 4 is the Paneitz operator. We next introduce some notation to enable us to write down the explicit form of the Paneitz operator. Let d denote the exterior derivative operator and δ its adjoint, the divergence operator. Next we denote by,

$$J = \frac{R_g}{6},$$

where R_g denotes the scalar curvature of the background metric. We also set

$$P = \frac{1}{2}(\mathrm{Ric}_g - Jg).$$

Ric_g denotes the Ricci curvature. The Paneitz operator in dimension 4 is defined as

$$P^g_2(\phi) = \Delta^2_g \phi + \delta(2Jg - 4P)d\phi.$$

In even dimensions the critical operators $P^g_{n/2}$ are all differential operators. Further details maybe found in [1].
In contrast to the situation in even dimensions where the critical operators $P^g_{n/2}$ are differential operators, the critical operators in odd dimensions are non-local pseudo-differential operators. This is seen easily because the leading term of $P^g_{n/2}$ is a fractional power of the Laplace operator given by $(-\Delta_g)^{n/2}$. One may consult [3] for the role in Conformal Geometry of the fractional operators that arise. All of these critical operators in all dimensions reduce to $(-\Delta)^{n/2}$ with no lower order terms in the case of the flat metric, see [11].
We have the following proposition whose proof follows the same scheme as the two dimensional case where instead we use (7). The proposition leads to a higher order free boundary problem involving now the critical GJMS operator. In dimension 4 the operator that arises is the Paneitz operator and for odd n the operator is a fractional operator leading to fractional free boundary problems which are unstable. The Paneitz operator and its conformal covariant properties seems to have been observed in the Physics literature also, see [8] eqn. (5.12) and [9], eqn. (28).

Proposition 4.

Consider (Ω^n, g_0), with metrics g conformal to g_0 via the relation (1) and which lie in class C. Then the problem,

$$\inf_{g \in C} \inf_{\phi \in H^{n/2}_0(\Omega)} \frac{\int_\Omega P^g_{n/2} \phi \phi \, dV_g}{\int_\Omega |\phi|^2 dV_g},$$

is equivalent to

$$\inf_{\int_\Omega \rho dV_{g_0} = M} \inf_{\phi \in H_0^{n/2}(\Omega)} \frac{\int_\Omega P_{n/2}^{g_0} \phi\phi \, dV_{g_0}}{\int_\Omega |\phi|^2 \rho dV_{g_0}}.$$

with given $M > 0$, $\lambda = e^{-nA} > 0$, $\Lambda = e^{nA} < \infty$ *and* $0 < \lambda \leq \rho \leq \Lambda < \infty$ *and where* $\rho = e^{nu}$.

We end with some questions that arise naturally in the higher dimensional case and which are natural analogs of the results in two dimensions.

1. Establish the existence of a minimizing pair (ϕ, ρ).
2. What effect does the symmetry of the domain have on the uniqueness and symmetry of the functions ϕ, ρ ?
3. Is there a natural free boundary problem associated to the higher dimensional case? If so what is the regularity of the free boundary and the associated eigenfunction ϕ ?
4. Is there a natural symmetry breaking phenomena in the higher dimensional case, analogous to the two dimensional case [4] ? Specifically, are there examples of domains which have a symmetry axis, for which the eigenfunctions are not symmetric with respect to this axis? This will establish that minimizers are not unique.

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